# Some Considerations on the State Equations of Linear Active Networks and the Network Topology

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**Abstract**

The state equations as a characterization of a linear active network involve two factors, the network topology and the properties of the network elements. In this paper the way these two factors influence the formulation of the state equations is studied. A set of state equations for the networks with certain topological restrictions is derived based on an overnormal tree. Furthermore the order of complexity of a linear active network is investigated in connection with the network topology, and an upper bound on the order of complexity is stated with respect to a particular common tree. Several examples are given to illustrate the investigation.

## 1. Introduction

The state equations as the description of a linear active network involve the network topology and the properties of the network elements. The process of deriving the state equations, however, are usually very complicated, and the relation between the state equations obtained and the original network is difficult to observe.[1),2),3),4)] One of the reasons for this difficulty is the fact that the order of complexity of an active network depends not only on the network topology but also on the element values such as resistances, capacitances, inductances or proportinality constants (mutual resistances or conductances) of the controlled sources. Moreover the kind of elements in a general active network is so great that the equations appearing in the process look complex, which prevents us from seeing the total view of the derivation. In this paper we investigate how the topological properties of the network and the properties of the network elements are combined to formulate the state equations. We try to make the equations which are used to derive the state equations, as simple as possible so that the relation between the relevant equations and the network topology can be observed.

The network we are considering contains only capacitors, inductors, resistors, independent sources, current-controlled voltage sources (current sensor-controlled

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voltage source pairs) and voltage-controlled current sources (voltage sensor-controlled current source pairs). The elements such as transformers, gyrators, negative-impedance converters which appear in a general active network can be replaced by proper controlled sources. Furthermore a current-controlled current source can be replaced by the cascade connection of a current-controlled voltage source and a voltage-controlled current source. A voltage-controlled voltage source can be replaced similarly. Thus we do not suffer loss of generality due to the above restriction on the element kind in the network we are considering. Later we also replace the resistors and inductors.

## 2. Overnormal Tree and Network Equations

We now try to find a tree denoted $T$ of the network having the following properties.[4),5)]

1. $T$ constains a maximum of independent voltage sources.
2. $T$ contains a maximum of current sensors and then controlled voltage sources consistent with Property 1.
3. $T$ contains a maximum of capacitors consitent with Properties 1 and 2.
4. $T$ contains a minimum of independent current sources.
5. $T$ contains a minimum of voltage sensors and then controlled current sources consistent with Property 4.
6. $T$ contains a minimum of inductors consistent with Properties 4 and 5.

Such a tree is called an overnormal tree.[5)] The cotree of $T$ is denoted $CT$.

For the usual network an overnormal tree contains all the independent voltage sources and the current sensors. (The voltage across a current sensor is zero). Its cotree contains all the independent current sources and voltage sensors. (The current through a voltage sensor is zero.) Moreover we assume that there is no voltage source in $CT$ or current source in $T$, and thus there is no voltage source only loop or current source only cutset in the network if the current sensors and voltage sensors are regarded as short and open circuits, respectively. There is no restriction such as a current sensor in series with or a voltage sensor in parallel with a passive element. The elements in $T$, that is, the independent voltage sources, current sensors, controlled voltage sources, capacitors, resistors and inductors in $T$ are denoted by $E$, $\alpha$, $\beta$, $C$, $G$ and $\Gamma$, respectively. The elements in $CT$, that is, the independent current sources, voltage sensors, controlled current sources, inductors, resistors and capacitors in $CT$ are denoted by $J$, $\gamma$, $\delta$, $L$, $R$ and $S$.

Now in order to make the form of the network equations simpler we replace $G$ and $R$ by equivalent controlled sources as shown in Fig. 1 (a) and (b).

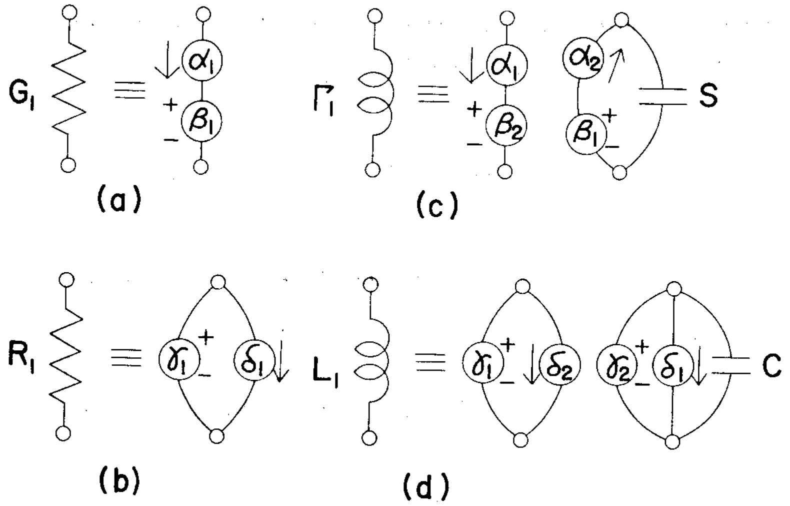


Fig. 1. Equivalent circuits for resistors and inductors.

