1/3-harmonic Oscillation in Three-phase Circuit with Series Condensers (II)

By

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Abstract

The authors deal with the nearly 1/3-harmonic oscillation (a kind of almost periodic oscillation) in three-phase circuit with series condensers. The fundamental equation becomes non-autonomous type with small parameter ε

$$\frac{dx_k}{d\tau} = \sum_{q=1}^{5} a_{kq}\, x_q + \varepsilon\, f_k\, (x_1, x_2, x_5, \tau) \qquad k=1, 2, 3, 4, 5$$

An analog computer is used for obtaining the parameter region where the nearly 1/3- harmonic oscillation is sustained. Furthermore, the asymptotic method of Bogoliubov and Mitropolsky is extended to analyse the behavior of the nearly 1/3-harmonic oscillation.

1. Introduction

The authors have reported the 1/3- harmonic oscillation in the three-phase circuit with series condensers. This paper describes the nearly 1/3- harmonic oscillation in the circuit. By means of an analog computer, the region is obtained where the nearly 1/3- harmonic oscillation is sustained. Making reference to it, the nearly 1/3- harmonic oscillation is analysed by the extention of the asymptotic method for the system with many degrees of freedom. The notations are the same as the paper already published.(1)

2. The region of the nearly 1/3-harmonic oscillation.

Making use of an analog computer, we obtain the region on E-η plane where the

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nearly 1/3-harmonic oscillation is sustained. The oscillation is maintained in the shaded part of Fig. 1. The typical wave forms are shown in Fig. 2 for parameters $\xi=0.15$, $\eta=0.20$, $E=0.40$. In the shaded region, the imaginary parts of the eigen values, ω_1 and ω_2, for the unpurturbed system of the fundametal equation, hold the relation $2\omega_1 \simeq \omega_2$

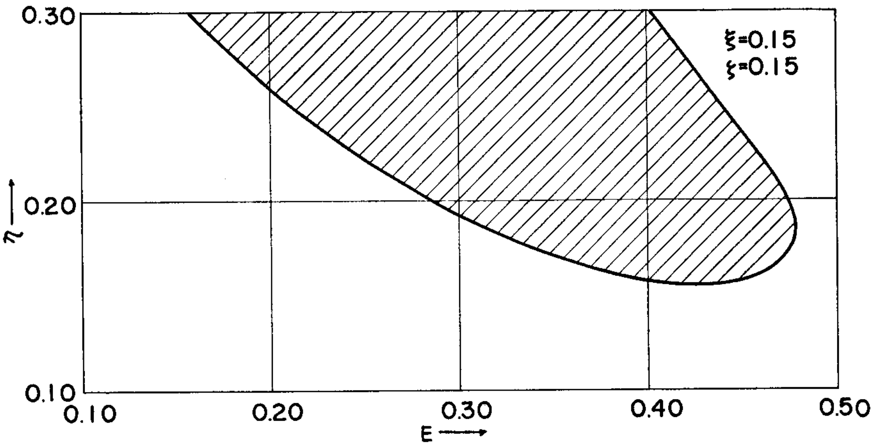

Fig. 1. Region where the nearly 1/3 harmonic oscillation is sustained (analog computer).

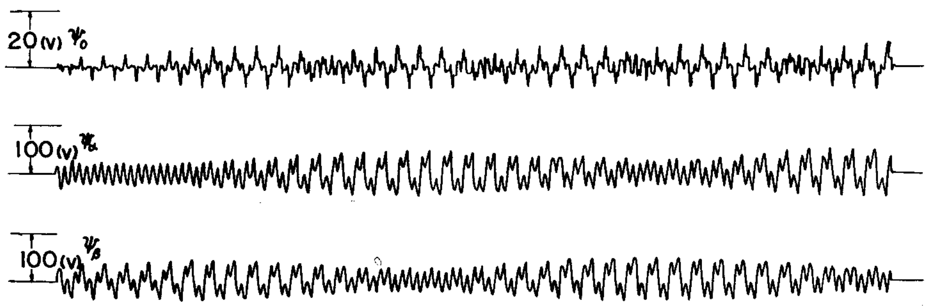

Fig. 2. Typical wave forms of the flux interlinkages for parameters $\xi=0.15$, $\eta=0.20$, $\zeta=0.15$, E=0.4.

Thus, we can modify the unpurturbed system in the same manner as the case of the pure 1/3-harmonic oscillation. Hereafter, notations ω_1 and ω_2 show the absolute values of the eigen values for the modified unpurturbed system.

