Generalized Hypo-Elasticity with Thermal Influence II. Failure Criteria and Failure Slips

By

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The fracture and yield criteria and the associated slips and flows of stretching and temperature rate are investigated theoretically on the basis of the thermo-hypo-elastic materials defined in the first paper of this series. A ten-dimensional $\mathcal{T}$-space is introduced. The failure criteria and the failure slips are classified naturally into three types: pure mechanical shear, mechanico-thermal normal and heat conduction types. The heat conduction failure is analysed in particular. Its failure surface in the principal stress space has three-fold symmetry and its failure slip is, in general, parallel to a principal axis of stress.

1. Introduction

In the preceding article[1] the author proposed the constitutive equations (I.2.16 b, c and d)+ of a *thermo-hypo-elastic material of free of past history*. In this paper the failure conditions, which were investigated by the author,[2),3),4)] are generalized to include thermal influence.

2. Ten-Dimensional Representation of Constitutive Equations

Consider six-, one- and three-dimensional inner-product spaces $\mathscr{S}_6$, $\mathscr{S}_1$ and $\mathscr{S}_3$ of symmetric tensors, scalars and vectors with the following inner-products:

$$(\boldsymbol{S}_1,\ \boldsymbol{S}_2)=\mathrm{tr}\,(\boldsymbol{S}_1\boldsymbol{S}_2), \tag{2.1a}$$

$$(s_1,\ s_2)=s_1s_2, \tag{2.1b}$$

$$(\boldsymbol{s}_1,\ \boldsymbol{s}_2)=\boldsymbol{s}_1\cdot\boldsymbol{s}_2, \tag{2.1c}$$

where $\boldsymbol{S}_\Gamma$, s_Γ and $\boldsymbol{s}_\Gamma$ $(\Gamma=1,2)$ are any elements in $\mathscr{S}_6$, $\mathscr{S}_1$ and $\mathscr{S}_3$, respectively.

Now consider a ten-dimensional inner-product space $\mathscr{S}_{10}$ defined by

$$\mathscr{S}_{10}=\mathscr{S}_6\oplus\mathscr{S}_1\oplus\mathscr{S}_3, \tag{2.2}$$

where the element of $\mathscr{S}_{10}$ are ordered set:

$$\mathfrak{S}=(\boldsymbol{S},\ s,\ \boldsymbol{s}). \tag{2.3}$$

The product $\alpha\mathfrak{S}=\mathfrak{S}\alpha$, where α is a scalar, and the sum $\mathfrak{S}_1+\mathfrak{S}_2$, where $\mathfrak{S}_\Gamma=$

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+ (I.2.16) denotes Equation (2.16) of the preceding paper.[1]

$(\boldsymbol{S}_\Gamma, s_\Gamma, \boldsymbol{s}_\Gamma)$ $(\Gamma=1,2)$, are defined, respectively, by

$$\alpha\mathfrak{S}=(\alpha\boldsymbol{S}, \alpha s, \alpha\boldsymbol{s}), \tag{2.4}$$

$$\mathfrak{S}_1+\mathfrak{S}_2=(\boldsymbol{S}_1+\boldsymbol{S}_2, s_1+s_2, \boldsymbol{s}_1+\boldsymbol{s}_2). \tag{2.5}$$

Then the inner-product over $\mathscr{S}_{10}$ is given by

$$(\mathfrak{S}_1, \mathfrak{S}_2)=\mathrm{tr}(\boldsymbol{S}_1\boldsymbol{S}_2)+s_1s_2+\boldsymbol{s}_1\cdot\boldsymbol{s}_2. \tag{2.6}$$

The ten-dimensional representation of the constitutive equations (I.2.16 b, c and d) reduces to the single form:

$$\mathring{\mathfrak{T}}=\mathfrak{H}(\mathfrak{T})\{\mathfrak{D}\}, \tag{2.7}$$

where

$$\mathring{\mathfrak{T}}=(\mathring{\boldsymbol{T}}, \dot{\eta}, \mathring{\boldsymbol{q}}), \tag{2.8a}$$

$$\mathfrak{T}=(\boldsymbol{T}, \theta, \boldsymbol{0}), \tag{2.8b}$$

$$\mathfrak{D}=(\boldsymbol{D}, \dot{\theta}, \boldsymbol{g}) \tag{2.8c}$$

and $\mathfrak{H}$ may be regarded as a linear operator on $\mathscr{S}_{10}$.