The network equations consist of the equilibrium equations obtained by the application of Kirchhoff's voltage and current laws to the network, and of the voltage-current relations of the elements. They are

$$\begin{pmatrix} v_\gamma \\ v_L \\ v_S \end{pmatrix} = -\begin{pmatrix} B_{\gamma\Gamma} & B_{\gamma C} & B_{\gamma\beta} & B_{\gamma E} \\ B_{L\Gamma} & B_{LC} & B_{L\beta} & B_{LE} \\ 0 & B_{SC} & B_{S\beta} & B_{SE} \end{pmatrix} \begin{pmatrix} v_\Gamma \\ v_C \\ v_\beta \\ v_E \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} i_\alpha \\ i_C \\ i_\Gamma \end{pmatrix} = -\begin{pmatrix} C_{\alpha S} & C_{\alpha L} & C_{\alpha\delta} & C_{\alpha J} \\ C_{CS} & C_{CL} & C_{C\delta} & C_{CJ} \\ 0 & C_{\Gamma L} & C_{\Gamma\delta} & C_{\Gamma J} \end{pmatrix} \begin{pmatrix} i_S \\ i_L \\ i_\delta \\ i_J \end{pmatrix} \tag{2}$$

$$p\begin{pmatrix} v_C \\ v_S \\ i_L \\ i_\Gamma \end{pmatrix} = \begin{pmatrix} C_C^{-1} & & & 0 \\ & C_S^{-1} & & \\ & & L_L^{-1} & \\ 0 & & & L_\Gamma^{-1} \end{pmatrix} \begin{pmatrix} i_C \\ i_S \\ v_L \\ v_\Gamma \end{pmatrix}, \tag{3}$$

and

$$\begin{bmatrix} v_\beta \\ i_\delta \end{bmatrix} = \begin{bmatrix} r_{\beta\alpha} & 0 \\ 0 & g_{\delta\gamma} \end{bmatrix} \begin{bmatrix} i_\alpha \\ v_\gamma \end{bmatrix}. \tag{4}$$

where $v.$ and $i.$ are the voltage and current vectors respectively associated with the elements as noted by the subscripts. $B..$ and $C..$ are the submatrices of the characteristic parts of the fundamental loop and cutset matrices respectively associated with the elements as noted by the subscripts. $C_C$, $C_S$, $L_L$, $L_\Gamma$, $r_{\beta\alpha}$, and

$g_{\delta\tau}$ are diagonal matrices of the element values (the capacitances, inductances and proportionality constants) as are respectively indicated by the subscripts.

Since we are mainly concerned with the state equations, we are satisfied without immediately knowing the currents through the voltage sources or the voltages across the current sources. Therefore the equations for these variables are not shown.

### 3. State Equations and the Order of Complexity

We first eliminate $i_\alpha$ and $v_\tau$ from the equations in (4) and the first equations in (1) and (2). Then we have

$$Pv_\beta = -C_{\alpha S}i_S - C_{\alpha L}i_L - C_{\alpha J}i_J + C_{\alpha\delta}g_{\delta\tau}(B_{\tau\Gamma}v_\Gamma + B_{\tau C}v_C + B_{\tau E}v_E) \tag{5}$$

$$Qi_\delta = -B_{\tau\Gamma}v_\Gamma - B_{\tau C}v_C - B_{\tau E}v_E + B_{\tau\beta}r_{\beta\alpha}(C_{\alpha S}i_S + C_{\alpha L}i_L + C_{\alpha J}i_J) \tag{6}$$

where

$$P = r_{\beta\alpha}{}^{-1} - C_{\alpha\delta}g_{\delta\tau}B_{\tau\beta} \tag{7}$$

and

$$Q = g_{\delta\tau}{}^{-1} - B_{\tau\beta}r_{\beta\alpha}C_{\alpha\delta}. \tag{8}$$