3. The analysis by the asymptotic method

Let's analyse the nearly 1/3-harmonic oscillation by the asymptotic method. The fundamental equation and the parameters refer to Reference (1). In the case of the nearly 1/3- harmonic oscillation, we can put the first approximate solution as

$$\left.\begin{array}{l} x_k^{(0)}(a, x, y, \psi) = a\varphi_k e^{j\psi} + a\varphi_k^* e^{-j\psi} + (x + jy)\chi_k e^{j2\psi} + (x - jy)\chi_k^* e^{-j2\psi} \\ \psi = \omega_1\tau \quad j=\sqrt{-1} \qquad\qquad (k = 1, 2, 3, 4) \end{array}\right\} \quad (1)$$

where variables a, x and y are real, and ψ is also real whose time derivative represents the angular frequency (hereafter, called frequency for short). In the steady state the variable x_5 which represents the zero phase fluxinterlinkage can be put approximately as

$$\begin{aligned} x_5 &\simeq x_5^{(0)}(a, x, y, \tau) \\ &= \sum_{l=1}^{L}(Z_l e^{j\Omega_l\tau} + z_l^* Z e^{-j\Omega_l\tau}) \end{aligned} \quad (2)$$

where $Z_l(l=1,\cdots,L)$ is complex function of the real variables a, x, and y and is written as

$$\begin{aligned} Z_l &= Z_l(a, x, y) \\ &= P_l(a, x, y) + jQ_l(a, x, y) \end{aligned} \quad (3)$$

Functions $k(x_1, x_2, \tau)$ and $h(x_1, x_2)$ can also approximately be written as

$$\begin{aligned} k(x_1, x_2, \tau) &\simeq k(x_1^{(0)}, x_2^{(0)}, \tau) \\ &= k(a, x, y, \tau) \\ &= \sum_{n=1}^{N}\{K_n e^{j\Omega n\tau} + K_n^* e^{-j\Omega n\tau}\} \end{aligned} \quad (4)$$

$$\begin{aligned} h(x_1, x_2; \tau) &\simeq h(x_1^{(0)}, x_2^{(0)}, \tau) \\ &= h(a, x, y, \tau) \\ &= H_0 + \sum_{m=1}^{M}(H_m e^{j\omega m\tau} + H_m^* e^{-j\omega m\tau}) \end{aligned} \quad (5)$$

where K_n, H_m are complex functions of the variables a, x and y written as

$$\begin{aligned} K_n &= K_n(a, x, y) \\ &= u_n(a, x, y) + jv_n(a, x, y) \qquad n = 1, 2, \cdots, N \end{aligned} \quad (6)$$

$$H_m = H_m(a, x, y) = p_m(a, x, y) + jq_m(a, x, y) \qquad m=1, 2, \cdots, M \tag{7}$$

and H_0 is real function written as

$$H_0 = H_0(a, x, y) \tag{8}$$

Frequencies Ω_n, ω_m are written as

$$\left.\begin{aligned} \Omega_n &= (2n-1)/3 \\ \omega_m &= 2m/3 \end{aligned}\right\} \tag{9}$$

In the first approximation, we obtain

$$N = 5, \ M = 2$$

and the variable $x_5^{(0)}$ can be considered to include approximately the same frequency components as those of $k(x_1^{(0)}, x_2^{(0)}, \tau)$. Thus we can approximately put

$$\left.\begin{aligned} L &= N \\ \Omega_l &\simeq (2l-1)/3 \end{aligned}\right\} \tag{10}$$

Let's define the two $2L$ real column vectors $\boldsymbol{u}$ and $\boldsymbol{Z}$ and $2L\times 2L$ real matrix $\boldsymbol{\Psi}$ a

$$\boldsymbol{u} \triangleq {}^t(u_1, v_1, \cdots, u_L, v_L)$$
$$\boldsymbol{Z} \triangleq {}^t(P_1, Q_1, \cdots, P_L, Q_L)$$

where the superscript 't' indicates the transposition and the components are as follows

$$u_1 = -12\zeta\Big[\frac{1}{2}\rho_0(x^2-y^2)\cos(3\theta_0) + \rho_0 xy\sin(3\theta_0) + (x^2-y^2)a\cos(3\theta_0) + 2axy\sin(3\theta_0) + a^2\{x\cos(3\theta_0) + y\sin(3\theta_0)\}\Big]$$

$$v_1 = -12\zeta\Big[\frac{1}{2}\rho_0(x^2-y^2)\sin(3\theta_0) - \rho_0 xy\cos(3\theta_0) - (x^2-y^2)a\sin(3\theta_0) + 2axy\cos(3\theta_0) + a^2\{x\sin(3\theta_0) - y\cos(3\theta_0)\}\Big]$$

$$u_2 = -4\zeta[3\rho_0 a\{x\cos(3\theta_0) + y\sin(3\theta_0)\} + (x^2-3y^2)x\cos(3\theta_0) + (3x^2-y^2)y\sin(3\theta_0) + a^3\cos(3\theta_0)]$$

$$v_2 = -4\zeta[-3\rho_0 a\{y\cos(3\theta_0) - x\sin(3\theta_0)\} - (y^2-3x^2)y\cos(3\theta_0) - (x^2-3y^2)x\sin(3\theta_0) + a^3\sin(3\theta_0)]$$

$$u_3 = -3\zeta\,\rho_0\{(2a^2+\rho_0 x)\cos(3\theta_0) + \rho_0 y\sin(3\theta_0)\}$$

$$v_3 = -3\zeta\,\rho_0\{(2a^2+\rho_0 x)\sin(3\theta_0) - \rho_0 y\cos(3\theta_0)\}$$