3. Failure Criteria and Failure Slips

For given values of $\mathfrak{T}$ and $\mathfrak{D}$ relation (2.7) assigns a unique value of $\mathring{\mathfrak{T}}$. On the other hand for given values of $\mathfrak{T}$ and $\mathring{\mathfrak{T}}$ there may or may not be assigned a unique value of $\mathfrak{D}$. Now we propose.

Definition. *The condition on $\mathfrak{T}$ that $\mathfrak{H}(\mathfrak{T})$ be singular is called a "failure criterion", and non-zero element of the null space of $\mathfrak{H}(\mathfrak{T})$ is called a "failure slip".*

Generalized stress $\mathfrak{T}$ is confined into a seven-dimensional inner-product space $\mathscr{S}_7=\mathscr{S}_6+\mathscr{S}_1$, then from the above Definition we can say, in general, that a failure criterion is expressed as a six-dimensional hypersurface, which is called the *failure surface* in $\mathscr{S}_7$.

4. $\mathscr{T}$-Space

Consider the six- and three-dimensional inner-product spaces $\mathscr{T}_6$ and $\mathscr{T}_3$, where $\mathscr{T}_6$ has the orthonormal basis $\{\boldsymbol{e}_\alpha\}$ $(\alpha=1,\ldots\ldots, 6)$:

$$\boldsymbol{e}_k=\frac{1}{\sqrt{2}}(\boldsymbol{v}_{k+1}\otimes\boldsymbol{v}_{k+2}+\boldsymbol{v}_{k+2}\otimes\boldsymbol{v}_{k+1}), \ (\text{mod.}=3), \tag{4.1a}$$

$$\boldsymbol{e}_{k+3}=\boldsymbol{v}_k\otimes\boldsymbol{v}_k, \qquad \text{(not summed on } k) \tag{4.1b}$$

where $\boldsymbol{v}_k$ $(k=1, 2, 3)$ are unit proper vectors of a given $\boldsymbol{T}$; and where $\mathscr{T}_3$ has the orthonormal basis $\{\boldsymbol{v}_k\}$. Then the inner-products on $\mathscr{T}_6$ and $\mathscr{T}_3$ can be given, respectively, by the usual scalar products of *image vectors* on $\mathscr{S}_6$ and $\mathscr{S}_3$. We can say that by the theorem of linear algebra $\mathscr{S}_6$ and $\mathscr{S}_3$ are, respectively, isometric with $\mathscr{T}_6$ and $\mathscr{T}_3$ and so $\mathscr{S}_{10}$ is also isometric with

$$\mathscr{T}_{10}=\mathscr{T}_6\oplus\mathscr{S}_1\oplus\mathscr{T}_3. \tag{4.2}$$

In $\mathscr{T}$-space representation the constitutive relations (2.7) are shown by

$$\mathring{\mathfrak{t}}=\mathfrak{h}(\mathfrak{t})\mathfrak{d}, \tag{4.3}$$

where $\mathring{\mathfrak{t}}$, $\mathfrak{t}$ and $\mathfrak{d}$ are, respectively, ten-dimensional image vectors on $\mathscr{T}_{10}$ of $\mathring{\mathfrak{T}}$, $\mathfrak{T}$ and $\mathfrak{D}$.