We observe that $P$ or $Q$ becomes singular if and only if the proportionality constants satisfy a certain relation specified by the network topology. The cases where this occurs are rather special and we distinguish them from the cases where the rank of a matrix decreases due to the network topology only. If the element values can be chosen arbitrarily, $P$ and $Q$ can be made nonsingular and we can solve (5) and (6) for $v_\beta$ and $i_\delta$. Then from the third equations in (1) and (2) we obtain the relations between $v_S$, $i_\Gamma$ and $i_S$, $v_\Gamma$, which can be written in the form

$$\begin{bmatrix} v_S \\ i_\Gamma \end{bmatrix} = H_{S\Gamma}\begin{bmatrix} i_S \\ v_\Gamma \end{bmatrix} + H_{CL}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + H_{EJ}\begin{bmatrix} v_E \\ i_J \end{bmatrix}. \tag{9}$$

In the above equations

$$H_{S\Gamma} = \begin{bmatrix} B_{S\beta} & 0 \\ 0 & C_{\Gamma\delta} \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} U & -C_{\alpha\delta}g_{\delta\tau} \\ -B_{\tau\beta}r_{\beta\alpha} & U \end{bmatrix}\begin{bmatrix} C_{\alpha S} & 0 \\ 0 & B_{\tau\Gamma} \end{bmatrix}, \tag{10}$$

and if this matrix is nonsingular, then (10) can be solved for $i_S$ and $v_\Gamma$. Eliminating these variables together with $i_C$ and $v_L$ using equations in (3), we obtain the state equations. The order of complexity for this case is the sum of the numbers of capacitors and inductors. If matrix $H_{S\Gamma}$ is singular and the nullity of the matrix is $\mu$, there must be $\mu$ relations between $v_S$, $i_\Gamma$, $v_C$ and $i_L$ (Possibly some of these variables become constant.). Thus an upper bound on the order of complexity is given by the sum of the rank of $H_{S\Gamma}$ and the numbers of $C$ and $L$. There are four matrices on the right-hand side of (10). The second and third matrices are square and become singular only if there are certain relations among the proportinality constants. The rank of the other two matrices can be determined only by considering the network topology. Now let us define two graphs. The voltage graph, denoted $G_v$, is the graph obtained from the graph $G$ representing the original network by short circuiting $E$ and $\alpha$, and open circuiting $J$ and $\delta$. The current graph denoted $G_i$ is defined as the graph obtained from $G$ by open circuiting $J$ and $\gamma$, and short circuiting $E$ and $\beta$. The replacement of the resistors in the original network induces the corresponding replacement in $G_v$ and $G_i$, also. Furthermore we define $G_{vS}$ as the graph obtained from $G_v$ by short circuiting $C$ and $\Gamma$, and open circuiting $\beta$, $\gamma$ and $L$, leaving $S$ only. The rank of $B_{S\beta}$ is equal to the rank of $G_{vS}$. (See Apendix I. The rank of $B_{S\beta}$ can be stated with respect to some other graph.) Similarly the rank of $C_{\alpha S}$ is equal to the rank of $G_{iS}$, the graph obtained from $G_i$ by short circuiting $C$ and $\Gamma$, and open circuiting $\alpha$, $\delta$ and $L$. We can also get $G_{v\Gamma}$ from $G_v$ by open circuiting $L$ and $S$, and short circuiting $\beta$, $\gamma$ and $S$, and graph $G_{i\Gamma}$ from $G_i$ by open circuiting $L$ and $S$, and short circuiting $\alpha$, $\delta$ and $C$. Then we have the ranks of $B_{\gamma\Gamma}$ and $C_{\Gamma\delta}$ are equal to the nullities of $G_{v\Gamma}$ and $G_{i\Gamma}$ respectively. Thus

$$\text{rank of } H_{S\Gamma} \leq \min\ (\text{rank of } G_{vS} + \text{nullity of } G_{i\Gamma},\ \text{rank of } G_{iS} + \text{nullity of } G_{v\Gamma}). \tag{11}$$

Now in order to make the following discussions simpler we replace all the inductors in the network by use of the equivalent circuits as shown in Fig. 1. (c) and (d). Then the terms with subscripts $L$ and $\Gamma$ in (1), (2), (3) and (4) disappear. From the second equation in (2) and also (5), (6) and (9) without the terms concerning inductors, we obtain the following equation.