$$u_4 = -3\zeta\,\rho_0^2\,a\cos(3\theta_0)$$

$$v_4 = -3\zeta\ \rho_0^2\ a\ sin\ (3\theta_0)$$

$$u_5 = -\tfrac{1}{2}\zeta\ \rho_0^3\ cos\ (3\theta_0)$$

$$v_5 = -\tfrac{1}{2}\zeta\ \rho_0^3\ sin\ (3\theta_0)$$

$2L\times 2L$ matrix $\boldsymbol{\Psi}$ is also defined as

$$\boldsymbol{\Psi} \triangleq \begin{pmatrix} H_0+p_1 & q_1-\frac{1}{3} & p_1+p_2 & q_1+q_2 & p_2 & q_2 & 0 & 0 & 0 & 0 \\ p_1+\frac{1}{3} & H_0-p_1 & q_2-q_1 & p_1-p_2 & -q_2 & p_2 & 0 & 0 & 0 & 0 \\ p_1+p_2 & q_2-q_1 & H_0 & -1 & p_1 & q_1 & p_2 & q_2 & 0 & 0 \\ q_1+q_2 & p_1-p_2 & 1 & H_0 & -q_1 & p_1 & -q_2 & p_2 & 0 & 0 \\ p_2 & -q_2 & p_1 & -q_1 & H_0 & -\frac{5}{3} & p_1 & q_1 & p_2 & q_2 \\ q_2 & p_2 & q_1 & p_1 & \frac{5}{3} & H_0 & -q_1 & p_1 & -q_2 & p_2 \\ 0 & 0 & p_2 & -q_2 & p_1 & -q_1 & H_0 & -\frac{7}{3} & p_1 & q_1 \\ 0 & 0 & q_2 & p_2 & q_1 & p_1 & \frac{7}{3} & H_0 & -q_1 & p_1 \\ 0 & 0 & 0 & 0 & p_2 & -q_2 & p_1 & -q_1 & H_0 & -3 \\ 0 & 0 & 0 & 0 & q_2 & p_2 & q_1 & p_1 & 3 & H_0 \end{pmatrix}$$

where

$$H_0 = 6\zeta\ \{\rho_0^2 + 4\ (x^2 + a^2 + y^2)\}$$
$$p_1 = 12\zeta\ (\rho_0 + 2x)\ a$$
$$p_2 = 12\zeta\ \rho_0\ x$$
$$q_1 = 24\zeta\ a\ y$$
$$q_2 = 12\zeta\ \rho_0\ y$$

By the method of harmonic balance for the equation which represents the zero phase component x_5, we have a linear equation

$$\boldsymbol{\Psi}\ \boldsymbol{Z} = \boldsymbol{u} \tag{11}$$

If the matrix $\boldsymbol{\Psi}$ is nonsingular,

$$\boldsymbol{Z} = \boldsymbol{\Psi}^{-1}\ \boldsymbol{u} \tag{12}$$

This equation gives the components for the frequencies Ω_l ($l=1,\ 2,\cdots,L$) of the approximate zero phase component $x_5^{(0)}$.

We shall obtain the approximate solution of the fundamental equation by the asymptotic expansion,

$$x_k\ (a, x, y, \phi) = x_k^{(0)}\ (a, x, y, \phi) + \varepsilon x_k^{(1)}\ (a, x, y, \phi) + \varepsilon^2\ x_k^{(2)}\ (a, x, y, \phi) + \cdots$$
$$k = 1, 2, 3, 4 \tag{13}$$

where the parameter ε is small and the functions $x_k^{(0)}, x_k^{(1)}$ ......are all periodic with respect to both ψ and 2ψ. The real variables a, x and y which characterise the amplitudes of the nearly 1/3-harmonic oscillation are assumed to be determined by the following set of autonomous differential equations

$$\left.\begin{aligned} \frac{da}{d\tau} &= \varepsilon A_1(a,x,y) + \varepsilon^2 A_2(a,x,y) + \cdots \\ \frac{dx}{d\tau} &= \varepsilon B_1(a,x,y) + \varepsilon^2 B_2(a,x,y) + \cdots \\ \frac{dy}{d\tau} &= \varepsilon C_1(a,x,y) + \varepsilon^2 C_2(a,x,y) + \cdots \end{aligned}\right\} \tag{14}$$

The real value ψ whose time derivative gives the frequency of the oscillation, is also assumed to be determined by the function of the real variables a, x and y and to be the solution of the equation

$$\frac{d\psi}{d\tau} = \omega_1 + \varepsilon D_1(a,x,y) + \varepsilon^2 D_2(a,x,y) + \cdots \tag{15}$$

