A Method of Evaluation of the Function $K_{is}(x)$

By

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The modified Bessel function of the second kind with imaginary order, which is important in the potential theory, is considered from the stand point of numerical computation.

A computing method based on the integral representation of this function is proposed. The accuracy and the computing time are also discussed.

A short table and a graphical representation of this function are given.

Introduction

A special form of the modified Bessel function $K_\nu(x)$ when ν is imaginary, plays important roles in the analysis of some kinds of boundary value problems in the potential theory[1].

Some works have been done to transform the infinite integrals involving the function $K_{is}(x)$ into computable forms. An example can be seen in K. Maeda's paper[2] in which a series expression for the integral

$$\int_0^\infty \frac{\sinh s\pi}{\sinh s\pi - k \sinh s(\pi - 2\alpha)} [\cosh s(\pi-\varphi) + k \cosh s(\pi - 2\alpha + \varphi)] K_{is}(\mu c) K_{is}(\mu r) ds$$

is obtained by the use of Cauchy's theorem.

However, the method to evaluate the function $K_{is}(x)$ itself has not been established yet. In this paper the authors discuss the possibilities of computing the values of $K_{is}(x)$ starting from an integral representation for this function. Then they propose a procedure based on such an algorithm.

1. Integral Representations of the Function $K_{is}(x)$

It is well known that the modified Bessel function of the second kind $K_\nu(x)$ can be expressed in the following forms[3]:

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$$K_\nu(x) = \frac{1}{\cos\frac{\nu\pi}{2}}\int_0^\infty \cos(x\sinh t)\cosh(\nu t)dt\,, \tag{1.1}$$

and

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t}\cosh(\nu t)dt\,, \tag{1.2}$$

where $x<0$ and $-1<R(\nu)<1$.

When $\nu=is$, where s is real, these two become as follows:

$$K_{is}(x) = \frac{1}{\cosh\frac{\pi s}{2}}\int_0^\infty \cos(x\sinh t)\cos(st)dt\,, \tag{1.3}$$

and

$$K_{is}(x) = \int_0^\infty e^{-x\cosh t}\cos(st)dt\,. \tag{1.4}$$

If we put $s=0$, we get

$$K_0(x) = \int_0^\infty \cos(x\sinh t)dt = \int_0^\infty \frac{\cos(xu)}{\sqrt{u^2+1}}du, \qquad (u=\sinh t) \tag{1.5}$$

and

$$K_0(x) = \int_0^\infty e^{-x\cosh t}dt\,. \tag{1.6}$$

2. Numerical Computation of the Function $K_{is}(x)$

It is clear that the representation (1.4) is easier to compute than (1.3), since the integrand of the latter oscillates rapidly as t increases and does not decay when t approaches infinity.

One of the methods based on (1.4) will be to use a quadrature formula of Gauss-Laguerre type, i.e.

$$\int_0^\infty e^{-t}f(t)dt = \sum_{i=1}^n a_i f(t_i)\,. \tag{2.1}$$

In order to apply this formula to (1.4), it is necessary to write

$$K_{is}(x) = \int_0^\infty e^{-t}\left(e^{-x\cosh t+t}\cos(st)\right)dt\,, \tag{2.2}$$

or, substituting $x\cosh t=v+x$,

$$K_{is}(x) = e^{-x}\int_0^\infty e^{-v} f(v) dv\,,$$

$$f(v) = \cos\left(s \log\left(\frac{1}{x}(v+x+\sqrt{v(v+2x)}\right)\right)\frac{1}{\sqrt{v(v+2x)}}\,. \tag{2.3}$$

The results obtained by these methods combined with the 10–point Gauss-Laguerre quadrature formula are shown in Table 1.

Table 1. $K_0(x)$ computed by (2.2) and (2.3).

x	$K_0(x)$ by (2.2)	$K_0(x)$ by (2.3)	$K_0(x)$ exact [4]
0.1	2.4395	1.8566	2.42707
0.2	1.7370	1.3943	1.75270
0.3	1.3689	1.1095	1.37246
0.4	1.1120	0.9092	1.11452
0.5	0.9316	0.7585	0.92442
1.0	0.4188	0.3501	0.42102
1.5	0.2123	0.1787	0.21381
2.0	0.1141	0.0955	0.11389

According to these results, the method using (2.2) yields much higher accuracy than (2.3) in the case of $s=0$. However, both methods yield very large errors when s is large and x is small. For example, when $s=2.00$, (2.2) gives the following values:

for x = 0.1 , 0.2 , 0.3 , 0.4 ,
$K_{is}(x)$ = −0.11508 , −0.20108 , −0.12946 , −0.050795;

while the exact values are:

$K_{is}(x)$ = −0.01229 , −0.07672 , −0.05473 , −0.01707 .

One of the reasons which cause such large errors is the fact that the "envelope" of the integrand of (1.4)

$$g(t) = e^{-x\cosh t} \tag{2.4}$$

decays so rapidly that some of the pivots t_i in the Gaussian formula (2.1) do not contribute to the quadrature. Therefore a considerable improvement should be expected if we apply the Gauss-Legendre or an adequate formula to a finite interval $(0,b)$, where b is the value of t for which $g(t)$ vanishes in the sense that

$$g(t)/g(0) = 10^{-N}\,. \tag{2.5}$$

The root of this equation can be obtained easily:

$$\left.\begin{aligned} b &= \log_e (u+\sqrt{u^2-1}) \,, \\ u &= (N/M)/x+1 \,, \quad 1/M = \log_e 10 \,. \end{aligned}\right\} \tag{2.6}$$

Thus the integral (1.4) becomes:

$$K_{is}(x) = \int_0^b e^{-x \cosh t} \cos (st) dt \,. \tag{2.7}$$

Table 2 shows the values of $K_0(x)$ computed by the Gauss-Legendre formula:

$$K_0(x) = \int_0^b e^{-x \cosh t}\, dt = b\,(0.5a_0 e^{-x} + \sum_{i=1}^{m} a_i \exp(-x \cosh (bt_i))) \,. \tag{2.8}$$

Table 2. Values of $K_0(x)$ computed by quadrature formula (2.8).

x	$m = 7$	$m = 12$	BASS (4)
0.01	4.7210	4.7212 4467 3	4.7212 447
.02	4.0284 1	4.0284 5734 9	4.0284 573
.03	3.6235 5	3.6235 2955 4	3.6235 295
.04	3.3365 9	3.3365 4146 5	3.3365 415
0.5	3.1142 8	3.1142 3403 2	3.1142 340
.06	2.9329 2	2.9328 7953 7	2.9328 795
.07	2.7798 6	2.7798 1776 5	2.7798 178
.08	2.6475 2	2.6474 8946 6	2.6474 895
.09	2.5310 4	2.5310 1710 0	2.5310 171
0.10	2.4270 9	2.4270 6902 3	2.4270 6902 47
.20	1.7526 98	1.7527 0385 6	1.7527 0385 57
.30	1.3724 56	1.3724 6006 1	1.3724 6006 05
.40	1.1145 28	1.1145 2913 5	1.1145 2913 45
.50	0.9244 1903	0.9244 1907 12	0.9244 1907 12
.60	0.7775 227	0.7775 2209 19	0.7775 2209 19
.70	0.6605 205	0.6605 1985 99	0.6605 1985 99
.80	0.5653 447	0.5653 4710 53	0.5653 4710 53
.90	0.4867 307	0.4867 3030 82	0.4867 3030 82
1.00	0.4210 247	0.4210 2443 82	0.4210 2443 82
1.50	0.2138 0550	0.2138 0556 26	0.2138 0556 265
2.00	0.1138 9383	0.1138 9387 27	0.1138 9387 275
2.50	0.0623 4755 1	0.0623 4755 320	0.0623 4755 320
3.00	0.0347 3951	0.0347 3950 439	0.0347 3950 439
3.50		0.0195 9889 717	0.0195 9889 7170
4.00	0.0111 5667 8	0.0111 5967 609	0.0111 5967 6086
4.50		0.0063 9985 7243	0.0063 9985 7243
5.00	0.0036 9109 77	0.0036 9109 8334	0.0036 9109 8334

3 An Exact Table of the Function $K_{is}(x)$

To make an exact table of a function, it is preferable to start directly from a formula which defines the function. In our case, the integral representation (2.7) which is equivalent to (1.4) seems to be most appropriate, because of its simpliicty.

The difficulties expected here are:

(i) the cancellation of significant digits due to the oscillation of the integrand, and

(ii) the computing time necessary to obtain exact values of the integral.

