Temperature Dependent Analysis of Elastoplastic Thermal Stresses by Finite Element Method

By

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The finite element formulation was developed for the elastoplastic thermal stress problem by employing a loading function composed of stress, plastic strain and temperature. Then, the technique of utilizing the formulation was discussed. The generalized plastic stress-strain matrix for the strain incremental theory was experssed explicitly in terms of the loading function ; and it was found that the additional nodal force due to the temperature dependence of the loading function should be considered in the matrix equilibrium equation.

It was also shown that the transformation stresses during quenching could be analysed by taking account of the dependence of both temperature and cooling velocity on the coefficient of thermal expansion.

The formulated analysis was applied to the elastoplastic thermal stress problem of the thick-walled cylinder subjected to an unsteady radial temperature gradient. Good agreement was obtained between the calculated residual stresses and experimental values measured by Sachs' boring-out technique.

1. Introduction

The exact estimation of thermal stresses in plastic range is, generally speaking, a matter of considerable difficulty, in spite of its extreme need not only for practical engineering problems but also for the study of the mechanism of welding, quenching, thermal fatigue and so on. The situation has, however, been generally improved by the development of the heat conduction and the thermal stress analysis by means of the finite element method[1)~3)]. Several problems in such fields have been treated

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(for example, thermal stress in the railroad wheel[4], elastoplastic analysis during quenching[5] and welding[6]). Summarizing the studies up to this time, the following problems may be pointed out :

(1) Temperature dependence of the stress-strain relation of material can not be introduced to the theories without difficulty ; and

(2) the concept of loading path, that is, strain history, is not taken into account.

The interest of the present authors has been in analyzing the elastoplastic response of the materials at elevated temperatures, especially the thermal stress and thermal deformation of the materials subjected to the cyclic varying temperatures which are ultimately related to the thermal fatigue phenomena. In order to formulate a rational analysis for such phenomena, as a matter of course, both the temperature dependence of stress-strain relation and the strain history must be taken into consideration.

From this point of view, the general method of elastoplastic thermal stress analysis is given in this paper and the emphasis is focused on the role of the strain history and the temperature dependent constitutive relation by using the loading function. The elastoplastic analysis based on the loading function which was determined from the parameters of the temperature, plastic strains and stresses was previously treated in the book by Boley and Weiner[7]. In the present study the fundamental equations by using the loading function were presented in matrix form. In other words, the general form of stress-strain relation was developed. It was also proved that the additional nodal force due to the temperature dependence of the loading runction[8] must be introduced. Moreover, several techniques to apply such fundamental relations to the finite element calculation were proposed. This theory was extended in Art. 2.4 to the analysis of quenching by taking account of the temperature and the cooling velocity dependence of the coefficient of thermal expansion.

An attempt of the application of the method was made to the analysis of thermal residual stresses in the thick-walled cylinder of aluminum-magnesium alloy subjected to the unsteady temperature field. The analytical results were compared with the experimental ones.

2. Analysis

2.1 Elastic thermal stress analysis

The fundamental formulation for the elastic thermal stress analysis is briefly presented below, as shown in some references[1),5)]. The basic equations are derived in the incremental forms for the advantage of plastic analysis discussed in the next section.

strain-displacement relation :

$$\{d\varepsilon\} = [B]\{du\}^e \tag{1}$$

sterss-strain relation :

$$\{d\sigma\} = [D^e](\{d\varepsilon\} - \{\alpha\}dT) \tag{2}$$

stiffness matrix of an element :

$$[K]^e = \int_v [B]^T[D^e][B]\,dv \tag{3}$$

equivalent nodal force due to thermal expansion :

$$\{dL\}_t^e = \int_v [B]^T[D^e]\{\alpha\}dT\ dv \tag{4}$$

equilibrium equation for the complete system :

$$\{dL\} + \{dL\}_t = [K]\{du\}. \tag{5}$$

The procedure of the calculation is as follows. First of all, Eq. (5) is solved for the applied external force and the temperature increment. This gives the displacements of each node. Then, by making use of Eqs. (1) and (2), the strain and stress increments $\{d\varepsilon\}$ and $\{d\sigma\}$ can be calculated. Therefore, the current values of strain $\{\varepsilon\}$ and stress $\{\sigma\}$ are obtained by adding these incremental values to the total ones at the last stage. By repeating this procedure, the elastic analysis can be advanced for a given temperature distribution which is a function of the time.