$$\begin{bmatrix} B_{S\beta}P^{-1}C_{\alpha S} & 0 \\ C_{CS}+C_{C\delta}Q^{-1}B_{\gamma\beta}r_{\beta\alpha}C_{\alpha S} & U \end{bmatrix}\begin{bmatrix} i_S \\ i_C \end{bmatrix}=\begin{bmatrix} U & B_{SC}+B_{S\beta}P^{-1}C_{\alpha\delta}g_{\delta\gamma}B_{\gamma C} \\ 0 & C_{C\delta}Q^{-1}B_{\gamma C} \end{bmatrix}\begin{bmatrix} v_S \\ v_C \end{bmatrix}$$
$$+\begin{bmatrix} B_{SE}+B_{S\beta}P^{-1}C_{\alpha\delta}g_{\delta\gamma}B_{\gamma E} & B_{S\beta}P^{-1}C_{\alpha J} \\ C_{C\delta}Q^{-1}B_{\gamma E} & C_{CJ}+C_{C\delta}Q^{-1}B_{\gamma\beta}r_{\beta\alpha}C_{\alpha J} \end{bmatrix}\begin{bmatrix} v_E \\ i_J \end{bmatrix} \tag{12}$$

We are now going to derive a set of state equations from (12). The rank of the matrix in the left-hand side of (12) is the sum of the rank, denoted $r$, of $B_{S\beta}P^{-1}C_{\alpha S}$ and of the number of $C$. If there exists $(r \times r)$ nonsingular principal minor matrix in $B_{S\beta}P^{-1}C_{\alpha S}$, we procede as follows. First dividing $S$ into two sets $s_1$ and $s_2$ such that the capacitors in $s_2$ correspond to the principal minor matrix, we can write (12) as (13), where $A_{22}$ is nonsingular.

$$\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & U \end{pmatrix}\begin{pmatrix} i_{s_1} \\ i_{s_2} \\ i_C \end{pmatrix}=\begin{pmatrix} U & 0 & B_{13} \\ 0 & U & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix}\begin{pmatrix} v_{s_1} \\ v_{s_2} \\ v_C \end{pmatrix}+\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{32} & C_{33} \end{pmatrix}\begin{bmatrix} v_E \\ i_J \end{bmatrix} \tag{13}$$

As the result of the elimination of $i_{s_1}$, $i_{s_2}$ and $i_C$ by use of (3), and then of $v_{s_1}$, we have

$$\begin{bmatrix} A_{22}C_{s_2}+A_{21}C_{s_1}A_{12}A_{22}^{-1} & A_{21}C_{s_1}(-B_{13}+A_{12}A_{22}^{-1}B_{23}) \\ A_{32}C_{s_2}+A_{31}C_{s_1}A_{12}A_{22}^{-1} & A_{31}C_{s_1}(-B_{13}+A_{12}A_{22}^{-1}B_{23})+C_G \end{bmatrix} \begin{bmatrix} pv_{s_2} \\ pv_G \end{bmatrix}$$
$$=\begin{bmatrix} U & B_{23} \\ 0 & B_{33} \end{bmatrix}\begin{bmatrix} v_{s_2} \\ v_G \end{bmatrix}+\begin{bmatrix} C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}\begin{bmatrix} v_E \\ i_J \end{bmatrix}$$
$$-\begin{bmatrix} A_{21}C_{s_1}(C_{11}-A_{12}A_{22}^{-1}C_{21}) & A_{21}C_{s_1}(C_{12}-A_{12}A_{22}^{-1}C_{22}) \\ A_{31}C_{s_1}(C_{11}-A_{12}A_{22}^{-1}C_{21}) & A_{31}C_{s_1}(C_{12}-A_{12}A_{22}^{-1}C_{22}) \end{bmatrix}\begin{bmatrix} pv_E \\ pi_J \end{bmatrix}. \tag{14}$$

The matrix on the left-hand side of (14) is nonsingular unless there exists a special relation among the element values. No further decrease in the order of complexity is caused by reason of the network topology only. (See Appendix II.) A set of state equations can be obtained by multiplying the inverse of the matrix to both side of the equation. Note that only the first derivative terms of the independent sources may appear in the state equations. If the matrix in the left-hand side of (14) is singular, the process used to derive (14) from (13) is repeated, and each repetition of the process may yield higher order derivatives of the independent sources.