Applying the Definition in section 3 on (4.3) we can easily say that the failure criterion is shown by

$$\det \mathfrak{h}(\mathfrak{t})=0 \tag{4.4}$$

and the failure slip is a nontrivial solution $\bar{\mathfrak{d}}$ of the homogeneous equation:

$$\mathfrak{h}(\mathfrak{t})\bar{\mathfrak{d}}=\mathbf{0}. \tag{4.5}$$

From the representations of isotropic functions (I.4.1)—(I.4.3) and $\mathscr{T}$-space representation we can easily obtain

$$\|\mathfrak{h}\| \equiv \left\| \begin{matrix} \|\mathfrak{h}_S\| & \|\mathbf{0}\| & \|\mathbf{0}\| \\ \|\mathbf{0}\| & \|\mathfrak{h}_N\| & \|\mathbf{0}\| \\ \|\mathbf{0}\| & \|\mathbf{0}\| & \|\mathfrak{h}_C\| \end{matrix} \right\|, \tag{4.6}$$

where

$$\|\mathfrak{h}_S\| \equiv \left\| \begin{matrix} S_{23} & 0 & 0 \\ 0 & S_{31} & 0 \\ 0 & 0 & S_{12} \end{matrix} \right\|, \tag{4.7a}$$

$$\|\mathfrak{h}_N\| \equiv \left\| \begin{matrix} S_{11}+N_{11} & N_{12} & N_{13} & H_1 \\ N_{21} & S_{22}+N_{22} & N_{23} & H_2 \\ N_{31} & N_{32} & S_{33}+N_{33} & H_3 \\ M_1 & M_2 & M_3 & M \end{matrix} \right\|, \tag{4.7b}$$

$$\|\mathfrak{h}_C\| \equiv \left\| \begin{matrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{matrix} \right\|, \tag{4.7c}$$

and where

$$S_{km}\equiv\alpha_1+\alpha_2(t_k+t_m)+\alpha_3(t_k^2+t_m^2), \tag{4.8a}$$

$$N_{kp}\equiv\alpha_4+\alpha_7 t_k+\alpha_5 t_p+\alpha_{10} t_k^2+\alpha_8 t_k t_p+\alpha_6 t_p^2+\alpha_9 t_k t_p^2+\alpha_{11} t_k^2 t_p$$
$$+\alpha_{12} t_k^2 t_p^2, \tag{4.8b}$$

$$H_k\equiv\alpha_{13}+\alpha_{14} t_k+\alpha_{15} t_k^2, \tag{4.8c}$$

$$M_k\equiv\beta_1+\beta_2 t_k+\beta_3 t_k^2, \tag{4.8d}$$

$$M\equiv\beta_4, \tag{4.8e}$$

$$P_k\equiv\gamma_1+\gamma_2 t_k+\gamma_3 t_k^2, \tag{4.8f}$$

t_k ($k=1, 2, 3$) are principal values of $\boldsymbol{T}$, and k, m and p are not summed.

Then the failure conditions (4.4) and (4.5) are separated into three types, *shear*, *normal* and *heat conduction failure conditions*, which are given, respectively, by

$$\det \mathfrak{h}_S=0, \qquad \mathfrak{h}_S\bar{\mathfrak{d}}_S=\mathbf{0}, \tag{4.9a}$$

$$\det \mathfrak{h}_N=0, \qquad \mathfrak{h}_N\bar{\mathfrak{d}}_N=\mathbf{0}, \tag{4.9b}$$

$$\det \mathfrak{h}_C=0, \qquad \mathfrak{h}_C\bar{\mathfrak{d}}_C=\mathbf{0}. \tag{4.9c}$$

The failure slip $\bar{\mathfrak{d}}_S$ consists in the pure mechanical shear components of stretching $\boldsymbol{D}$ while $\bar{\mathfrak{d}}_N$ consists in the pure mechanical normal components of $\boldsymbol{D}$ and the temperature rate slip. These two failure conditions were discussed by Tokuoka.[4] In the next section we will discuss briefly the heat conduction failure.

5. Heat Conduction Failure

From the definition of $\mathscr{T}_6$-space and (2.8b) $\mathfrak{t}$ is a vector on the four-dimensional space $\mathscr{T}_N$, which is a direct sum of a three-dimensional space defined by $\{\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6\}$ and $\mathscr{S}_1$, while $\bar{\mathfrak{d}}_N$ is a vector on $\mathscr{T}_3$-space. Then from (4.9c) we can say that

Theorem 1. *The heat conduction failure surface is at most a three-dimensional manifold on $\mathscr{T}_N$, while the heat conduction slip is a vector on $\mathscr{T}_3$.*

From (4.7c) and (4.9c) we have the failure surface:

$$P_k\equiv\gamma_1+\gamma_2 t_k+\gamma_3 t_k^2=0 \tag{5.1}$$

and the associated failure slip:

$$\boldsymbol{g}=\psi\boldsymbol{v}_k, \tag{5.2}$$

where k is not summed and ψ is any scalar.