2.2 Elastoplastic thermal stress analysis

2.2.1 Incremental stress-strain relation

As the deformation of the materials progresses, the plastic region begins to appear in some parts of the body. It is necessary to obtain the incremental stress-strain relation corresponding to Eq. (2) held in plastic range to perform the elastoplastic analysis. Supposing that the loading function F is given as a function of the stress σ_{ij}, the plastic strain ε_{ij}^p and the temperature T, there will be the following equation :

$$F = F(\sigma_{ij}, \varepsilon_{ij}^p, T) = 0. \tag{6}$$

As the stress point stays on the yield surface during plastic deformation process, the following equation should be obtained :

$$dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p + \frac{\partial F}{\partial T} dT = 0. \tag{7}$$

Here (and henceforth) repeated indices are summed up over the range (1, 2, 3). Using the principle of the maximum plastic work leads to the results that the loading function F may be regarded as the plastic potential function ; and the increment of plastic strain can be given as

$$d\varepsilon_{ij}^{p} = \Lambda \frac{\partial F}{\partial \sigma_{ij}}, \tag{8}$$

where, Λ is a constant which depends on the stress, the plastic strain and the temperature. By substituting Eq. (8) into Eq. (7) and solving for Λ, the following relation holds :

$$\Lambda = \hat{G} \left(\frac{\partial F}{\partial \sigma_{pq}} d\sigma_{pq} + \frac{\partial F}{\partial T} dT\right), \tag{9}$$

with

$$\hat{G} = - \frac{1}{\dfrac{\partial F}{\partial \sigma_{mn}} \dfrac{\partial F}{\partial \varepsilon_{mn}^{p}}}. \tag{10}$$

Hence, the increment of plastic strain can be expressed as

$$d\varepsilon_{ij}^{p} = \hat{G} \left(\frac{\partial F}{\partial \sigma_{pq}} d\sigma_{pq} + \frac{\partial F}{\partial T} dT\right) \frac{\partial F}{\partial \sigma_{ij}} \tag{11}$$

and the increment of the total strain $d\varepsilon_{ij}$ including thermal expansion will be

$$\begin{aligned} d\varepsilon_{ij} = & \frac{1}{2(1+\nu)G} [(1+\nu)\, d\sigma_{ij} - \delta_{ij}\, \nu\, d\sigma_{kk}] + \delta_{ij}\, \alpha\, dT \\ & + \hat{G} \left(\frac{\partial F}{\partial \sigma_{pq}} d\sigma_{pq} + \frac{\partial F}{\partial T} dT\right) \frac{\partial F}{\partial \sigma_{ij}}, \end{aligned} \tag{12}$$

where δ_{ij} is Kronecker's delta and G, ν and α are shear modulus, Poisson's ratio and coefficient of thermal expansion, respectively.

Considering the fact that the plastic strain is kept incompressible, that is,

$$d\varepsilon_{ii}^{p} = \Lambda \frac{\partial F}{\partial \sigma_{ii}} = 0, \tag{13}$$

Eq. (12) is solved for the stress increment

$$\begin{aligned} d\sigma_{ij} = 2G \Bigg[& d\varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij}\, d\varepsilon_{mm} - \frac{1+\nu}{1-2\nu} \delta_{ij}\, \alpha\, dT \\ & - \frac{\dfrac{\partial F}{\partial \sigma_{kl}} d\varepsilon_{kl}}{S} \frac{\partial F}{\partial \sigma_{ij}} \Bigg] - \frac{1}{S} \frac{\partial F}{\partial T} dT \frac{\partial F}{\partial \sigma_{ij}}, \end{aligned} \tag{14}$$

where S is a function of the stress, the plastic strain and the temperature, and it is given as

$$S = \frac{\partial F}{\partial \sigma_{mn}} \frac{\partial F}{\partial \sigma_{mn}} \left(1 + \frac{1}{2G\, \hat{G}\, \frac{\partial F}{\partial \sigma_{pq}} \frac{\partial F}{\partial \sigma_{pq}}}\right). \tag{15}$$