Even if the rank of $B_{S\beta}P^{-1}C_{\alpha S}$ is $r$, there does not always exist $(r\times r)$ nonsingular principal minor matrix. For example, if $C_2$ in the network shown in Fig. 2. (a) is included in an overnormal tree, $B_{S\beta}P^{-1}C_{\alpha S}$ becomes

$$\begin{pmatrix} 0 & r_2 & r_2+r_3 & r_2+r_3+r_4 \\ 0 & 0 & 0 & 0 \\ 0 & r_1 & r_1 & r_1 \\ 0 & r_1 & r_1+r_3 & r_1+r_3 \end{pmatrix}.$$

The rank of this matrix is easily seen to be 3, but there is no $(3\times 3)$ nonsingular principal minor matrix. The largest nonsingular principal minor matrix is a $(1\times 1)$ matrix. There is a capacitor in the tree, and the order of complexity of this network is 2. There is no $(r\times r)$ nonsingular principal minor matrix for any overnormal tree of the network. For this kind of network the derivation

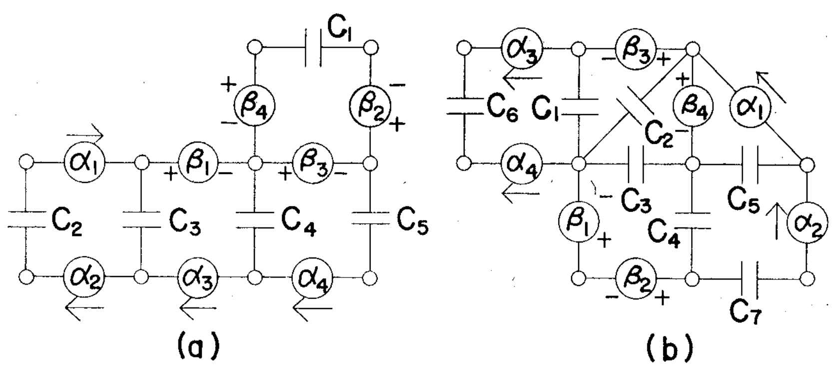


Fig. 2. (a) Example 1, (b) Example 2.

of state equations needs more complex process. For some networks the existence of a $(r \times r)$ nonsingular principal minor matrix depends on the choice of the overnormal tree. If $C_1$ in the network shown in Fig. 2. (b) (Example 2) is included in T, the rank of $B_{S\beta}P^{-1}C_{\alpha S}$ is 2, but there is no $(2 \times 2)$ nonsingular principal minor matrix. Some other overnormal tree, however, say a tree with $C_7$ as a tree-element, leads to the matrix with a $(2 \times 2)$ nonsingular principal minor matrix. The order of complexity of this network is 3. It may be interesting to note that for the network obtained from the network in Fig. 2. (a) by exchanging $\alpha_3$ and $\beta_3$, we can get a $(r \times r)$ nonsingular principal minor matrix for any possible overnormal tree. The network is identified as Example 3 for later comments.

Let us now mention the cases where matrix $P$ or $Q$ becomes singular. In these cases we go back to the original network equations. It may or may not be possible to get a set of state equations depending on the network topology and the element values, as are illustrated by the following examples.

*Example* 4. (Fig. 3. (a)) We replace the resistor in the network by the circuit shown in Fig. 1. (b). If $r_1 = R$, $P$ is singular, but a set of state equations can be obtained.

*Example* 5. (Fig. 3. (b)) If $r_1 g_1 = 1$ for the network, $P$ is singular and $v_{\beta 1}$ is indeterminate. No set of state equations exists.

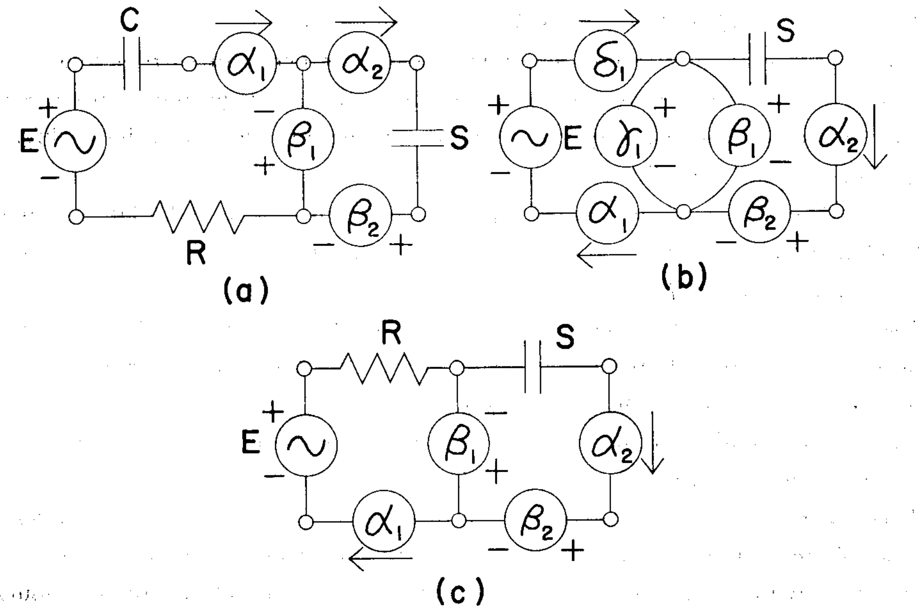


Fig. 3. (a) Example 4, (b) Example 5, (c) Example 6.