In order to estimate the amount of cancellation, we write

$$K_{is}(x) = \sum_{n=0}^{n_c} (-1)^n G_n(x, s) , \tag{3.1}$$

where

$$\left.\begin{aligned} G_0(s, x) &= \frac{1}{2}\int_{-\pi/(2s)}^{\pi/(2s)} f(t)dt , \\ G_n(s, x) &= \int_{(2n-1)\pi/(2s)}^{(2n+1)\pi/(2s)} f(t)dt , \\ f(t) &= e^{-x\cosh t}\cos(st) , \end{aligned}\right\} \tag{3.2}$$

and

$$n_c = [[2sb/\pi]/2+0.5] \tag{3.3}$$

means the number of zeros of $f(t)$ between $0<t<b$.

The cancellation between two adjacent terms in the series of (3.1) can be roughly estimated by computing the values of the integrand at $t=n\pi/s$, $n=0, 1, \cdots$, or

$$g(n) = f(n\pi/s) = \exp(-x\cosh(n\pi/s)) . \tag{3.4}$$

Table 3. The values of $g(n)$. ($s=5\pi$, $x=0.01$)

n	0	1	2	3	4	5	6
$g(n)$	0.9900	0.9899	0.9893	0.9882	0.9867	0.9847	0.9821

n	7	8	9	10	11	12	13
$g(n)$	0.9787	0.9746	0.9694	0.9631	0.9554	0.9460	0.9436

n	14	15	16	17	18	19	20
$g(n)$	0.9208	0.9042	0.8793	0.8607	0.8327	0.7996	0.7610

n	21	22	23	24	25	26	27
$g(n)$	0.7164	0.6654	0.6081	0.5447	0.4761	0.4040	0.3304

n	28	29	30	35	40
$g(n)$	0.2587	0.1918	0.1330	0.00416	0.00000034

In Table 3, the values of $g(n)$ are shown for the case where $s=5\pi$ and $x=0.01$. It can be seen from this table that considerable cancellations between G_n and G_{n+1} will occur when $n<20$. The number of decimal places lost in the computation of the series (3.1), however, will not exceed three in case $s\leqslant 15$ and $x\geqslant 0.01$.

The amount of computation strongly depends on the values of x and s, since the former determines the upper bound of the integral and the latter changes the shape of the integrand or the number of zeros of the integrand included in the interval $(0, b)$.

In the procedure *Kitc*(s, x) shown in Appendix 1, the trapezoidal rule

$$K_{is}(x) \simeq K(n) = h_n(0.5e^{-x}+\sum_{j=1}^{p} \exp(-x\cosh(jh_n)\cos(sjh_n)) , \tag{3.5}$$

where

$$ph_n \leqslant b \quad \text{and} \quad h_n = 2^{-n}h_0 ,$$

is used, and n is increased until the condition

$$\begin{aligned} &|K(n)-K(n+1)|/|K(n+1)|<10^{-11} \quad \text{if } |K(n+1)|\geqslant 0.5\times 10^{-3}, \\ \text{or} \quad &|K(n)-K(n+1)|<10^{-14} \quad \text{if } s<x \text{ and } |K(n+1)|<0.5\times 10^{-3} \end{aligned} \tag{3.6}$$

is satisfied.

A very large number of iterations was observed in case s is large and x is small. But the results obtained here seem to be accurate to ten decimal places excluding the case where $K_{is}(x)$ is very small.

In Appendix 3, a part of the numerical results is shown, being rounded to eight decimal places.

Fig. 1 illustrates the behavior of the function $K_{is}(x)$ in a perspective form. It should be noted that the function $K_{is}(x)$, $s<0$, oscillates very rapidly when x approaches zero, keeping its amplitude nearly constant (see Fig. 2).

In spite of the property of $K_0(x)$:

$$\lim_{x\to 0} K_0(x) = \infty, \quad \lim_{x\to 0} K_0'(x) = -\infty,$$

the values of $K_{is}(x)$, where s is positive and x is nearly zero, are finite but indeterminate, although it can be shown from (1.3) and (1.4) that $K_{is}(0)=0$.

4. A More Practical Procedure

Although the procedure *Kitc*(s, x) yields very accurate values of the function $K_{is}(x)$, it is too time-consuming to be used for some practical purposes. In order

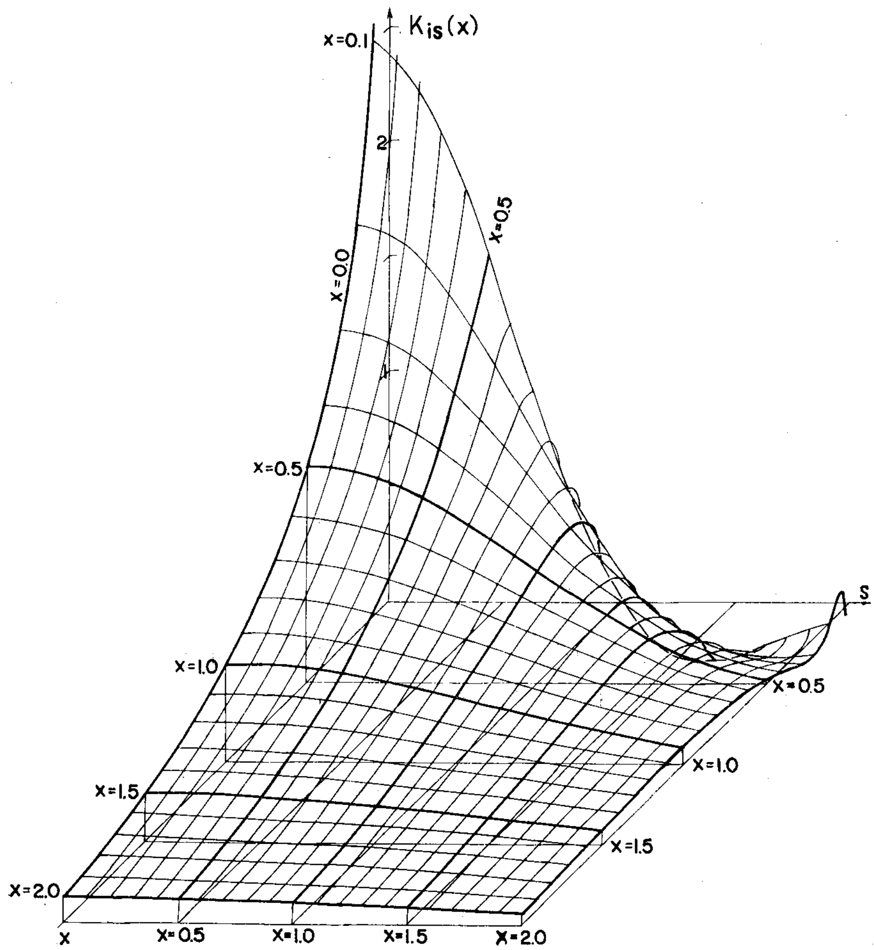

Fig. 1. $K_{is}(x)$ as function of s and x.

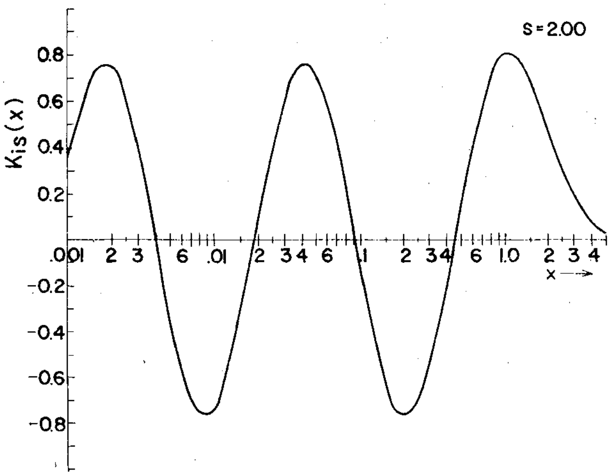

Fig. 2. Oscillation of $K_{is}(x)$ near $x=0$.