The necessary and sufficient conditions that $x_k^{(1)}(k=1,2,3,4)$ is periodic with ψ and 2ψ respectively determine the following set of equations in complex forms

$$\left.\begin{aligned} &\varepsilon A_1(a,x,y) + ja\,\varepsilon D_1(a,x,y) = \frac{1}{2\pi}\left\{\sum_{k=1}^{4}\bar{\varphi}_k^*\int_0^{2\pi}\varepsilon X_k(x_1^{(0)},x_2^{(0)},x_5^{(0)},\tau)\,e^{-j\psi}\,d\psi\right\} \\ &\qquad\qquad /\sum_{k=1}^{4}\bar{\varphi}_k^*\,\varphi_k \\ &\{\varepsilon B_1(a,x,y) - 2y\,\varepsilon D_1(a,x,y)\} + j\{\varepsilon C_1(a,x,y) + 2x\,\varepsilon D_1(a,x,y)\} \\ &\qquad = \frac{1}{2\pi}\left\{\sum_{k=1}^{4}\bar{\chi}_k^*\int_0^{2\pi}\varepsilon X_k(x_1^{(0)},x_2^{(0)},x_5^{(0)},\tau)\,e^{-j2\psi}\,d\psi\right\}/\sum_{k=1}^{4}\bar{\chi}_k^*\,\chi_k \end{aligned}\right\} \tag{16}$$

where complex functions $\bar{\varphi}_k^*$, $\bar{\chi}_k^*$ are the complex conjugate functions correspon-ing to the adjoint system of the unpurturbed system. By equating the real and imaginary part of Eq.(16), we have the following :

$$\left.\begin{aligned} \frac{da}{d\tau} &= \varepsilon A_1(a,x,y) \\ &= -a_{11}\,a - \xi\{a_{13}a^2 + a_{14}(x^2+y^2)\}a - \rho_0\{\xi a_{12}\,x - \eta b_{12}\,y\} + R_e(U) \\ \frac{dx}{d\tau} &= \varepsilon B_1(a,x,y) \\ &= -a_{21}x - (b_{21}+2b_{11})y - 2\rho_0(\xi a_{12}y + \eta b_{12}x)y \\ &\quad -\{\xi a_{23}x + \eta(b_{23}+2b_{14})y\}(x^2+y^2) - \{\xi(a_{22}\rho_0 + a_{24}x) + \eta(b_{24}+2b_{13})y\}a^2 \\ &\quad + R_e(V) + 2y\,I_m(U)/a \end{aligned}\right\} \tag{17}$$

$$\begin{aligned}\frac{dy}{d\tau} &= \varepsilon C_1(a, x, y)\\ &= -a_{21}y+(b_{21}+2b_{11})x+2\rho_0(\xi a_{12}y+\eta b_{12}x)x\\ &\quad -\{\xi a_{23}y-\eta(b_{23}+2b_{14})x\}(x^2+y^2)-\{\xi(a_{24}y-b_{22}\rho_0)-\eta(b_{24}+2b_{13})x\}a^2\\ &\quad +I_m(V)-2x\ I_m(U)/a\end{aligned}$$

$$\begin{aligned}\frac{d\phi}{d\tau} &= \omega_1+\varepsilon D_1(a, x, y)\\ &=\omega_1-b_{11}-\eta b_{13}a^2-\eta b_{14}(x^2+y^2)-\rho_0(\xi a_{12}y+\eta b_{12}x)+I_m(U)/a \end{aligned} \tag{18}$$

where the coefficients a_{11}, a_{13}, $\cdots$, b_{11}, b_{21}, $\cdots$ are all real constants and the funcions $Re()$, $Im()$ represent the real and imaginary parts of the complex functions $U(a, x, y)$ and $V(a, x, y)$ respectively. These are the modified terms resulting from the existence of the zero phase component x_5 and are expressible by components of vector $\boldsymbol{Z}$ as

$$\begin{aligned}U(a, x, y) &= -(\xi+j3\eta)(Z_1S_0+Z_1^*S_1+Z_2^*S_2+Z_3^*S_3+Z_4^*S_4)\\ V(a, x, y) &= -(\xi-j3\eta)(Z_1S_1+Z_2S_0+Z_1^*S_2+Z_2^*S_3+Z_3^*S_4)\end{aligned} \tag{19}$$

$$\begin{aligned}S_0 &= 2(x^2-y^2)cos(3\theta_0)+4xy\ sin(3\theta_0)+j\{2(x^2-y^2)sin(3\theta_0)\\ &\quad -4xy\ cos(3\theta_0)\}\\ S_1 &= 4a\{x\ cos(3\theta_0)+y\ sin(3\theta_0)\}+j4a\{x\ sin(3\theta_0)-y\ cos(3\theta_0)\}\\ S_2 &= 2a^2cos(3\theta_0)+2\rho_0\{x\ cos(3\theta_0)+y\ sin(3\theta_0)\}\\ &\quad +j[2a^2sin(3\theta_0)+2\rho_0\{x\ sin(3\theta_0)-y\ cos(3\theta_0)\}\\ S_3 &= 2\rho_0a\ cos(3\theta_0)+j2\rho_0a\ sin(3\theta_0)\\ S_4 &= \tfrac{1}{2}\rho_0^2\{cos(3\theta_0)+j\ sin(3\theta_0)\}\end{aligned}$$

We can consider the two types of the steady state in the autonomous system of Eq.(17). One corresponds to the singular points and the other to the limit cycles. In our case, only the former case is considered, not the latter. If the real roots (a_0, x_0, y_0) of the simultaneous algebraic equations

$$\left.\begin{aligned}\varepsilon A_1\ (a, x, y) &= 0\\ \varepsilon B_1\ (a, x, y) &= 0\\ \varepsilon C_1\ (a, x, y) &= 0\end{aligned}\right\} \tag{20}$$

are obtained, then the phase angle ϕ is determined by the integration of Eq.(18) as

$$\begin{aligned}\phi &= \int(\omega_1+\Delta_0)d\tau+\phi_0\\ \Delta_0 &= \varepsilon D_1(a_0, x_0, y_0)\end{aligned} \tag{21}$$

where ϕ_0 is an arbitrary constant.