The matrix form of Eq. (14) is also represented in the Cartesian coordinate system (0-xyz) as

$$\{d\sigma\} = [D^p] (\{d\varepsilon\} - \{\alpha\}\, dT) - \frac{1}{S} \left\{\frac{\partial F}{\partial \sigma}\right\} \frac{\partial F}{\partial T}\, dT, \tag{16}$$

where the engineering notation of strains $d\gamma_{xy}$, $d\gamma_{yz}$ and $d\gamma_{zx}$ is used and the following matrix notation is adopted ;

$$\{d\sigma\} = \begin{Bmatrix} d\sigma_x \\ d\sigma_y \\ d\sigma_z \\ d\tau_{xy} \\ d\tau_{yz} \\ d\tau_{zx} \end{Bmatrix}, \ \{d\varepsilon\} = \begin{Bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_z \\ d\gamma_{xy} \\ d\gamma_{yz} \\ d\gamma_{zx} \end{Bmatrix}, \ \{\alpha\} = \begin{Bmatrix} \alpha \\ \alpha \\ \alpha \\ o \\ o \\ o \end{Bmatrix} \text{ and } \left\{\frac{\partial F}{\partial \sigma}\right\} = \begin{Bmatrix} \frac{\partial F}{\partial \sigma_x} \\ \frac{\partial F}{\partial \sigma_y} \\ \frac{\partial F}{\partial \sigma_z} \\ \frac{\partial F}{\partial \tau_{xy}} \\ \frac{\partial F}{\partial \tau_{yz}} \\ \frac{\partial F}{\partial \tau_{zx}} \end{Bmatrix}. \tag{17}$$

Moreover, the matrix $[D^p]$ in Eq. (16) is written as

$$[D^p] = 2G \cdot \begin{bmatrix} \frac{1-\nu}{1-2\nu} - \frac{1}{S}\left(\frac{\partial f}{\partial \sigma_x}\right)^2 & & & & & \\ \frac{\nu}{1-2\nu} - \frac{1}{S}\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_y} & \frac{1-\nu}{1-2\nu} - \frac{1}{S}\left(\frac{\partial f}{\partial \sigma_y}\right)^2 & & & \text{SYM.} & \\ \frac{\nu}{1-2\nu} - \frac{1}{S}\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_z} & \frac{\nu}{1-2\nu} - \frac{1}{S}\frac{\partial f}{\partial \sigma_y}\frac{\partial f}{\partial \sigma_z} & \frac{1-\nu}{1-2\nu} - \frac{1}{S}\left(\frac{\partial f}{\partial \sigma_z}\right)^2 & & & \\ -\frac{1}{S}\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \tau_{xy}} & -\frac{1}{S}\frac{\partial f}{\partial \sigma_y}\frac{\partial f}{\partial \tau_{xy}} & -\frac{1}{S}\frac{\partial f}{\partial \sigma_z}\frac{\partial f}{\partial \tau_{xy}} & \frac{1}{2} - \frac{1}{S}\left(\frac{\partial f}{\partial \tau_{xy}}\right)^2 & & \\ -\frac{1}{S}\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \tau_{yz}} & -\frac{1}{S}\frac{\partial f}{\partial \sigma_y}\frac{\partial f}{\partial \tau_{yz}} & -\frac{1}{S}\frac{\partial f}{\partial \sigma_z}\frac{\partial f}{\partial \tau_{yz}} & -\frac{1}{S}\frac{\partial f}{\partial \tau_{xy}}\frac{\partial f}{\partial \tau_{yz}} & \frac{1}{2} - \frac{1}{S}\left(\frac{\partial f}{\partial \tau_{yz}}\right)^2 & \\ -\frac{1}{S}\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \tau_{zx}} & -\frac{1}{S}\frac{\partial f}{\partial \sigma_y}\frac{\partial f}{\partial \tau_{zx}} & -\frac{1}{S}\frac{\partial f}{\partial \sigma_z}\frac{\partial f}{\partial \tau_{zx}} & -\frac{1}{S}\frac{\partial f}{\partial \tau_{xy}}\frac{\partial f}{\partial \tau_{zx}} & -\frac{1}{S}\frac{\partial f}{\partial \tau_{yz}}\frac{\partial f}{\partial \tau_{zx}} & \frac{1}{2} - \frac{1}{S}\left(\frac{\partial f}{\partial \tau_{zx}}\right)^2 \end{bmatrix} \tag{18}$$

The stress-strain matrix shown above corresponds to the the explicit form which Zienkiewicz et al.[9] have derived implicitly. If the von Meses criterion which is expressed in terms of the deviatric stress σ'_{ij}, the equivalent stress $\bar{\sigma}$ and the equivalent plastic strain $\bar{\varepsilon}^p$ is employed for a special case of the loading function F as

$$\left.\begin{aligned} F &= \frac{1}{2}\sigma'_{ij}\,\sigma'_{ij} - \frac{1}{3}\bar{\sigma}^2 = 0 \\ \bar{\sigma} &= \bar{\sigma}\,(\bar{\varepsilon}^p) \end{aligned}\right\}, \tag{19}$$

the matrix $[D^p]$ is reduced to the same one proposed by Yamada et al.[10] This verifies that Eq. (18) presents the stress-strain matrix for the general case of loading.