Table 4. Comparison of trapezoidal and Gauss-Legendre quadratures.

s	x	method		P	abs. error	rel. error
0.01	0.01	trapez.	h = 0.4	22	$.24\times10^{-10}$	$<10^{-11}$
			0.5	18	$.37\times10^{-8}$	$.78\times10^{-9}$
		Gauss	m= 17	18	$.17\times10^{-10}$	$<10^{-11}$
			12	13	$.51\times10^{-7}$	$.11\times10^{-7}$
	0.05	trapez.	h = 0.4	18	$.24\times10^{-10}$	$<10^{-11}$
			0.5	14	$.15\times10^{-8}$	$.47\times10^{-9}$
		Gauss	m= 17	18	$<10^{-11}$	$<10^{-11}$
			12	13	$.24\times10^{-8}$	$.77\times10^{-9}$
	0.10	trapez.	h = 0.4	16	$<10^{-11}$	$<10^{-12}$
			0.5	13	$.34\times10^{-8}$	$.14\times10^{-8}$
		Gauss	m= 17	18	$<10^{-11}$	$<10^{-12}$
			12	13	$.19\times10^{-8}$	$.79\times10^{-9}$
	0.50	trapez.	h = 0.4	12	$<10^{-11}$	$<10^{-11}$
			0.5	10	$.96\times10^{-9}$	$.10\times10^{-8}$
		Gauss	m= 17	18	$<10^{-12}$	$<10^{-12}$
			12	13	$.17\times10^{-10}$	$.18\times10^{-10}$
	1.00	trapez.	h = 0.3	14	$<10^{-12}$	$<10^{-12}$
			0.4	10	$.24\times10^{-10}$	$.57\times10^{-10}$
			0.5	8	$.17\times10^{-8}$	$.33\times10^{-8}$
		Gauss	m= 17	18	$<10^{-12}$	$<10^{-12}$
			12	13	$<10^{-12}$	$<10^{-11}$
0.05	0.01	trapez.	h = 0.4	22	$.22\times10^{-10}$	$<10^{-11}$
			0.5	18	$.34\times10^{-8}$	$.73\times10^{-9}$
		Gauss	m= 17	18	$.15\times10^{-10}$	$<10^{-11}$
			12	13	$.44\times10^{-7}$	$.94\times10^{-8}$
	0.05	trapez.	h = 0.4	18	$.22\times10^{-10}$	$<10^{-11}$
			0.5	14	$.68\times10^{-9}$	$.22\times10^{-9}$
		Gauss	m= 17	18	$<10^{-11}$	$<10^{-12}$
			12	13	$.11\times10^{-8}$	$.36\times10^{-9}$
	0.10	trapez.	h = 0.4	16	$<10^{-11}$	$<10^{-12}$
			0.5	13	$.32\times10^{-8}$	$.13\times10^{-8}$
		Gauss	m= 17	18	$<10^{-11}$	$<10^{-12}$
			12	13	$.19\times10^{-8}$	$.77\times10^{-9}$
	0.50	trapez.	h = 0.4	12	$<10^{-11}$	$<10^{-11}$
			0.5	10	$.10\times10^{-8}$	$.11\times10^{-8}$
		Gauss	m= 17	18	$<10^{-12}$	$<10^{-12}$
			12	13	$.17\times10$	$.18\times10^{-10}$
	1.00	trapez.	h = 0.3	14	$<10^{-12}$	$<10^{-12}$
			0.4	10	$.24\times10^{-10}$	$.57\times10^{-10}$
			0.5	8	$.17\times10^{-8}$	$.40\times10^{-8}$
		Gauss	m= 17	18	$<10^{-12}$	$<10^{-12}$
			12	13	$<10^{-12}$	$<10^{-11}$

Table 4. Continued

s	x	method		p	abs. error	rel. error
0.10	0.01	trapez.	h = 0.4	22	$.17 \times 10^{-10}$	$< 10^{-11}$
			0.5	18	$.23 \times 10^{-8}$	$.51 \times 10^{-9}$
		Gauss	m= 17	18	$.11 \times 10^{-10}$	$< 10^{-11}$
			12	13	$.25 \times 10^{-7}$	$.55 \times 10^{-8}$
	0.05	trapez.	h = 0.4	18	$.18 \times 10^{-10}$	$< 10^{-11}$
			0.5	14	$.89 \times 10^{-9}$	$.29 \times 10^{-9}$
		Gauss	m= 17	18	$< 10^{-11}$	$< 10^{-12}$
			12	13	$.25 \times 10^{-7}$	$.82 \times 10^{-10}$
	0.10	trapez.	h= 0.4	16	$< 10^{-11}$	$< 10^{-11}$
			0.5	13	$.28 \times 10^{-8}$	$.12 \times 10^{-8}$
		Gauss	m= 17	18	$< 10^{-11}$	$< 10^{-12}$
			12	13	$.17 \times 10^{-8}$	$.70 \times 10^{-9}$
	0.50	trapez.	h = 0.4	12	$< 10^{-11}$	$< 10^{-11}$
			0.5	10	$.11 \times 10^{-9}$	$.12 \times 10^{-9}$
		Gauss	m= 17	18	$< 10^{-12}$	$< 10^{-12}$
			12	13	$.16 \times 10^{-10}$	$.18 \times 10^{-10}$
	1.00	trapez.	h = 0.3	14	$< 10^{-12}$	$< 10^{-12}$
			0.4	10	$.23 \times 10^{-10}$	$.55 \times 10^{-10}$
			0.5	8	$.15 \times 10^{-8}$	$.36 \times 10^{-8}$
		Gauss	m= 17	18	$< 10^{-12}$	$< 10^{-12}$
			12	13	$< 10^{-12}$	$< 10^{-11}$
0.50	0.01	trapez.	h = 0.4	22	$.21 \times 10^{-10}$	$.19 \times 10^{-10}$
			0.5	18	$.41 \times 10^{-8}$	$.36 \times 10^{-8}$
		Gauss	m= 17	18	$.10 \times 10^{-10}$	$< 10^{-11}$
			12	13	$.57 \times 10^{-8}$	$.51 \times 10^{-8}$
	0.05	trapez.	h = 0.4	18	$.32 \times 10^{-10}$	$.19 \times 10^{-10}$
			0.5	14	$.21 \times 10^{-8}$	$.13 \times 10^{-8}$
		Gauss	m= 17	18	$< 10^{-11}$	$< 10^{-12}$
			12	13	$.42 \times 10^{-8}$	$.25 \times 10^{-8}$
	0.10	trapez.	h = 0.4	16	$< 10^{-11}$	$< 10^{-11}$
			0.5	13	$.48 \times 10^{-8}$	$.30 \times 10^{-8}$
		Gauss	m= 17	18	$< 10^{-11}$	$< 10^{-12}$
			12	13	$.28 \times 10^{-8}$	$.18 \times 10^{-8}$
	0.50	trapez.	h = 0.4	12	$.17 \times 10^{-10}$	$.21 \times 10^{-10}$
			0.5	10	$.26 \times 10^{-8}$	$.33 \times 10^{-8}$
		Gauss	m= 17	18	$< 10^{-12}$	$< 10^{-12}$
			12	13	$< 10^{-11}$	$< 10^{-11}$
	1.00	trapez.	h = 0.3	14	$< 10^{-12}$	$< 10^{-12}$
			0.4	10	$.53 \times 10^{-11}$	$.14 \times 10^{-10}$
			0.5	8	$.31 \times 10^{-8}$	$.81 \times 10^{-8}$
		Gauss	m= 17	18	$< 10^{-12}$	$< 10^{-12}$
			12	13	$< 10^{-12}$	$< 10^{-11}$