Eq. (21) gives the modification of the frequency ω_1 in the steady state.

We are now in a position to investigate the stability condition in order to confirm the actual existence of the oscillations corresponding to the singular points. Here, the investigation is limited. See Appendix.

4. Representation of the nearly 1/3-harmonic oscillation in the original circuit.

If the singular points are obtained by the above procedures, the first approximate solution becomes

$$\left.\begin{aligned} x_1^{(0)} &= 2a_0cos(\phi+\alpha_1)+2x_0cos(2\phi+\alpha_2)-2y_0sin(2\phi+\alpha_2)\\ x_2^{(0)} &= -2a_0sin(\phi+\alpha_1)-2x_0sin(2\phi+\alpha_2)-2y_0cos(2\phi+\alpha_2)\\ x_3^{(0)} &= -\frac{2}{3}a_0sin(\phi+\alpha_1)-\frac{2}{3}x_0sin(2\phi+\alpha_2)-\frac{2}{3}y_0cos(2\phi+\alpha_2)\\ x_4^{(0)} &= -\frac{2}{3}a_0cos(\phi+\alpha_1)-\frac{2}{3}x_0cos(2\phi+\alpha_2)+\frac{2}{3}y_0sin(2\phi+\alpha_2)\\ x_5^{(0)} &= 2\sum_{l=1}^{5}\{P_lcos(\Omega_l\tau)-Q_lsin(\Omega_l\tau)\} \end{aligned}\right\} \quad (22)$$

where values α_1 and α_2 are the phase angles of the eigen functions of the unpurturbed system.

o-, d-, q- components of the flux interlinkages and condenser voltages are written as

$$\begin{aligned} \Psi_d &= \rho_0cos(\theta_0)+2a_0cos(\phi-\theta_0+\alpha_1)+2x_0cos(2\phi-\theta_0+\alpha_2)-2y_0sin(2\phi-\theta_0+\alpha_2)\\ \Psi_q &= \rho_0sin(\theta_0)-2a_0sin(\phi-\theta_0+\alpha_1)-2x_0sin(2\phi-\theta_0+\alpha_2)-2y_0cos(2\phi-\theta_0+\alpha_2)\\ \Psi_0 &= 2\sum_{l=1}^{5}\{P_lcos(\Omega_l\tau)-Q_lsin(\Omega_l\tau)\} \end{aligned} \quad (23)$$

$$\left.\begin{aligned} v_d &= \eta\rho_0^3sin(\theta_0)-\frac{2}{3}a_0sin(\phi-\theta_0+\alpha_1)-\frac{2}{3}x_0sin(2\phi-\theta_0+\alpha_2)\\ &\quad -\frac{2}{3}y_0cos(2\phi-\theta_0+\alpha_2)\\ v_q &= -\eta\rho_0^3cos(\theta_0)-\frac{2}{3}a_0cos(\phi-\theta_0+\alpha_1)-\frac{2}{3}x_0cos(2\phi-\theta_0+\alpha_2)\\ &\quad +\frac{2}{3}y_0sin(2\phi-\theta_0+\alpha_2) \end{aligned}\right\} \quad (24)$$

The α-, β- components of the flux interlinkages and the terminal voltages of the series condensers can be written as

$$\left.\begin{aligned} \Psi_\alpha &= \rho_0cos(\tau+\theta_0)+2a_0cos(\overline{\phi-1}\tau-\theta_0+\alpha_1)+2x_0cos(\overline{2\phi-1}\tau-\theta_0+\alpha_2)\\ &\quad -2y_0sin(\overline{2\phi-1}\tau-\theta_0+\alpha_2)\\ \Psi_\beta &= \rho_0sin(\tau+\theta_0)-2a_0sin(\overline{\phi-1}\tau-\theta_0+\alpha_1)-2x_0sin(\overline{2\phi-1}\tau-\theta_0+\alpha_2)\\ &\quad -2y_0cos(\overline{2\phi-1}\tau-\theta_0+\alpha_2) \end{aligned}\right\} \quad (25)$$

$$v_\alpha = \eta\rho_0^3 sin(\tau+\theta_0) - \frac{2}{3}a_0 sin(\overline{\phi-1}\tau-\theta_0+\alpha_1) - \frac{2}{3}x_0 sin(\overline{2\phi-1}\tau-\theta_0+\alpha_2)$$
$$-\frac{2}{3}y_0 cos(\overline{2\phi-1}\tau-\theta_0+\alpha_2)$$
$$v_\beta = -\eta\rho_0^3 cos(\tau+\theta_0) - \frac{2}{3}a_0 cos(\overline{\phi-1}\tau-\theta_0+\alpha_1) - \frac{2}{3}x_0 cos(\overline{2\phi-1}\tau-\theta_0+\alpha_2)$$
$$+\frac{2}{3}y_0 sin(\overline{2\phi-1}\tau-\theta_0+\alpha_2) \tag{26}$$