2.2.2 Equilibrium equation

Introducing the incremental stress-strain relation of Eq. (16), the equilibrium equation of the complete system is given by the principle of virtual work :

$$\{dL\} + \{dL\}_t + \{dL\}_F = [K]\{du\}. \tag{20}$$

$\{du\}$ in this equation is the increment of the nodal displacement and the stiffness matrix $[K]$ for the comptete system is the assembly of the elastic expressions for the elastic elements presented in Eq. (3) and plastic ones for the plastic elements in the next equation,

$$[K]^e = \int_v [B]^T [D^p] [B]\, dv. \tag{21}$$

$\{dL\}$ and $\{dL\}_t$ are increments of external force and that of an equivalent nodal force due to thermal expansion, respectively. And then, $\{dL\}_F$ in Eq. (20) is the increment of the equivalent nodal force due to the stress increment of the second term in Eq. (16), corresponding to the temperature dependence of the loading function F. The terms of $\{dL\}_t$ and $\{dL\}_F$ for the plastic elements are given as

$$\{dL\}_t^e = \int_v [B]^T [D^p] \{\alpha\}\, dT\ dv \tag{22}$$

and

$$\{dL\}_F^e = \int_v [B]^T \left\{\frac{\partial F}{\partial \sigma}\right\} \frac{1}{S} \frac{\partial F}{\partial T}\, dT\ dv, \tag{23}$$

respectively. On the other hand, Eq. (4) must be used as $\{dL\}_t^e$ in place of Eq. (22) and the term $\{dL\}_F^e$ is taken as zero for the elastic elements.

Once the loading function F is determined as the form of Eq. (6), the nodal displacements $\{du\}$ are given by solving Eq. (20). Therefore, it can be seen that the procedure developed here gives the general solution of history dependent elastoplastic thermal stresses if the loading function F is given as the function of strain history. Morerver, as the equivalent nodal force represented by Eq. (23) is introduced, the effect of the temperature dependence of the stress-strain relation can be properly taken into consideration. It is worth noting that the persent method of analysis facilitates an orderly tracing the behavior of the generic point in the stress-strain-temperature field during the deformation.

2.3 Treatments of the elements in the elastoplastic transitional region and in the process of unloading

Complications arise with the next temperature increment, when the elastic elements of the body with the stresses near yield become plastic or the plastic elements begin to unload, although the others continue to hold elastic or plastic states.

Firstly, the technique to treat the elements in the transition region is discussed as follows. As the data of unsteady temperature distribution is generally given as a function of time, the distribution of temperature can not be determined arbitrarily. This means the measure of the deformation to be employed is not the temperature but the time itself. Furthermore, the effect of the temperature dependence of the stress-strain relation of the material should be taken into the analysis. The method developed here is the extension of the case using a general loading function instead of the von Mises criterion[10], and includes the above-mentioned effects.

Suppose the temperature increment $DT=T_2-T_1$ is applied to a material point. Figure 1 shows the behavior of the yield curve and the stress state on π-plane corresponding to the stress point. The figure represents only the case that the initial yield curve shrinks as the temperature increases from T_1 to T_2. As a matter of course, the following discussion is available without any difficulty for the case that the initial yield curve expands with temperature. The broken lines in the figure are the initial yield curve at the temperature level of T_1, T_x and T_2, and

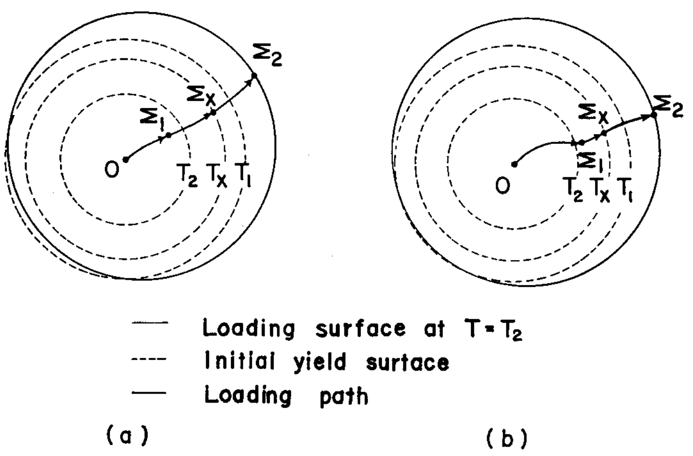

Fig. 1 Behavior of the stress point in π-plane.