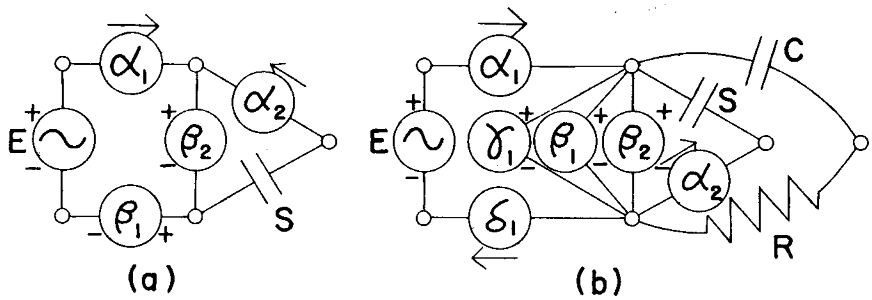


Fig. 4. (a) Example 7, (b) Example 8.

*Example* **6**. (Fig. 3. (c)) If $r_1=R$ for the network, no solution of the network equations is possible except for $v_E=0$.
The order of complexity may or may not decrease compared with the network with the same topology and element kind but with nonsingular $P$ or $Q$. Note that Example 4 serves as a counter example to Theorem 3 of DeClaris[5] if the series connection of $\alpha_2$ and $\beta_2$ is replaced by a resistor.

Now we consider the networks with voltage sources in $CT$ or current sources in $T$. Such networks contain voltage source only loops or current source only cutsets if the current sensors and voltage sensors in the network are regarded as short and open circuits respectively. There are several cases:
*Case* **1**. The state equations can be obtained without special relations among the element values. (Fig. 4. Example 7 and Example 8)
*Case* **2**. Some of the voltages across the elements other than current sources, or some of the currents through the elements other than voltage sources become indeterminate, if there exist special relations among the element values. (Fig. 5. Example 9 with $r_1g_1=1$, $r_1=r_2$)
*Case* **3**. No solution of the network equations is possible for special relations among the element values. (Fig. 4. (c). Example 6 with the resistor replaced by a series connection of $\alpha$ and $\beta$)
*Case* **4**. A set of state equations can be obtained with special relations among the element values. If the special relations are not satisfied, no solution is possible. (Fig. 6. Example 10 with $r_1=r_2$)
*Case* **5**. No solution of the network equations is possible except for special independent sources. (Fig. 7. Example 11)
Cases where some of the voltages other than those across the current sources or some of the currents other than those through the voltage sources become indeterminate without special relations among the element values are also found. (Note that we assume there is no current sensor only loop or voltage sensor only

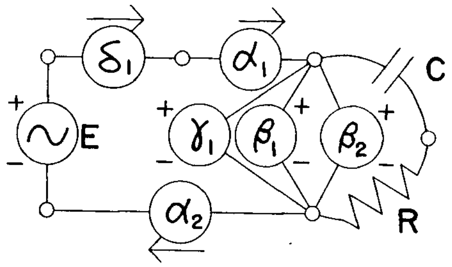


Fig. 5. Example 9.

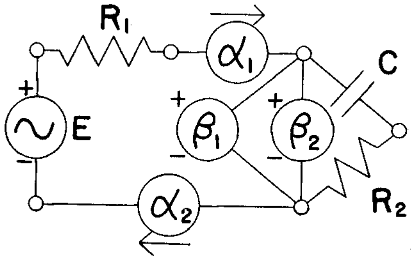


Fig. 6. Example 10.

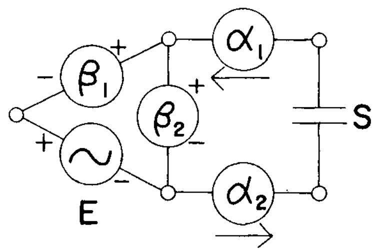


Fig. 7. Example 11.

cutset in the network.) In cases 1 and 4 the order of complexity may decrease or increase if certain special relations among the element values exist. For Example 8 the order of complexity is 2 if $r_1 g_1 = 1$, otherwise it is 1.