Table 4. Continued

s	x	method		p	abs. error	rel. error
1.00	0.01	trapez.	h = 0.4	22	$.85 \times 10^{-11}$	$.17 \times 10^{-10}$
			0.5	18	$.25 \times 10^{-8}$	$.50 \times 10^{-8}$
		Gauss	m= 17	18	$.43 \times 10^{-10}$	$.86 \times 10^{-10}$
			12	13	$.25 \times 10^{-6}$	$.50 \times 10^{-6}$
	0.05	trapez.	h = 0.4	18	$.61 \times 10^{-10}$	$.48 \times 10^{-9}$
			0.5	14	$.47 \times 10^{-8}$	$.37 \times 10^{-7}$
		Gauss	m= 17	18	$.16 \times 10^{-11}$	$.13 \times 10^{-10}$
			12	13	$.19 \times 10^{-7}$	$.15 \times 10^{-6}$
	0.10	trapez.	h = 0.4	16	$.31 \times 10^{-10}$	$.14 \times 10^{-9}$
			0.5	13	$.91 \times 10^{-9}$	$.40 \times 10^{-8}$
		Gauss	m= 17	18	$.16 \times 10^{-11}$	$< 10^{-11}$
			12	13	$.49 \times 10^{-8}$	$.22 \times 10^{-7}$
	0.50	trapez.	h = 0.3	16	$< 10^{-12}$	$< 10^{-12}$
			0.4	12	$.52 \times 10^{-10}$	$.11 \times 10^{-9}$
			0.5	10	$.77 \times 10^{-8}$	$.16 \times 10^{-7}$
		Gauss	m= 17	18	$< 10^{-12}$	$< 10^{-12}$
			12	13	$.79 \times 10^{-10}$	$.16 \times 10^{-9}$
	1.00	trapez.	h = 0.3	14	$< 10^{-12}$	$< 10^{-12}$
			0.4	10	$.60 \times 10^{-10}$	$.21 \times 10^{-9}$
			0.5	8	$.42 \times 10^{-8}$	$.15 \times 10^{-7}$
		Gauss	m= 17	18	$< 10^{-12}$	$< 10^{-12}$
			12	13	$.41 \times 10^{-11}$	$.14 \times 10^{-10}$
2.00	0.01	trapez.	h = 0.3	29	$< 10^{-13}$	$< 10^{-12}$
			0.4	22	$.29 \times 10^{-9}$	$.39 \times 10^{-8}$
			0.5	18	$.44 \times 10^{-7}$	$.60 \times 10^{-6}$
		Gauss	m= 17	18	$.79 \times 10^{-9}$	$.11 \times 10^{-7}$
			12	13	$.42 \times 10^{-5}$	$.57 \times 10^{-4}$
	0.05	trapez.	h = 0.3	24	$< 10^{-13}$	$< 10^{-12}$
			0.4	18	$.12 \times 10^{-10}$	$.16 \times 10^{-9}$
			0.5	14	$.30 \times 10^{-7}$	$.42 \times 10^{-6}$
		Gauss	m= 17	18	$.27 \times 10^{-11}$	$.37 \times 10^{-10}$
			12	13	$.28 \times 10^{-6}$	$.39 \times 10^{-5}$
	0.10	trapez.	h = 0.3	21	$.43 \times 10^{-12}$	$.35 \times 10^{-10}$
			0.4	16	$.28 \times 10^{-9}$	$.23 \times 10^{-7}$
			0.5	13	$.16 \times 10^{-7}$	$.13 \times 10^{-5}$
		Gauss	m= 17	18	$.33 \times 10^{-11}$	$.27 \times 10^{-9}$
			12	13	$.35 \times 10^{-7}$	$.28 \times 10^{-5}$
	0.50	trapez.	h = 0.3	16	$< 10^{-13}$	$< 10^{-11}$
			0.4	12	$.27 \times 10^{-9}$	$.16 \times 10^{-7}$
			0.5	10	$.48 \times 10^{-7}$	$.29 \times 10^{-5}$
		Gauss	m= 17	18	$.37 \times 10^{-12}$	$.22 \times 10^{-10}$
			12	13	$.54 \times 10^{-9}$	$.33 \times 10^{-7}$
	1.00	trapez.	h = 0.3	14	$< 10^{-13}$	$< 10^{-17}$
			0.4	10	$.26 \times 10^{-9}$	$.32 \times 10^{-8}$
			0.5	8	$.22 \times 10^{-7}$	$.27 \times 10^{-6}$
		Gauss	m= 17	18	$< 10^{-13}$	$< 10^{-12}$
			12	13	$.37 \times 10^{-10}$	$.46 \times 10^{-7}$

to reduce the computing time, it will be effective to use the trapezoidal rule with a predetermined step width h or the Gauss-Legendre formula with an appropriate number of pivots.

4.1. Trapezoidal rule and Gaussian quadrature.

A series of numerical experiments was made to compare the speed and accuracy of two quadrature methods. The results are shown in Table 4.

It can be seen from these results that in some cases the Gauss-Legendre formula will yield more accurate values than the trapezoidal rule, if the same number of pivots is used.

For example, if we choose the cases where $p=18$, the errors of both methods become as follows:

$p = 18$		rel. error (trapez.)	rel. error (Gauss)
$(s, x) =$	(0.01, 0.01)	$.78\times10^{-9}$	$.36\times10^{-11}$
	(0.01, 0.05)	$.77\times10^{-11}$	$.32\times10^{-12}$
	(0.05, 0.01)	$.73\times10^{-9}$	$.32\times10^{-12}$
	(0.05, 0.05)	$.71\times10^{-11}$	$.32\times10^{-12}$
	(0.10, 0.01)	$.51\times10^{-9}$	$.24\times10^{-11}$
	(0.10, 0.05)	$.62\times10^{-11}$	$.33\times10^{-12}$
	(0.50, 0.01)	$.36\times10^{-8}$	$.90\times10^{-11}$
	(0.50, 0.05)	$.19\times10^{-10}$	$.60\times10^{-12}$
	(1.00, 0.01)	$.50\times10^{-8}$	$.86\times10^{-10}$
	(1.00, 0.05)	$.48\times10^{-9}$	$.13\times10^{-10}$
	(2.00, 0.01)	$.60\times10^{-6}$	$.11\times10^{-7}$
	(2.00, 0.05)	$.16\times10^{-9}$	$.37\times10^{-10}$
	(5.00, 0.01)	$.16\times10^{-1}$	$.20\times10^{-2}$
	(5.00, 0.05)	$.33\times10^{-3}$	$.11\times10^{-3}$

However, if we take the case $p=13$, the error of the Gaussian quadrature exceeds that of the trapezoidal rule when s is large:

$p = 13$		rel. error (trapez.)	rel. error (Gauss)
$(s, x) =$	(0.10, 0.10)	$.12\times10^{-8}$	$.70\times10^{-9}$
	(0.50, 0.10)	$.30\times10^{-8}$	$.18\times10^{-8}$
	(1.00, 0.10)	$.40\times10^{-8}$	$.22\times10^{-7}$
	(2.00, 0.10)	$.13\times10^{-5}$	$.28\times10^{-5}$
	(5.00, 0.10)	$.60\times10^{-1}$	$.32\times10^{+1}$
	(5.00, 1.00)	$.16\times10^{-7}$	$.40\times10^{-3}$

Although the Gaussian quadrature yields better results than the trapezoidal rule under some conditions, the program to take advantage of the former will become very lengthy, since it must be provided with a large number of sets of constants, abscissa and weights. Besides, the trapezoidal rule allows one to estimate the error very easily, and the program can be written very simply.

4.2. Error estimation for the trapezoidal quadratures.

According to the theory developed by Takahashi and Mori[5], the error of the trapezoidal rule can be estimated by the integral of the form:

$$\Delta I = \frac{1}{2\pi i} \oint f(\zeta)\Phi(\zeta)d\zeta\,, \tag{4.1}$$

where

$$\left.\begin{aligned} \Phi(\zeta) &= -\frac{2\pi i}{1-\exp\left(\frac{2\pi\eta}{h}-i\frac{2\pi\xi}{h}\right)}, & \eta>0 \\ &= +\frac{2\pi i}{1-\exp\left(-\frac{2\pi\eta}{h}+i\frac{2\pi\xi}{h}\right)}, & \eta<0 \end{aligned}\right\} \tag{4.2}$$

and

$$f(\zeta) = \exp(-x\cosh\zeta + is\zeta)\,. \tag{4.3}$$

If we take the ptah of integral such as:

$$\oint d\zeta = \int_{-b-ic}^{b-ic} d\xi + \int_{b-ic}^{b+ic} id\eta + \int_{b+ic}^{-b+ic} d\xi + \int_{-b+ic}^{-b-ic} id\eta$$

and put $c=\pi/2$, then we have:

$$|\Delta I| \simeq C\exp(-\pi^2/h+\pi s/2)\,, \tag{4.4}$$

where C is a constant which can be set to 1.0 for a rough estimation. Thus the order of magnitude of the error can be evaluated by:

$$\begin{aligned} |\Delta I| &\simeq 10^{-M(\pi^2/h-\pi s/2)} \\ &\simeq 10^{4.29/h+0.682s} \end{aligned} \tag{4.5}$$

or

$$|\Delta I| \simeq 10^{-4.29/h} \qquad \text{when} \quad h \ll 2\pi/s. \tag{4.6}$$

As shown in Table 5, the formula (4.5) is very useful for error estimation, while the simpler formula (4.6) can not be applied when s is large.

4.3. Estimation of relative error.

The formula (4.5) enables us to evaluate the optimal step width of trapezoidal quadrature for a given pair of (s, x). Solving the inequality:

Table 5. Observed and estimated errors of the trapezoidal. rule.