the fine solid line indicates the yield curve (subsequent yield curve) at $T=T_2$. Thick solid lines with arrows show the locus of the stress point and Σ_1, Σ_x and Σ_2 are the stress points at $T=T_1$, T_x and T_2, respectively. So long as the value of the temperature increment DT is finite, the behavior of the stress point in the element which shifts from the elastic state to the plastic at this stage may be classified, as shown in Fig. 1, into

(a) the stress point at $T=T_1$ exists inside the initial yield curve at $T=T_2$, or

(b) out side the curve.

In both cases, the stress point separates from the origin O and at the same time the initial yield curve contracts as the temperature changes. Hence, it may be regarded that the stress point reaches the yield surface at $T=T_x$ after the element behaves as elastically due to the early part of the temperature increment $DT^e=T_x-T_1$, and then the plastic process advances due to the remaining part of $DT^p=T_2-T_x$. The temperature T_x at which the yield condition is satisfied is determined as follows ; Adopting the parameter $r(0\leq r\leq 1)$, T_x can be expressed as

$$T_x = T_1 + r\,(T_2 - T_1). \tag{24}$$

If we assume the linear change of the stresses and the temperature increment DT, the parameter r can be expressed by noting

$$F_{T=T_x} = 0 \tag{25}$$

$$r = -\left[\frac{F}{\dfrac{\partial F}{\partial \sigma_{mn}} D\sigma_{mn} + \dfrac{\partial F}{\partial T} DT}\right]_{T=T_1}, \tag{26}$$

where $D\sigma_{ij}$ is the stress increment due to DT for the elastic deformation. Equation (26) shows the parameter r only in the first order, but of course it is possible to indicate the higher order expression.

Details of the procedure of elastoplastic calculation during a typical temperature increment $DT=T_2-T_1$ are presented as follows :

1. Apply the prescribed temperature increments to each element and carry out the calculation. For some elements, for example n elements, stress state exceeds the yield point at $T=T_2$. For these elements the following relation holds ;

$$F\,(\sigma_{ij}, \varepsilon_{ij}^p, T)_{T=T_2} > 0. \tag{27}$$

2. For these n elements the parameter r's are calculated and put in order ; $(r_1<r_2<\cdots<r_n)$.

3. Calculate the elastoplastic stresses at each step of successive temperature increments $DT\cdot r, DT\cdot(r_2-r_1), \cdots, DT\cdot(1-r_n)$. At each process, n elements enter

into the plastic region one by one.

4. After computing the temperature increment at $n+1$ stage, return to step 1. and continue the calculation.

The second focussed emphasis in the present method of thermal stress analysis is on the treatment of unloading elements presented as below. The behavior of the element in the stage is classified into the three cases at the temperature T_1.

$$\left(\frac{\partial F}{\partial \sigma_{ij}}\right)_{T=T_1} D\sigma_{ij} + \left(\frac{\partial F}{\partial T}\right)_{T=T_1} DT \begin{cases} >0 & \text{loading,} \\ =0 & \text{neutral loading,} \\ <0 & \text{unloading.} \end{cases} \tag{28}$$

Namely, at the above-mentioned stage 1, Eq. (28) is examined and for the unloading elements, which have been in the plastic state, elastic calculation should be carried out after replacing the matrix $[D^p]$ by $[D^e]$.

2.4 Elastoplastic analysis of quenching

In the case of the elastoplastic analysis of quenching we should take into account the transformation stresses in addition to the ordinary thermal stresses. The transformation stresses are induced by the effect that the coefficient of thermal expansion has different values at each point of the body when materials are cooled immediately after having been heated up beyond the transformation temperature.