A permutation in t_1, t_2 and t_3 causes the same interchanges of the failure surfaces $P_1=0$, $P_2=0$ and $P_3=0$. Then we can assert that

Theorem 2. *The failure surface consists in three branches and has three-fold symmetry around the pressure axis in the normal stress space $\mathscr{T}_N'$.*

Here $\mathscr{T}_N'=\mathscr{T}_N\ominus S_1$ and the pressure axis is a line passing through the origin and making an equal angle with three coordinate axes of the principal stress.

When a state of $\boldsymbol{T}$ and θ satisfies simultaneously

$$P_k=P_m=0 \qquad (k\neq m) \tag{5.3}$$

or

$$P_1=P_2=P_3=0, \tag{5.4}$$

the associated failure slip is given by

$$\boldsymbol{g}=\psi\boldsymbol{v}_k+\phi\boldsymbol{v}_m \tag{5.5}$$

or

$$\boldsymbol{g}=\text{any vector}, \tag{5.6}$$

respectively, where ψ and ϕ are any scalar. Then we can say that

Theorem 3. *The heat conduction failure slip may be on one-, two- and three-dimensional spaces spaned by one, two and three principal axes of stress according as the state situates* *on a single failure surface, and on cross-sections of two and three failure surfaces, respectively.*

Expanding the scalar functions γ_1, γ_2 and γ_3 of (5.1) with respect to three invariants of stress

$$p, \quad II^* \equiv t_2^* t_3^* + t_3^* t_1^* + t_1^* t_2^*, \quad III^* \equiv t_1^* t_2^* t_3^*, \tag{5.7}$$

where

$$t_k^* = t_k + p, \qquad p \equiv -\frac{1}{3}\mathrm{tr}(\boldsymbol{T}) \tag{5.8}$$

are, respectively, the deviatoric stress and the pressure, and restricting P_k to grade two in the stress, we have

$$\gamma_1 = a_1 + a_2 p + a_3 II^*, \tag{5.9a}$$

$$\gamma_2 = a_4 + a_5 p, \tag{5.9b}$$

$$\gamma_3 = a_6, \tag{5.9c}$$

where a's are scalar functions of θ. Substituting (5.9) into P_1, we have

$$P_1 = a_1 + (a_2 - a_4) p + (-a_5 + a_6) p^2 + (a_5 - 2a_6) p t_1^* + a_4 t_1^* + a_3 II^* + a_6 s_1^{*2}. \tag{5.10}$$

Now let us introduce rectangular coordinates x, y and z on $\mathscr{T}_N'$ such that

$$x = \frac{1}{\sqrt{2}}(t_3^* - t_2^*), \tag{5.11a}$$

$$y = \sqrt{\frac{3}{2}}\, t_1^*, \tag{5.11b}$$

$$z = -\sqrt{3}\, p. \tag{5.11c}$$

Substituting (5.11) into (5.10), we have

$$c_1 x^2 + c_2 y^2 + c_3 y + c_4 yz + c_5 z^2 + c_6 z + c_7 = 0, \tag{5.12}$$

where

$$c_1 \equiv \frac{a_3}{2}, \qquad c_2 \equiv \frac{a_3}{2} - \frac{2}{3} a_6, \qquad c_3 \equiv \sqrt{\frac{2}{3}}\, a_4, \qquad c_4 \equiv \frac{\sqrt{2}}{3}(a_5 - 2a_6),$$

$$c_5 \equiv \frac{1}{3}(a_5 - a_6), \qquad c_6 \equiv \frac{1}{\sqrt{3}}(a_2 - a_4), \qquad c_7 \equiv -a_1. \tag{5.13}$$

Failure surface (5.12) represents a quadric on $\mathscr{T}_N'$. When the failure criterion is pressure-insensitive, we have

$$c_1 (x^2 + y^2) + c_3 y + c_7 = 0. \tag{5.14}$$

References

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