$(s, x) = (1.00, 0.10)$, $K_{is}(x) = +.22538\ 18853 \times 10^{0}$

h	abs. error observed	abs. error estimated by (4.6)	by (4.5)
0.2	–	$.37\times10^{-21}$	$.18\times10^{-20}$
0.3	–	$.52\times10^{-14}$	$.25\times10^{-13}$
0.4	–	$.19\times10^{-10}$	$.33\times10^{-10}$
0.5	$.91\times10^{-8}$	$.27\times10^{-8}$	$.13\times10^{-7}$

$(s, x) = (2.00, 0.10)$, $K_{is}(x) = -.12290\ 33496 \times 10^{-1}$

h	abs. error observed	abs. error estimated by (4.6)	by (4.5)
0.2	–	$.37\times10^{-21}$	$.86\times10^{-20}$
0.3	–	$.52\times10^{-14}$	$.12\times10^{-12}$
0.4	$.28\times10^{-9}$	$.19\times10^{-10}$	$.45\times10^{-9}$
0.5	$.16\times10^{-7}$	$.27\times10^{-8}$	$.62\times10^{-7}$

$(s, x) = (3.00, 0.10)$, $K_{is}(x) = -.75188\ 38870_5 \times 10^{-2}$

h	abs. error observed	abs. error estimated by (4.6)	by (4.5)
0.2	$.5\times10^{-12}$	$.37\times10^{-21}$	$.41\times10^{-19}$
0.3	$.2\times10^{-12}$	$.52\times10^{-14}$	$.57\times10^{-12}$
0.4	$.14\times10^{-8}$	$.19\times10^{-10}$	$.21\times10^{-8}$
0.5	$.14\times10^{-6}$	$.27\times10^{-8}$	$.30\times10^{-6}$

$(s, x) = (4.00, 0.10)$, $K_{is}(x) = +.23123\ 93456 \times 10^{-2}$

h	abs. error observed	abs. error estimated by (4.6)	by (4.5)
0.2	–	$.37\times10^{-21}$	$.20\times10^{-18}$
0.3	$.1\times10^{-11}$	$.52\times10^{-14}$	$.28\times10^{-11}$
0.4	$.33\times10^{-8}$	$.19\times10^{-10}$	$.10\times10^{-7}$
0.5	$.12\times10^{-5}$	$.27\times10^{-8}$	$.14\times10^{-5}$

$(s, x) = (5.00, 0.10)$, $K_{is}(x) = -.23714\ 18700 \times 10^{-4}$

h	abs. error observed	abs. error estimated by (4.6)	by (4.5)
0.2	–	$.37\times10^{-21}$	$.95\times10^{-18}$
0.3	$.77\times10^{-11}$	$.52\times10^{-14}$	$.13\times10^{-10}$
0.4	$.15\times10^{-7}$	$.19\times10^{-10}$	$.50\times10^{-7}$
0.5	$.14\times10^{-5}$	$.27\times10^{-8}$	$.69\times10^{-5}$

$$10^{-N} \leqslant 10^{-M(\pi^2/h-\pi s/2)}$$

we have

$$h \leqslant 4.286/(N+0.682s) . \tag{4.7}$$

However, the value of h thus obtained guarantees only that the absolute error is less than 10^{-N}.

In order to estimate the optimal value of h to keep the relative error less than a required value, we have to know the approximate value of the function itself.

For such a purpose, the following formula[6] will be useful:

$$K_{is}(x) \sim \sqrt{\frac{\pi}{2}}(x^2-s^2)^{-1/4}\exp(-(s^2-x^2)^{1/2}-s\sin^{-1}(s/x)) , \qquad x>s>0; \quad (4.8a)$$

$$K_{is}(x) \sim \sqrt{2\pi}(s^2-x^2)^{-1/4}\exp(-\pi s/2)\sin(s\cosh^{-1}(s/x)-(s^2-x^2)^{1/2}+\pi/4), \qquad s>x>0; \quad (4.8b)$$

$$K_{is}(x) \sim \frac{\pi}{3}\Gamma\left(\frac{1}{3}\right)\sin\left(\frac{\pi}{3}\right)\exp(-\pi s/2)(x/6)^{-1/3} , \qquad s \simeq x. \quad (4.8c)$$

Considering the case where $K_{is}(x) \sim 0$ when $s>x$, we put $\sin(\cdots) \sim 1.0$ in (4.8b), and we put $\sin^{-1}(s/x) \sim \pi/2$ in (4.8a) for simplicity.

Thus we have:

$$K_{is}(x) \sim 1.2533(x^2-s^2)^{-1/4}\exp(-(x^2-s^2)^{1/2}-1.5708s) , \qquad x>s; \quad (4.9a)$$

$$K_{is}(x) \sim 2.5066(s^2-x^2)^{-1/4}\exp(-1.5708s) , \qquad x<s; \quad (4.9b)$$

$$K_{is}(x) \sim 2.4316\exp(-1.5708s)(x/6)^{-1/3} , \qquad x \simeq s. \quad (4.9c)$$

The equation to estimate the value h becomes:

$$10^{-N} = 10 \uparrow (-M(\pi^2/h-\pi s/2))/K_a , \tag{4.10}$$

where K_a is the approximate value of $K_{is}(x)$ given by (4.9). Solving this equation we get:

$$h = \pi^2/(N/M-\log_e K_a+\pi s/2) ,$$

or

$$h = 9.8696/(2.3026N+1.5708s-\log_e K_a) . \tag{4.11}$$

4.4. Procedure *Kitr*(*s*, *x*).

The procedure *Kitr*(*s*, *x*) given in Appenxdi 2 is based on the trapezoidal quadrature using the optimal value of h estimated by the method described above. The numerical results are shown in Table 6.

It can be seen from this table that the procedure *Kitr*(*s*, *x*) gives fairly good results using relatively a small number of pivots.

Table 6. Results obtained by the procedure Kitr(s, x).

s	x	$K_{is}(x)$ computed	p	rel. error
0.01	0.01	.47191 42928x10^1	17	.42x10^{-9}
	0.02	.40270 76814x10^1	17	.28
	0.05	.31135 15031x10^1	15	.096
	0.10	.24266 71649x10^1	14	.17
	0.20	.17525 09486x10^1	13	.26
	0.50	.92436 25418	11	.15
	1.00	.42100 90479	10	.040
	2.00	.11389 15117	9	.11
	5.00	.36910 64499x10^{-2}	7	.024
0.02	0.01	.47128 41352x10^1	18	.31x10^{-9}
	0.02	.40229 37234x10^1	16	.57
	0.05	.31113 58758x10^1	15	.13
	0.10	.24254 79811x10^1	14	.16
	0.20	.17519 26478x10^1	13	.39
	0.50	.92419 29705	11	.14
	1.00	.42096 28799	10	.30
	2.00	.11388 44288	9	.020
	5.00	.36909 62998x10^{-2}	7	.026
0.05	0.01	.46688 90408x10^1	18	.14x10^{-9}
	0.02	.39940 43025x10^1	17	.20
	0.05	.30962 95230x10^1	15	.22
	0.10	.24171 49208x10^1	14	.14
	0.20	.17478 49659x10^1	13	.066
	0.50	.92300 66937	11	.12
	1.00	.42063 98075	10	.006
	2.00	.11383 48597	9	.11
	5.00	.36902 52559x10^{-2}	8	.042
0.10	0.01	.45141 92445x10^1	19	.019x10^{-9}
	0.02	.38920 25367x10^1	17	.020
	0.05	.30429 25339x10^1	15	.043
	0.10	.23875 71605x10^1	14	.0036
	0.20	.17333 49929x10^1	13	.077
	0.50	.91878 02976	11	.076
	1.00	.41948 78299	10	.024
	2.00	.11365 79873	9	.092
	5.00	.36877 16340x10^{-2}	8	.050
0.20	0.01	.39297 80697x10^1	19	.033x10^{-9}
	0.02	.35018 80122x10^1	18	.038
	0.05	.28360 37596x10^1	16	.057
	0.10	.22719 52757x10^1	14	.10
	0.20	.16762 84852x10^1	12	.12
	0.50	.90203 48225	11	.024
	1.00	.41490 72556	10	.041
	2.00	.11295 29949	9	.058
	5.00	.36775 88033x10^{-2}	8	.051