## 4. Normal Common Tree and the Order of Complexity

A normal common tree is defined to be a common tree of $G_v$ and $G_i$ such that the sum of the numbers of capacitors in the common tree and inductors in its cotree is maximum. It can be shown that an upper bound on the order of complexity is this maximaum sum, denoted $\sigma_{max}$, for a network without voltage source only loops or current source only cutsets. (A current sensor and a voltage sensor are regarded as a short circuit and an open circuit repectively.) This upper bound is related only to the network topology. The voltages across the capacitors in the normal common tree and the currents through the inductors in its cotree are the possible candidates for the state variables. Now $\sigma_{max}$ is the possible highest degree of the characteristic polynominal of the network given by

$$\sum_{\sigma=0}^{\sigma_{max}} \sum_{T_c^\sigma} (\text{sign of } T_c^\sigma)\,(C_t g_t L_c R_c r_c)\, p^\sigma \tag{15}$$

where $T_c^\sigma$ denotes a common tree such that the sum of the numbers of capacitors in the common tree and inductors in its cotree is $\sigma$, $C_t$, $g_t$, $L_c$, $R_c$ and $r_c$ are the products of the capacitances in $T_c^\sigma$, of the proportionality constants of the voltage-controlled current sources in $T_c^\sigma$, of the inductances in $CT_c^\sigma$ ($CT_c^\sigma$ is the cotree of $T_c^\sigma$.), of the resistances in $CT_c^\sigma$, and of the proportionality constants of the voltage-controlled current sources in $CT_c^\sigma$, respectively. Thus the order of complexity becomes less than $\sigma_{max}$ if there are more than one normal common tree and if the products of the element values in the summation in (15) corresponding to these normal common trees cancel out. Similar observations can be made for the coefficients of $p^\sigma$ for $\sigma<\sigma_{max}$. In many cases the common trees contributing to the coefficient of some degree of $p$ have a part in common with the common trees contributing to the coefficients of the other degrees of $p$, and thus the coefficients become zero at the same time.

Although an overnormal tree always induces a common tree, no simple algorithm to get a normal common tree from an overnormal tree by the tree transformation has been found. Thus the relation between these two trees is not clear, but we can comment on the examples we have given. Capacitor $C_1$ in the network shown as Example 2 cannot be in any normal common tree, but every other capacitor in the network can be in a normal common tree. As for Example 3 the capacitors in any over-normal tree are all included in a normal common tree. As far as the networks we have examined with the above-mentioned topological restrictions, the capacitors in a normal common tree consist of the capacitors in a properly-chosen overnormal tree and those corresponding to a largest nonsingular principal minor matrix of $B_{S\beta}P^{-1}C_{\alpha S}$ with arbitrary element values. Now for Example 4 the maximum degree of the characteristic polynominal decreases from $\sigma_{max}$ by 1, and for Example 5 and 6 the characteristic polynomial becomes zero if $r_1 g_1=1$ and $r_1=\mathrm{R}$ respectively.

The situation is more complicated for the networks with voltage source only loops or current source only cutsets. Extending the definitions of $G_v$ and $G_i$ for these networks, we see that there exists a common tree for Example 7, but no common tree for Examples 8–11. In the network shown as Example 7 there is a current sensor in the loop of the voltage sources. However this is not the necessary condition nor sufficient condition for a set of state equations to exist without special relations among the element values. Note that $\alpha_1$ and $\beta_1$ in Example 7 are connected in such a way that they actually constitute a resistor, and thus effectively there is no voltage source only loop in the network. In general the network topology must be examined in detail to find such a connection. This is apparently related to finding a normal common tree. It seems that in cases where a normal common tree exists the controlled current sources in the common tree and the controlled voltage sources in its cotree are regarded as resistors and otherwise as independent sources concerning the problems of the order of complexity and the solvability.

A normal common tree is a simplified form of Tow's maximal common tree.[6] No efficient algorithm to obtain such a tree has been found, although an overnormal tree can be easily found. Algorithms using Minty's method[7] or some other methods to list all trees modified for the two-graphs have been tried. They are simple but time-consuming.