Figure 2 is a schematic representation of the dilatation curve for several cooling velocities $\dot{T}$. This indicates the coefficient of thermal expansion α is represented

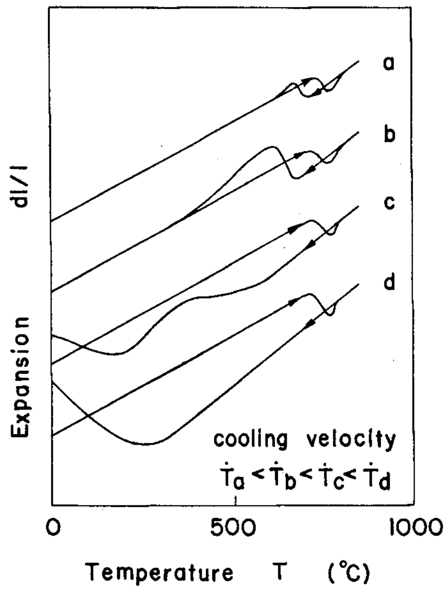

Fig. 2 Relation between expansion and temperature.

as a function of the temperature T and the cooling velocity $\dot{T}$;

$$\alpha = \alpha\,(T, \dot{T}). \tag{29}$$

If Eq. (29) is given explicitly, elastoplastic analysis, in which the transformation stress is taken into consideration, is formulated by generalizing the before-mentioned elastoplastic thermal stress analysis as follows.

Equation (29) is reduced to

$$d(\alpha T) = \alpha\, dT + \left(\frac{\partial \alpha}{\partial T}\, dT + \frac{\partial \alpha}{\partial \dot{T}}\, d\dot{T}\right)T, \tag{30}$$

then, total incremental strain $d\varepsilon_{ij}$ is expressed in the same way as Eq. (12)

$$d\varepsilon_{ij} = \frac{1}{2G}\,(d\sigma_{ij} - \delta_{ij}\,\frac{\nu}{1+\nu}\,d\sigma_{kk}) + \delta_{ij}\,\alpha dT + \delta_{ij}\,T\left(\frac{\partial \alpha}{\partial T}\,dT + \frac{\partial \alpha}{\partial \dot{T}}\,d\dot{T}\right)$$
$$+ \hat{G}\left(\frac{\partial F}{\partial \sigma_{mn}}\,d\sigma_{mn} + \frac{\partial F}{\partial T}\,dT\right)\frac{\partial T}{\partial \sigma_{ij}}. \tag{31}$$

The incremental stress $d\sigma_{ij}$ is obtained from Eq. (31) by noting the incompressibility of the plastic strain,

$$d\sigma_{ij} = 2G\left[d\varepsilon_{ij} + \delta_{ij}\,\frac{\nu}{1-2\nu}\,d\varepsilon_{mm} - \delta_{ij}\,\frac{1-\nu}{1-2\nu}\,\alpha dT - \delta_{ij}\,\frac{1-\nu}{1-2\nu}\,T\left(\frac{\partial \alpha}{\partial T}\,dT + \right.\right.$$
$$\left.\left.\frac{\partial \alpha}{\partial \dot{T}}\,d\dot{T}\right) - \frac{1}{S}\,\frac{\partial F}{\partial \sigma_{kl}}\,d\varepsilon_{kl}\,\frac{\partial F}{\partial \sigma_{ij}}\right] - \frac{1}{S}\,\frac{\partial F}{\partial T}\,dT\,\frac{\partial F}{\partial \sigma_{ij}}. \tag{32}$$

It should be noted that the fourth term in the bracket in the above equation corresponds to the transformation stress increment. The matrix form of Eq. (32) is represented as

$$\{d\sigma\} = [D^p]\left[\{d\varepsilon\} - \{\alpha\}\,dT - T\left(\left\{\frac{\partial \alpha}{\partial T}\right\}dT + \left\{\frac{\partial \alpha}{\partial \dot{T}}\right\}d\dot{T}\right)\right]$$
$$- \frac{1}{S}\left\{\frac{\partial F}{\partial \sigma}\right\}\frac{\partial F}{\partial T}dT. \tag{33}$$

Using this incremental stress-strain relation, we can obtain the final equilibrium equation of the entire domain as

$$\{dL\} + \{dL\}_t + \{dL\}_F + \{dL\}_q = [K]\,\{du\}, \tag{34}$$

where $\{dL\}_q$ is a newly derived equivalent nodal force obtained by summarizing the following equivalent nodal force for the element ;

$$\{dL\}_q^e = \int_v [B]^T\,[D^p]\,T\left(\left\{\frac{\partial \alpha}{\partial T}\right\}dT + \left\{\frac{\partial \alpha}{\partial \dot{T}}\right\}d\dot{T}\right)dv. \tag{35}$$

If the concrete form of α in Eq. (29) is obtained from the dilatation curves at various cooling velocities as shown in Fig. 2, we can carry out the elastoplastic analysis in which the transformation stress is taken into consideration by solving the equiliurium equation (34).