Table 6. Continued

s	x	$K_{is}(x)$ computed	p	rel. error
0.50	0.01	.11098 86091x10^{1}	20	.045x10^{-9}
	0.02	.14597 74241x10^{1}	19	.016
	0.05	.16524 45846x10^{1}	17	.062
	0.10	.15736 89487x10^{1}	15	.081
	0.20	.13162 51439x10^{1}	13	.074
	0.50	.79173 43053	11	.20
	1.00	.38404 30169	10	.029
	2.00	.10812 83324	9	.020
	5.00	.36074 27131x10^{-2}	8	.027
1.00	0.01	−.50063 37168	22	.022x10^{-9}
	0.02	−.47860 84238	20	.066
	0.05	−.12703 35077	18	.14
	0.10	.22538 18853	16	.071
	0.20	.47533 34599	14	.094
	0.50	.48339 60900	12	.017
	1.00	.28942 80370	10	.21
	2.00	.92385 45989x10^{-1}	10	.002
	5.00	.33670 99989x10^{-2}	8	.016
2.00	0.01	−.73834 84194x10^{-1}	25	.003x10^{-9}
	0.02	.64838 68854x10^{-2}	23	.69
	0.05	.72056 07945x10^{-1}	20	.009
	0.10	−.12290 33497x10^{-1}	18	.90
	0.20	−.76721 62242x10^{-1}	16	.061
	0.50	.16502 01895x10^{-1}	14	.22
	1.00	.80616 99762x10^{-1}	12	.017
	2.00	.47997 99085x10^{-1}	10	.059
	5.00	.25494 65278x10^{-2}	9	.002
5.00	0.01	−.38948 30912x10^{-3}	34	.34x10^{-9}
	0.02	.43102 57512x10^{-3}	31	.13
	0.05	−.11577 03973x10^{-3}	27	34.
	0.10	−.23714 18703x10^{-4}	25	18.
	0.20	.16035 12892x10^{-3}	22	.09
	0.50	−.42411 71460x10^{-3}	18	5.0
	1.00	.38046 18280x10^{-3}	16	.24
	2.00	−.34633 78807x10^{-3}	13	.31
	5.00	.31859 10253x10^{-3}	10	.22

4.5. Computing time.

The algorithm proposed here still takes considerable time to get a value of $K_{is}(x)$, due mainly to the large number of values of the integrand.

The time t_f necessary to compute one value of the integrand can be roughly estimated by:

$$t_f = t_c + 2t_x\,,$$

where t_c and t_x are the times necessary to compute cos (x) and exp (x) respectively.

Taking into account the preparatory computations to evaluate the approximate value of $K_{is}(x)$ and to determine the values of h and b for given values of s and x, the approximate total time T necessary to get a value of $K_{is}(x)$ can be estimated by:

$$T = t_n + t_q + t_h + pt_f\,, \tag{4.12}$$

$$\begin{aligned} t_h &= t_n + 2t_q + t_x \quad &\text{when} \quad s<x \text{ or } s<x, \\ &= 2t_n + 2t_x \quad &\text{when} \quad s\sim x, \end{aligned} \tag{4.13}$$

where t_n and t_q are computing time for $\ln(x)$ and $\mathrm{sqrt}(x)$ respectively.

The integer p is given by:

$$p = [b/h]\,. \tag{4.14}$$

If we assume a case where

$$t_q = 100\ \mu\text{s}\,, \quad t_c = t_x = 250\ \mu\text{s}\,, \quad t_n = 200\ \mu\text{s}\,,$$

the time T becomes:

$$\begin{aligned} T &= 950 + 750p\ \mu\text{s} \quad &\text{when} \quad s>x \text{ or } x<s, \\ &= 1200 + 750p\ \mu\text{s} \quad &\text{when} \quad s\sim x. \end{aligned} \tag{4.15}$$

Therefore this method might find its application in a range where $p<10$ (i.e. $T<2$ ms.) and when other methods such as series expansion will take longer computing time.

5. Conclusion

The procedure $Kitr(s, x)$ proposed here is capable of yielding very accurate values of the function $K_{is}(x)$, as verified by the comparison with the results of a more elaborated method.

However, such an algorithm based on the integral representation of this function is sometimes too time-consuming to be used in the numerical analysis of some kinds of problems containing this function.

A more efficient computing method seems to be obtained by the use of the series expansions of this function. The authors are now considering means to establish a practical algorithm, the results of which will be published in near future.

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Appendix 1. Procedure *Kitc*(*s*, *x*)

```
real procedure Kitc(s, x); value s, x; real s, x;
begin
    real b, h, hh, K, KK, t, u;
    h: =hh: =0.4;
    K: =exp(-x)×h;
    u: =25.0/x+1.0;
    b: =ln(u+sqrt(u×u-1.0));
LLL:
    KK: =0.0;
    t: =h;
LL:
    if t>b then goto CC;
    u: =exp(t);
    KK: =KK+exp(-x×0.5×(u+1.0/u))×cos(s×t);
    t: =t+hh;
    goto LL;
CC:
    KK: =0.5×K+h×KK;
    if abs((K-KK)/KK)<₁₀-11 then goto FIN;
    if s>x∧abs(KK)<0.5₁₀-3∧abs(K-KK)<₁₀-14
    then goto FIN;
```

```
      K: =KK;
      hh: =if h=0.4 then 0.4 else 0.5×hh;
      h: =0.5×h;
      goto LLL;
FIN:
      Kitc: =KK
end of Kitc;
```

Appendix 2. Procedure *Kitr*(*s*, *x*)

```
real procedure Kitr(s, x); value s, x; real s, x;
begin
      real b, h, K, t, u; integer N;
      N: =10; comment N should b be adjusted depending on the required accuracy;
      if s/x<0.9 then
            begin
            u: =sqrt (x×x—s×s);
            K: =(1.2533/sqrt (u))×exp (—u—1.5708×s)
            end else
      if x/s<0.9 then
            begin
            u: =sqrt (s×s—x×x);
            K: =(2.5066/sqrt (u))×exp (—1.5708×s)
            end else
            K: =2.4316×exp (—1.5708×s)×(6.0/x) ↑ (1/3);
      h: =9·8696/(2.3026×N+1.5708×s—ln (K));
      u: =1.0+2.3026×N/x;
      b: =ln (u+sqrt (u×u—1.0));
      K: =0.5×exp (—x);
      for t:=h step h until b do
            begin
            u: =exp (t)
            K: =K+exp (—x×0.5×(u+1.0/u))×cos (s×t)
            end
      Kitr: =h×k
end of Kitr;
```

Appendix 3. Table of the Function $K_{is}(x)$

x	s = 0.01	s = 0.02	s = 0.03	s = 0.04	s = 0.05
0.01	4.7191 429	4.7128 414	4.7023 514	4.6876 923	4.6688 904
2	4.0270 768	4.0229 372	4.0160 445	4.0064 085	3.9940 430
3	3.6224 790	3.6193 287	3.6140 825	3.6067 468	3.5973 307
4	3.3356 885	3.3331 304	3.3288 700	3.3229 121	3.3152 631
0.05	3.1135 150	3.1113 588	3.1077 674	3.1027 444	3.0962 950
6	2.9322 584	2.9303 956	2.9272 928	2.9229 529	2.9173 800
7	2.7792 717	2.7776 339	2.7749 059	2.7710 901	2.7661 896
8	2.6470 030	2.6455 440	2.6431 137	2.6397 141	2.6353 479
9	2.5305 793	2.5292 661	2.5270 786	2.5240 184	2.5200 879
0.10	2.4266 716	2.4254 798	2.4234 944	2.4207 168	2.4171 492
20	1.7525 095	1.7519 265	1.7509 551	1.7495 960	1.7478 497
30	1.3723 417	1.3719 867	1.3713 953	1.3705 676	1.3695 040
40	1.1144 496	1.1142 112	1.1138 139	1.1132 579	1.1125 433
0.50	0.9243 6254	0.9241 9297	0.9239 1041	0.9235 1494	0.9230 0669
60	0.7774 8034	0.7773 5510	0.7771 4640	0.7768 5430	0.7764 7887
70	0.6604 8818	0.6603 9314	0.6602 3477	0.6600 1311	0.6597 2820
80	0.5653 2257	0.5652 4897	0.5651 2632	0.5649 5464	0.5647 3398
90	0.4867 1100	0.4866 5308	0.4865 5656	0.4864 2147	0.4862 4782
1.00	0.4210 0905	0.4209 6288	0.4208 8594	0.4207 7825	0.4206 3983
1.50	0.2137 9992	0.2137 8299	0.2137 5479	0.2137 1530	0.2136 6455
2.00	0.1138 9151	0.1138 8443	0.1138 7262	0.1138 5610	0.1138 3486
2.50	0.0623 4648 7	0.0623 4328 9	0.0623 3795 9	0.0623 3049 8	0.0623 2090 6
3.00	0.0347 3899 8	0.0347 3748 1	0.0347 3495 1	0.0347 3141 0	0.0347 2685 8
3.50	0.0195 9864 8	0.0195 9790 2	0.0195 9665 9	0.0195 9491 8	0.0195 9268 1
4.00	0.0111 5955 1	0.0111 5917 4	0.0111 5854 7	0.0111 5766 8	0.0111 5653 9
4.50	0.0063 9979 26	0.0063 9959 87	0.0063 9927 55	0.0063 9882 31	0.0063 9824 15
5.00	0.0036 9106 45	0.0036 9096 30	0.0036 9079 38	0.0036 9055 70	0.0036 9025 26