Note that the zero phase component φ_0 is the same for the transformation from o-, d-, q- coordinates to o-, α-, β- coordinates. Thus, the three phase components of the above values can be written as

$$\left.\begin{aligned}
\Psi_a &= \rho_0 cos(\tau+\theta_0) + \Psi_1 cos(O_1\tau+\theta_0+\alpha_1) + \Psi_2 s\ n(O_2\tau-\theta_0+\alpha_2+\beta_0) \\
&\qquad + \sum_{l=1}^{5} r_l cos(\Omega_l\tau+\delta_l) \\
\Psi_b &= \rho_0 cos(\tau+\theta_0-\frac{2}{3}\pi) + \Psi_1 cos(O_1\tau+\theta_0+\alpha_1-\frac{2}{3}\pi) \\
&\quad + \Psi_2 sin(O_2\tau-\theta_0+\alpha_2+\beta_0+\frac{2}{3}\pi) + \sum_{l=1}^{5} r_l cos(\Omega_l\tau+\delta_l) \\
\Psi_c &= \rho_0 cos(\tau+\theta_0+\frac{2}{3}\pi) + \Psi_1 cos(O_1\tau+\theta_0+\alpha_1+\frac{2}{3}\pi) \\
&\quad + \Psi_2 sin(O_2\tau-\theta_0+\alpha_2+\beta_0-\frac{2}{3}\pi) + \sum_{l=1}^{5} r_l cos(\Omega_l\tau+\delta_l) \\
v_a &= V_1 sin(\tau+\theta_0) + V_2 sin(O_1\tau+\theta_0+\alpha_1) - V_3 cos(O_2\tau+\theta_0+\alpha_2+\beta_0) \\
v_b &= V_1 sin(\tau+\theta_0+\frac{2}{3}\pi) + V_2 sin(O_1\tau+\theta_0+\alpha_1-\frac{2}{3}\pi) \\
&\quad - V_3 cos(O_2\tau+\theta_0+\alpha_2+\beta_0+\frac{2}{3}\pi) \\
v_c &= V_1 sin(\tau+\theta_0-\frac{2}{3}\pi) + V_2 sin(O_1\tau+\theta_0+\alpha_1+\frac{2}{3}\pi) \\
&\quad - V_3 cos(O_2\tau+\theta_0+\alpha_2+\beta_0-\frac{2}{3}\pi)
\end{aligned}\right\} \tag{27}$$

where

$$\Psi_1 = 2a_0,\quad \Psi_2 = 2\sqrt{x_0^2+y_0^2},\quad r_l = 2\sqrt{P_l^2+Q_l^2}$$
$$V_1 = \eta\rho_0^3,\quad V_2 = \frac{2}{3}a_0,\quad V_3 = \frac{2}{3}\sqrt{x_0^2+y_0^2},\quad \beta_0 = tan^{-1}\left(\frac{y_0}{x_0}\right)+\phi_0$$
$$\delta_l = tan^{-1}\left(\frac{Q_l}{P_l}\right),\quad O_1 = \frac{1}{3}-\triangle_0,\quad O_2 = \frac{1}{3}+2\triangle_0$$

Eq. (27) shows that when the nearly 1/3-harmonic oscillation occurs in the three phase circuit, the flux interlinkages of the transformer mainly consist of two frequency components Ψ_1 and Ψ_2 and zero phase components r_l, but the terminal

Mode	x_k	$a_0\varphi_k$	$(x_0+jy_0)\chi_k$	$\lvert(x_0+jy_0)\chi_k\rvert$	$arg\{(x_0+jy_0)\chi_k\}$		Z_l	P_l+jQ_l	r_l	Stability
A_1	x_1	$-\ 0.2580\varphi$	$(-\ 0.1017+j0.1199)\chi$	0.1573	−0.867	$x_5^{(0)}$	Z_1	$-\ 0.0102+j0.0364$	0.0756	stable
	x_2	$-j0.2580\varphi$	$(-\ 0.1199-j0.1017)\chi$	0.1573	0.703		Z_2	$0.0020-j0.0074$	0.0153	
	x_3	$j0.0860\varphi$	$(0.0400+j0.0339)\chi$	0.0524	0.703		Z_3	$-\ 0.0070-j0.0009$	0.0141	
	x_4	$-\ 0.0860\varphi$	$(-\ 0.0339+j0.0400)\chi$	0.0524	−0.867		Z_4	$0.0039+j0.0012$	0.0081	
	$\triangle_0$	0.0121					Z_5	$-\ 0.0004+j0.0000$	0.0008	
A_2	x_1	0.2237φ	$(-\ 0.1302+j0.1308)\chi$	0.1846	−0.788	$x_5^{(0)}$	Z_1	$0.0188+j0.0018$	0.0378	stable
	x_2	$j0.2237\varphi$	$(-\ 0.1308-j0.1302)\chi$	0.1846	0.783		Z_2	$0.0000+j0.0179$	0.0358	
	x_3	$j0.0746\varphi$	$(0.0436+j0.0434)\chi$	0.0615	0.783		Z_3	$-\ 0.0061-j0.0004$	0.0122	
	x_4	$-\ 0.0746\varphi$	$(-\ 0.0434+j0.0436)\chi$	0.0615	−0.788		Z_4	$-\ 0.0030-j0.0013$	0.0065	
	$\triangle_0$	0.0062					Z_5	$-\ 0.0005+j0.0001$	0.0010	
A_3	x_1	$-\ 0.1544\varphi$	$(0.0615+j0.1399)\chi$	0.1528	1.157	$x_5^{(0)}$	Z_1	$-\ 0.0053+j0.0101$	0.0228	unstable
	x_2	$-j0.1544\varphi$	$(-\ 0.1399+j0.0615)\chi$	0.1528	−0.414		Z_2	$0.0098-j0.0069$	0.0240	
	x_3	$-j0.0515\varphi$	$(0.0466-j0.0205)\chi$	0.0509	−0.414		Z_3	$-\ 0.0058+j0.0006$	0.0117	
	x_4	0.0515φ	$(0.0205+j0.0466)\chi$	0.0509	1.157		Z_4	$0.0023+j0.0011$	0.0051	
	$\triangle_0$	−0.0079					Z_5	$-\ 0.0005-j0.0002$	0.0010	