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### References

1) S. R. Parker and V. T. Barmes: "Existence of Numerical Solutions and the Order of Linear Circuits with Dependent Sources," IEEE trans. on C.T., vol. CT-18, no. 3, 1971.
2) A. Dervisoglu: "State Equations and Initial Values in Active RLC Networks," IEEE trans. on C.T. Sept. 1971.
3) E. J. Purslow: "Solvability and Analysis of Linear Active Networks by Use of the State Equations," IEEE trans. on C.T., vol. CT-17 no. 4, 1970.
4) K. Hirano, F. Nishi and S. Toyama: CT. Monograph of IECEJ. no. CT71-66, Feb. 1972.
5) N. DeClaris and R. Saeks: "Theoretic Foundations of Finite Network and System Theory," Aspects of Network and Systems Theory, Holt, Rinehart and Winston, Inc., 1971.
6) J. Tow, "Order of Complexity of Linear Active Networks," Proc. IEE vol, 115, no. 9, Sept. 1968.
7) G. J. Minty: "A Simple Algorithm for Listing all the Trees of a Graph," IEEE trans. on C.T. vol. CT-12, 1965.

## Appendix I

Submatrix $B_{S\beta}$ of the fundamental loop matrix corresponds to the graph, denoted $G_{vS\beta}$, obtained from $G_v$ by short circuiting all the tree-elements except $\beta$ and open circuiting all the link-elements except $S$. A loop of $G_{vS}$ is the linear combination of the fundamental loops of $G_{vS\beta}$ and contains no $\beta$. Thus the linear combination of the rows of $B_{S\beta}$ corresponding to the fundamental loops is zero, that is these rows are linearly dependent. This means the linearly independent rows of $B_{S\beta}$ should correspond to a tree or a part of a tree in $G_{vS}$. Therefore, rank of $B_{S\beta} \leq$ rank of $G_{vS}$. Now a tree of $G_{vS}$ can be a tree or a part of a tree of $G_{vS\beta}$. Note that $\beta$ form a tree of $G_{vS\beta}$, and thus there must be a nonsingular (rank of $G_{vS} \times$ rank of $G_{vS}$) submatrix in $B_{S\beta}$ corresponding to the tree transformation. Thus rank of $B_{S\beta} \geq$ rank of $G_{vS}$, and with the previous result we have that rank of $B_{S\beta}$ = rank of $G_{vS}$.

Let us consider the graph, $G_{v\beta}$, obtained from $G_{vS\beta}$ by short circuiting $S$. Since

$$\text{rank of } G_{v\beta} + \text{rank of } G_{vS} = \text{rank of } G_{vS\beta} = \text{number of } \beta$$

and

$$\text{rank of } G_{v\beta} + \text{nullity of } G_{v\beta} = \text{number of } \beta,$$

we have that rank of $G_{vS}$ = nullity of $G_{v\beta}$. Thus we can state the rank of $B_{S\beta}$ with respect to $G_{v\beta}$ instead of $G_{vS}$. Similar considerations can be made for the ranks of the other submatrices utilizing duality between $v$ and $i$, $S$ and $\Gamma$, and $\alpha$ and $\beta$.

## Appendix II

We write the matrix in the left-hand side of (14) as

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{A-1}$$

where

$$D_{11} = A_{22}C_{s2} + A_{21}C_{s1}A_{12}A_{22}^{-1} \tag{A-2}$$

$$D_{12} = A_{21}C_{s1}(-B_{13} + A_{12}A_{22}^{-1}B_{23}) \tag{A-3}$$

$$D_{21} = B_{32}C_{s2} + A_{31}C_{s1}A_{12}A_{22}^{-1} \tag{A-4}$$

$$D_{33} = A_{31}C_{s1}(-B_{13} + A_{12}A_{22}^{-1}B_{23}) + C_G. \tag{A-5}$$

The first term in the right-hand side of (A-2) is nonsingular and the second term is not dependent on $C_{s2}$ which is in the first term. Therefore matrix $D_{11}$ is nonsingular unless there exists a special relation among the element values in $C_{s1}$ and $C_{s2}$. Multiplying (A-2) by the following nonsingular matrix

$$\begin{bmatrix} U & 0 \\ -D_{21}D_{11}^{-1} & U \end{bmatrix}, \tag{A-6}$$

we obtain

$$\begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} - D_{21}D_{11}^{-1}D_{12} \end{bmatrix}. \tag{A-7}$$

$D_{22} - D_{21}D_{11}^{-1}D_{12}$ consists of $C_G$ and the terms not dependent on $C_G$ but on some other element values. Thus it is nonsingular unless there exists a special relation among the element values. The nonsingularity of the matrix given by (A-7) follows immediately.