x	s = 0.10	s = 0.20	s = 0.30	s = 0.40	s = 0.50
0.01	4.5141 924	3.9297 807	3.0698 503	2.0783 686	1.1098 861
2	3.8920 254	3.5018 8011	2.9119 673	2.2012 569	1.4597 742
3	3.5195 361	3.2201 279	2.7610 211	2.1954 485	1.5859 465
4	3.2520 096	3.0075 497	2.6292 608	2.1565 009	1.6364 071
0.05	3.0429 253	2.8360 376	2.5137 600	2.1067 995	1.6524 458
6	2.8712 390	2.6919 523	2.4112 338	2.0539 088	1.6504 480
7	2.7255 990	2.5675 788	2.3191 327	2.0008 486	1.6382 143
8	2.5991 698	2.4581 048	2.2355 521	1.9489 176	1.6199 033
9	2.4875 109	2.3603 157	2.1590 575	1.8986 756	1.5979 095
0.10	2.3875 716	2.2719 528	2.0885 485	1.8503 386	1.5736 895
20	1.7333 499	1.6762 849	1.5844 273	1.4624 096	1.3162 514
30	1.3606 662	1.3257 704	1.2692 014	1.1932 480	1.1009 282
40	1.1066 033	1.0831 012	1.0448 360	0.9931 1783	0.9296 8957
0.50	0.9187 8030	0.9020 3482	0.8746 8770	0.8375 5619	0.7917 3431
60	0.7733 5628	0.7609 7115	0.7406 9923	0.7130 8025	0.6788 4015
70	0.6573 5807	0.6479 4971	0.6325 2301	0.6114 4934	0.5852 3002
80	0.5628 9803	0.5556 0530	0.5436 3058	0.5272 3742	0.5067 8268
90	0.4848 0282	0.4790 5987	0.4696 1884	0.4566 7137	0.4404 7762
1.00	0.4194 8783	0.4149 0726	0.4073 6964	0.3970 1711	0.3840 4302
1.50	0.2132 4199	0.2115 5924	0.2087 8101	0.2049 4625	0.2001 0833
2.00	0.1136 5799	0.1129 5299	0.1117 8684	0.1101 7262	0.1081 2833
2.50	0.0622 4102 7	0.0619 2245 2	0.0613 9483 0	0.0606 6311 3	0.0597 3413 3
3.00	0.0346 8894 5	0.0345 3767 7	0.0342 8692 7	0.0339 3871 9	0.0334 9585 3
3.50	0.0195 7404 2	0.0194 9965 4	0.0193 7626 2	0.0192 0474 2	0.0189 8630 5
4.00	0.0111 4713 2	0.0111 0958 1	0.0110 4726 1	0.0109 6056 6	0.0108 5004 2
4.50	0.0063 9339 67	0.0063 7405 22	0.0063 4193 45	0.0062 9722 71	0.0062 4018 47
5.00	0.0036 8771 63	0.0036 7758 80	0.0036 6076 63	0.0036 3733 88	0.0036 0742 71

x	s = 0.60	s = 0.70	s = 0.80	s = 0.90	s = 1.00
0.01	0.2974 7093	−0.2720 4894	−0.5700 6828	−0.6236 5445	−0.5006 3372
2	0.7733 3270	+0.2100 2906	−0.1889 6241	−0.4128 2424	−0.4786 0842
3	0.9950 4998	0.4765 3852	+0.0687 0154 5	−0.2093 6784	−0.3580 6366
4	1.1175 429	0.6436 9282	0.2487 5463	−0.0465 5127 9	−0.2357 8658
0.05	1.1899 357	0.7558 0181	0.3798 7162	+0.0824 4481 0	−0.1270 3351
6	1.2334 296	0.8339 1897	0.4782 5145	0.1856 2172	−0.0332 5508 4
7	1.2588 290	0.8895 0528	0.5536 2884	0.2690 9121	+0.0470 1706 6
8	1.2722 557	0.9293 9606	0.6122 1646	0.3372 9649	0.1157 2325
9	1.2774 453	0.9579 4421	0.6581 7846	0.3934 8134	0.1746 6625
0.10	1.2768 049	0.9780 6227	0.6944 1819	0.4400 5228	0.2253 8189
20	1.1529 420	0.9799 6794	0.8048 2530	0.6345 5426	0.4753 3346
30	0.9958 1750	0.8818 5149	0.7631 1347	0.6436 2299	0.5271 3838
40	0.8566 4235	0.7763 1581	0.6911 8939	0.6037 7069	0.5164 8739
0.50	0.7385 4609	0.6794 8980	0.6161 7620	0.5502 6406	0.4833 9609
60	0.6388 6318	0.5941 5839	0.5458 2210	0.4949 9824	0.4428 3818
70	0.5544 7882	0.5199 0069	0.4822 6787	0.4423 9426	0.4011 0918
80	0.4827 0506	0.4555 1106	0.4257 5920	0.3940 4285	0.3609 7256
90	0.4213 5847	0.3996 8603	0.3758 7284	0.3503 6006	0.3236 0524
1.000	0.3686 8651	0.3512 2596	0.3319 7136	0.3112 5605	0.2894 2804
1.50	0.1943 3388	0.1877 0142	0.1802 9972	0.1722 2602	0.1635 8399
2.00	0.1056 7660	0.1028 4427	0.0996 6193 4	0.0961 6347 6	0.0923 8546 0
2.50	0.0586 1650 2	0.0573 2048 8	0.0558 5787 0	0.0542 4176 8	0.0524 8646 1
3.00	0.0329 6186 7	0.0323 4099 5	0.0316 3811 2	0.0308 5867 6	0.0300 0865 9
3.50	0.0187 2248 6	0.0184 1512 7	0.0180 6635 3	0.0176 7855 5	0.0172 5435 7
4.00	0.0107 1638 4	0.0105 6042 4	0.0103 8312 8	0.0101 8558 3	0.0099 6898 73
4.50	0.0061 7113 06	0.0060 9045 41	0.0059 9860 69	0.0058 9609 89	0.0057 8349 40
5.00	0.0035 7118 61	0.0035 2880 21	0.0034 8049 20	0.0034 2650 07	0.0033 6710 00

x	s = 1.10	s = 1.20	s = 1.30	s = 1.40	s = 1.50
0.01	−0.2874 8367	−0.0664 9504 4	+0.1019 5904	+0.1883 7470	0.1935 2416
2	−0.4239 3795	−0.2969 1556	−0.1456 2949	−0.0094 4330 86	+0.0864 1546 8
3	−0.3944 1680	−0.3469 3317	−0.2492 9551	−0.1341 8611	−0.0283 1297 8
4	−0.3257 3428	−0.3335 5401	−0.2827 9793	−0.1990 5719	−0.1059 8424
0.05	−0.2497 3831	−0.2958 4943	−0.2819 3332	−0.2278 1816	−0.1534 8467
6	−0.1763 1702	−0.2492 4473	−0.2636 9719	−0.2350 2632	−0.1798 4510
7	−0.1086 1496	−0.2004 8679	−0.2367 3456	−0.2292 5110	−0.1917 7113
8	−0.0473 8800 4	−0.1526 7368	−0.2056 9269	−0.2156 5012	−0.1938 7787
9	+0.0075 0436 27	−0.1072 2447	−0.1731 6417	−0.1974 1419	−0.1893 1505
0.10	0.0565 2881 5	−0.0647 3886 1	−0.1406 2454	−0.1765 6411	−0.1802 4888
20	0.3321 6211	+0.2086 4979	+0.1069 2333	+0.0276 4915 2	−0.0298 4021 0
30	0.4169 8546	0.3159 2231	0.2260 4555	0.1487 4507	0.0847 0067 2
40	0.4315 8848	0.3510 5986	0.2765 5835	0.2093 6650	0.1503 6960
0.50	0.4171 3854	0.3529 2714	0.2920 2183	0.2354 7202	0.1840 9341
60	0.3904 6201	0.3389 2280	0.2891 7546	0.2420 5108	0.1982 3785
70	0.3592 3167	0.3175 4617	0.2767 8069	0.2375 8810	0.2005 3129
80	0.3271 5839	0.2931 9305	0.2596 3633	0.2270 0141	0.1957 4341
90	0.2960 6990	0.2682 0760	0.2404 5263	0.2132 0992	0.1868 4630
1.00	0.2668 4112	0.2438 4624	0.2207 8323	0.1979 7325	0.1757 1212
1.50	0.1544 8179	0.1450 2991	0.1353 3916	0.1255 1867	0.1156 7397
2.00	0.0883 6656 4	0.0841 4696 1	0.0797 6769 5	0.0752 7006 5	0.0706 9501 7
2.50	0.0506 0719 1	0.0486 1995 5	0.0465 4129 3	0.0443 8807 2	0.0421 7727 1
3.00	0.0290 9447 8	0.0281 2291 7	0.0271 0104 9	0.0260 3615 4	0.0249 3563 7
3.50	0.0167 9659 5	0.0163 0828 1	0.0157 9257 7	0.0152 5275 9	0.0146 9218 4
4.00	0.0097 3464 03	0.0094 8392 88	0.0092 1831 43	0.0089 3931 93	0.0086 4851 42
4.50	0.0056 6140 49	0.0055 3048 82	0.0053 9143 81	0.0052 4498 11	0.0050 9186 99
5.00	0.0033 0258 57	0.0032 3327 58	0.0031 5950 76	0.0030 8163 51	0.0030 0002 65