Table 1. Stationary solutions for parameters $E=0.30$, $\xi=0.15$, $\eta=0.20$, $\zeta=0.15$.

5. Stability of singular points

For parameters $\xi=0.20$, $\eta=0.15$, $\zeta=0.15$, $E=0.30$, three types of singular points can be found. Two of them are stable and the other unstable. They are listed in Table 1. The singular points A_1 and A_2 are stable and A_3 unstable, which predicts the physical existence of two types of the pearly 1/3- harmonic oscillations in the original circuit. Let's examine the zero phase components $x_5^{(0)}$. Values r_1, r_2, r_3 are greater than values r_4, r_5. Value r_5 which represents the amplitude of the third harmonics of the souce frequency is very small. Consequently, this shows that if there occurs the nearly 1/3-harmonic oscillation in the original circuit, then the zero phase components of the flux interlinkages of the transformer consist of mainly both one-third multiples of the source frequency and the same.

6. Conclusion

By making use of an analog computer, we show the building up of the nearly 1/3- harmonic oscillation and obtain the region where it is sustained on E-η plane for the parameters ξ and ζ. Furthermore, the extended asymptotic method for the system with many degrees of freedom is applied to obtain the stationary solutions corresponding to the nearly 1/3- harmonic oscillations. From the results of the analysis we can see that there exist two types of the nearly 1/3- harmonic oscillation for some parameters.

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Refference

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2) C. Hayashi : "Nonlinear Oscillations in Physical Systems" McGraw-Hill Book Co. 1964

Appendix

Here, we describe the method of the stability investigation of the singular points (a_0, x_0, y_0) based on the Routh-Hurwitz's theorem. Let's consider the variational equation for Eq. (17), which can be written by vector notation as

$$\frac{d\delta \boldsymbol{x}}{d\tau} = \boldsymbol{J}\,(a_0, x_0, y_0)\;\delta \boldsymbol{x} \qquad (\mathrm{I})$$

where vector $\delta \boldsymbol{x}$ is the variational vector from the singular point $(a_0, x_0, y_0,)$ and $\boldsymbol{J}$ $(a_0, x_0, y_0,)$ is the Jacobi matrix whose elements J_{pq} $(p, q=1, 2, 3)$ are

$$J_{11} = -a_{11}-\xi\{3a_{13}a_0^2+a_{14}(x_0^2+y_0^2)\}-(\xi a_{12}x_0-\eta b_{12}y_0)\rho_0+R_e\left(\frac{\partial U}{\partial a}\right)_0$$

$$J_{12} = -\xi(2a_{14}x_0+a_{12}\rho_0)a_0+R_e\left(\frac{\partial U}{\partial x}\right)_0$$

$$J_{13} = -(2\xi a_{14}y_0-\eta b_{12}\rho_0)a_0+R_e\left(\frac{\partial U}{\partial y}\right)_0$$

$$J_{21} = -2\{\xi(a_{24}x_0+a_{22}\rho_0)+\eta(b_{24}+2b_{13})y_0\}a_0+R_e\left(\frac{\partial V}{\partial a}\right)_0-2y_0\left\{\frac{I_m(U)_0}{a_0}-I_m\left(\frac{\partial U}{\partial a}\right)_0\right\}/a_0$$

$$J_{22} = -a_{21}-2\eta b_{12}\rho_0 y_0-\xi a_{23}(x_0^2+y_0^2)-2\{\xi a_{23}x_0+\eta(b_{23}+2b_{14})y_0\}x_0-\xi a_{24}a_0^2 +R_e(V)_0+2y_0\ I_m\left(\frac{\partial U}{\partial x}\right)_0/a_0$$