3. Example of Application of The Theory

An application of the developed elastoplastic thermal stress analysis is made, in this section, to a cylinder with the radial temperature gradient.

The specimen employed is the aluminum-magnesium alloy tube with 35mm and 19mm of outer and inner diameters. The lengh of the specimen is taken to be long enough to exclude the end effect. The specimen was at first heated uniformly by the high frequency inductoɪ and then, cooled from the inner surface by jet flow of water.

During the cooling process, the unsteady temperatbre distribution along the radius was induced to the specimen, which can be seen in Fig. 3 with the parameter of cooling time t. This temperature distribution was ontained by the finite element heat conduction analysis. The temperature levels determined experimetally both inside and outside of the cylinder were adopted as the initial and the boundary conditions.

As the strict form of the loading function for the material has not been obtained, the following expression[11] is employed in this analysis,

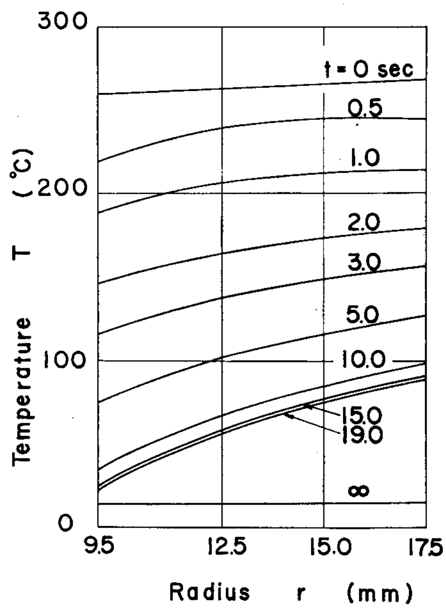

Fig. 3 Temperature distribution in the cylinder.

$$F = \frac{1}{2}\sigma'_{ij}\sigma'_{ij} - \kappa(\bar{\varepsilon}^p, T) = 0. \tag{36}$$

where strain hardening parameter κ can be estimated from the stress-strain curves at elevated temperatures. Within the range of the equivalent plastic strain in the problem treated here, κ can be represented as

$$\kappa(\bar{\varepsilon}^p, T) = \frac{1}{3}\{30.3(\bar{\varepsilon}^p + 3.30 \times 10^{-3})^{0.292} - 0.586 \times 10^{-4} T^2 + 0.150 \times 10^{-2} T - 0.02\}^2. \tag{37}$$

Values of Young's modulus, Poisson's ratio and the coefficient of thermal expansion are determined by the experiments :

$$E = 7200\, kg/mm^2, \quad \nu = 0.353$$
$$\alpha = 2.7 \times 10^{-5}\ 1/°C. \tag{38}$$

As the first approximation, these values are assumed to be temperature independent.

The variation of the equivalent stress distribution with the parameter of time t after cooling is presented in Fig. 4. The figure indicates that after an elapse of

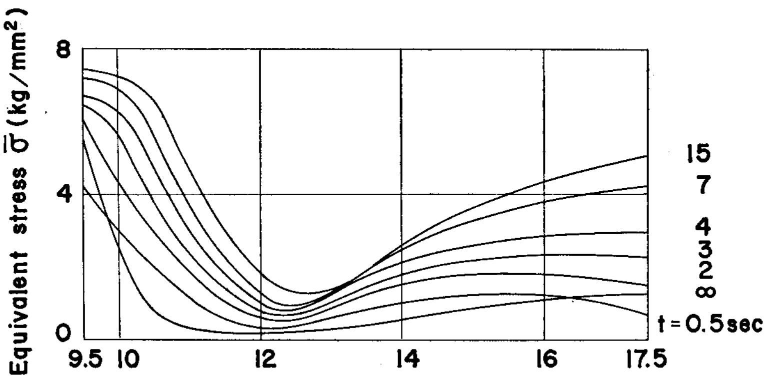

Fig. 4 Distribution of equivalent stress.

three minutes, the plastic region begins to appear from the inside and then spreads outwards. The distribution labeled $t=\infty$ represents that of $\bar{\sigma}$ for the flat temperature distribution at the last stage, which is equal to the water temperature ; and this may be regarded as the distribution of the residual stresses.