$$J_{23} = -(b_{21}+2b_{11})-4\xi a_{12}\rho_0 y_0-2\eta b_{12}\rho_0 x_0-\eta(b_{23}+2b_{14})(x_0^2+y_0^2) -2\{\xi a_{23}x_0+\eta(b_{23}+2b_{14})y_0\}y_0-\eta(b_{24}+2b_{13})a_0^2+R_e\left(\frac{\partial V}{\partial y}\right)_0 +2\left\{I_m(U)_0+y_0\ I_m\left(\frac{\partial U}{\partial y}\right)_0\right\}/a_0$$

$$J_{31} = -2\{\xi a_{24}y_0-\eta b_{22}\rho_0-\eta(b_{24}+2b_{13})x_0\}a_0+2x_0\left\{I_m(U)_0/a_0-I_m\left(\frac{\partial U}{\partial a}\right)_0\right\}/a_0 +I_m\left(\frac{\partial U}{\partial a}\right)_0$$

$$J_{32} = b_{21}+2b_{11}+2\rho_0(\xi a_{12}y_0+2\eta b_{12}x_0)+\eta(b_{23}+2b_{14})(x_0^2+y_0^2) -2\{\xi a_{23}y_0-\eta(b_{23}+2b_{14})x_0\}x_0+\eta(b_{23}+2b_{13})a_0^2 +I_m\left(\frac{\partial V}{\partial x}\right)_0-2\left\{I_m(U)+x_0\ I_m\left(\frac{\partial U}{\partial x}\right)_0\right\}/a_0$$

$$J_{33} = -a_{21}+2\xi a_{12}\rho_0 x_0-\xi a_{23}(x_0^2+y_0^2)-2\{\xi a_{23}y_0-\eta(b_{23}+2b_{14})x_0\}y_0 -\xi a_{24}a_0^2+I_m\left(\frac{\partial V}{\partial y}\right)_0-2x_0\ I_m\left(\frac{\partial V}{\partial y}\right)_0/a_0$$

where

$$\frac{\partial U}{\partial a} = -(\xi+j3\eta)\left(\frac{\partial Z_1}{\partial a}S_0+Z_1\frac{\partial S_0}{\partial a}+\frac{\partial Z_1^*}{\partial a}S_1+Z_1^*\frac{\partial S_1}{\partial a}+\frac{\partial Z_2^*}{\partial a}S_2+Z_2^*\frac{\partial S_2}{\partial a} +\frac{\partial Z_3^*}{\partial a}S_3+Z_3^*\frac{\partial S_3}{\partial a}+\frac{\partial Z_4^*}{\partial a}S_4+Z_4^*\frac{\partial S_4}{\partial a}\right)$$

$$\frac{\partial V}{\partial a} = -(\xi - j3\eta)\Big(\frac{\partial Z_1}{\partial a}S_1 + Z_1\frac{\partial S_1}{\partial a} + \frac{\partial Z_2}{\partial a}S_0 + Z_2\frac{\partial S_0}{\partial a} + \frac{\partial Z_1^*}{\partial a}S_2 + Z_1^*\frac{\partial S_2}{\partial a}$$
$$+\frac{\partial Z_2^*}{\partial a}S_3 + Z_2^*\frac{\partial S_3}{\partial a} + \frac{\partial Z_3^*}{\partial a}S_4 + S_3^*\frac{\partial S_4}{\partial a}\Big)$$

$\frac{\partial U}{\partial x}, \frac{\partial V}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial V}{\partial y}$ are obtained in like manner.

$\frac{\partial Z_i}{\partial a}, \frac{\partial Z_i}{\partial x}, \frac{\partial Z_i}{\partial y}$ must be obtained numerically since they are not explicitly expressible as the functions of real variables a, x and y. Partial differentiation of Eq. (12) by variables a, gives

$$\frac{\partial \boldsymbol{Z}}{\partial a} = -\boldsymbol{\varPsi}^{-1}\Big(\frac{\partial \boldsymbol{\varPsi}}{\partial a}\boldsymbol{\varPsi}^{-1}\ \boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial a}\Big) \tag{II}$$

$\frac{\partial \boldsymbol{Z}}{\partial x}$ and $\frac{\partial \boldsymbol{Z}}{\partial y}$ are obtained in like manner.

By the above procedures, the Jacobi matrix $\boldsymbol{J}(a_0, x_0, y_0)$ is numerically obtained and the characteristic equation becomes

$$det\{\lambda \mathbf{1} - \mathbf{J}(a_0, x_0, y_0)\} = 0 \qquad (\mathbf{1} : \text{unit matrix}) \tag{III}$$

Eq. (III) is the third order algebraic equation

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \tag{IV}$$

The real coefficients c_1, c_2 and c_3 can be obtained by the Frame method and if

$$\left.\begin{array}{l} c_i(i=1,2,3)>0 \\ c_1c_2 - c_3^2 > 0 \end{array}\right\} \tag{V}$$

then the real parts of the roots for Eq. (IV) are all negative, meaning the singular point (a_0, x_0, y_0) is asymptotically stable. The singular points for Eq. (17) are obtained by Newton's method which is often effective for the solution of the nonlinear algebraic equations.