The analytical results of the generic point in the stress-strain-temperature field are shown in Fig. 5. The fine solid lines in the figure represent the stress-strain curves at various temperatures. The thick solid line and broken line show the stress of the points of $r=9.58$mm and r=9.75mm, and the numbers beside the

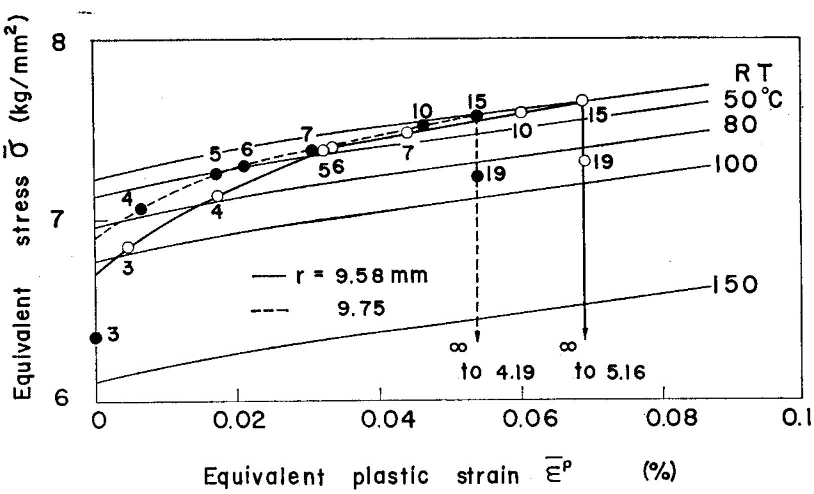

Fig. 5 Stress locus of the generic point in stress-strain-tempertaure field.

points indicate the cooling time in seconds. We can see from the figure that this analysis can give the rational behavior of the generic points in the stress-strain-temperature field.

The analytical stress distributions at $t=\infty$, that is the residual stress distributions are represented in Fig. 6 in comparison with the experimental results. The figure

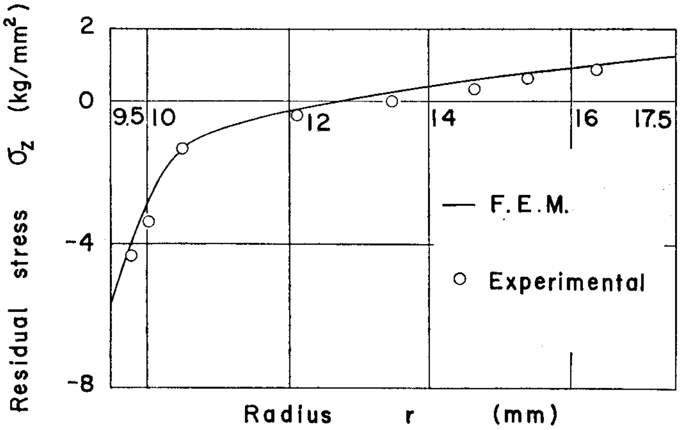

Fig. 6 Distribution of residual stress σ_z.

shows the distribution of the axial residual stress. The open circles in the figure are the experimental results determined by Sach's boring-out technique. From the figure it is found that the analytical results are in good agreement with the experimental ones.

4. Conclusions

A general elastoplastic thermal stress analysis by means of the finite element method was formulated by using a loading function. The method of the analysis is widely applicable to the thermal stress problem including the effects of strain history and the temperature dependence of the stress-strain relation. Simultaneously with this, the newly developted technique for procedures of the calculation was proposed. The followings were emphasized in the present investigation. Firstly, the general form of the stress-strain matrix $[D^p]$ was given by using loading function F. This means that, if the loading function is obtained by experiments, a similar method may be applicable to the general stress analysis besides thermal stress problems.

It was also pointed out that the additional equivalent nodal force due the temperature dependence of the loading function should be introduced into the equilibrium equation, when the stress-strain curve is affected by temperature.

Moreover, it was shown that the elastoplastic problem of quenching could be analyzed by taking account of the transformation stresses in addition to the ordinary stresses.

As an example of the application of the present formulation, the elastoplastic thermal stress analysis in the thick-walled cylinder subjected to an unsteady radial temperature distribution was presented ; and it was found that the calculated residual stresses were in good agreement with the experimental ones measured by Sachs' boring-out technique.

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