x	s = 1.60			s = 1.70			s = 1.80			s = 1.90			s = 2.00		
0.01	0.1402	6356		0.0614	3879	5	−0.0120	0166	1	−0.0594	2289	5	−0.0738	3484	2
2	0.1323	8656		0.1325	8759		+0.1003	8894		+0.0530	0058	5	+0.0064	8386	89
3	+0.0506	4704	2	0.0949	4133	9	0.1056	7858		0.0903	6675	6	0.0597	9290	0
4	−0.0222	8122	7	+0.0399	9979	0	0.0757	7548	4	0.0860	5255	4	0.0762	1924	0
0.05	−0.0764	3175	4	−0.0098	4346	95	0.0382	8778	9	0.0651	0333	2	0.0720	5607	9
6	−0.1138	0335		−0.0498	5245	4	+0.0028	2837	31	0.0391	9476	7	0.0580	5409	9
7	−0.1380	1414		−0.0801	2010	3	−0.0275	4314	9	+0.0135	1781	6	0.0401	6018	8
8	−0.1522	8753		−0.1019	9014		−0.0522	3220	1	−0.0097	3643	17	0.0215	3126	1
9	−0.1591	5764		−0.1169	7727		−0.0715	6438	0	−0.0297	7976	0	+0.0037	8810	67
0.10	−0.1605	4570		−0.1264	4789		−0.0861	7173	0	−0.0464	8186	0	−0.0122	9033	5
20	−0.0673	3522	7	−0.0874	7930	9	−0.0934	6788	8	−0.0887	4482	1	−0.0767	2162	2
30	+0.0339	2355	7	−0.0041	6749	66	−0.0306	4223	7	−0.0469	3712	1	−0.0547	2560	6
40	0.1000	5447		+0.0585	2849	9	+0.0255	5625	4	+0.0006	0991	235	−0.0170	7050	1
0.50	0.1384	5684		0.0988	8890	2	0.0654	8330	5	0.0381	2169	9	+0.0165	0201	9
60	0.1582	6890		0.1225	1711		0.0911	9680	1	0.0643	7150	8	0.0419	6722	8
70	0.1660	7233		0.1345	6614		0.1062	5841		0.0812	8765	4	0.0596	9099	4
80	0.1662	5059		0.1388	3822		0.1137	4538		0.0911	3446	2	0.0710	9327	6
90	0.1616	8345		0.1379	9258		0.1159	9108		0.0958	4101	6	0.0776	4941	4
1.00	0.1542	6473		0.1338	6062		0.1146	9089		0.0969	0634	9	0.0806	1699	8
1.50	0.1059	0534		0.0963	0620	4	0.0869	6185	3	0.0779	4838	0	0.0693	3185	7
2.00	0.0660	8257	0	0.0614	7127	8	0.0568	9774	0	0.0523	9616	3	0.0479	9799	1
2.50	0.0399	2577	4	0.0376	5017	1	0.0353	6656	4	0.0330	9039	4	0.0308	3628	5
3.00	0.0238	0695	0	0.0226	5750	7	0.0214	9461	6	0.0203	2539	9	0.0191	5672	8
3.50	0.0141	1425	9	0.0135	2240	8	0.0129	2003	9	0.0123	1051	4	0.0116	9711	9
4.00	0.0083	4750	21	0.0080	3790	60	0.0077	2135	38	0.0073	9946	58	0.0070	7384	10
4.50	0.0049	3287	69	0.0047	6878	85	0.0046	0039	83	0.0044	2850	15	0.0042	5388	88
5.00	0.0029	1506	15	0.0028	2712	80	0.0027	3661	97	0.0026	4393	33	0.0025	4946	53

x	s = 2.50	s = 3.00	s = 3.50	s = 4.00	s = 4.50
0.01	+0.0293 5520 2	−0.0122 9729 4	+0.0053 4323 16	−0.0023 3642 73	+0.0009 9088 329
2	−0.0152 7068 2	+0.0096 7287 75	−0.0048 5352 39	+0.0022 2890 68	−0.0009 8672 579
3	−0.0312 0296 9	0.0115 0382 2	−0.0032 6524 90	+0.0005 9947 569	+0.0000 5754 8803
4	−0.0244 3056 0	+0.0028 7743 41	+0.0019 8240 23	−0.0018 2123 35	0.0009 8207 112
0.05	−0.0104 2464 5	−0.0056 1058 60	0.0050 1047 27	−0.0022 8748 76	+0.0007 1167 080
6	+0.0035 9916 15	−0.0108 9174 6	0.0053 5852 27	−0.0013 7551 39	−0.0000 3486 3360
7	0.0149 9775 9	−0.0129 1580 5	0.0039 8858 91	−0.0000 2727 2531	−0.0006 6966 931
8	0.0231 4802 7	−0.0124 7778 9	+0.0018 6294 06	+0.0011 6789 59	−0.0009 7691 493
9	0.0282 4095 5	−0.0104 4276 6	−0.0003 6099 132	0.0019 6132 35	−0.0009 6468 328
0.10	0.0307 4813 2	−0.0075 1883 89	−0.0023 1033 23	+0.0023 1239 35	−0.0007 2778 300
20	+0.0006 0459 304	+0.0129 3483 4	−0.0015 3410 54	−0.0020 2434 02	+0.0007 4452 346
30	−0.0261 9556 4	+0.0058 8109 74	+0.0049 7803 85	−0.0010 7951 35	−0.0008 4166 032
40	−0.0311 3459 6	−0.0049 5673 34	0.0046 5140 34	+0.0014 5300 37	−0.0007 6479 604
0.50	−0.0244 5093 2	−0.0113 6253 1	+0.0012 6544 49	0.0023 4887 60	+0.0001 3881 860
60	−0.0135 3020 1	−0.0131 1692 9	−0.0021 4808 25	0.0017 8875 02	0.0008 2340 337
70	−0.0021 5440 56	−0.0117 1968 1	−0.0044 4116 53	+0.0005 9397 008	0.0010 0958 19
80	+0.0079 9820 90	−0.0086 0126 40	−0.0054 6267 61	−0.0006 3159 530	0.0007 9994 326
90	0.0163 2968 0	−0.0047 7788 32	−0.0054 4258 56	−0.0015 8277 76	+0.0003 8463 288
1.00	0.0227 6353 2	−0.0008 8614 792	−0.0046 9892 29	−0.0021 6071 36	−0.0000 7699 5652
1.50	0.0337 6741 5	+0.0120 1643 3	+0.0019 0704 85	−0.0010 5930 99	−0.0009 9886 901
2.00	0.0284 3237 6	0.0142 3804 1	0.0056 2186 34	+0.0013 9516 24	−0.0001 1460 025
2.50	0.0203 3529 8	0.0119 2433 8	0.0060 6268 94	0.0025 3954 19	+0.0007 6298 170
3.00	0.0135 3739 3	0.0087 3048 13	0.0050 7486 16	0.0026 0364 65	0.0011 3275 30
3.50	0.0086 7853 93	0.0059 7134 47	0.0037 8285 83	0.0021 8223 38	0.0011 2594 09
4.00	0.0054 4264 70	0.0039 2638 59	0.0026 4360 67	0.0016 5019 56	0.0009 4560 270
4.50	0.0033 6674 60	0.0025 1847 16	0.0017 7518 75	0.0011 7376 83	0.0007 2359 722
5.00	0.0020 6396 17	0.0015 8910 29	0.0011 6094 87	0.0008 0234 103	0.0005 2238 971

x	s = 5.00			s = 5.50			s = 6.00		
0.01	−0.0003	8948	309	+0.0001	3175	219	−0.0000	3117	8953
2	+0.0004	3102	575	−0.0001	8759	173	+0.0000	8144	1967
3	−0.0001	3617	953	+0.0000	9556	7273	−0.0000	5292	3071
4	−0.0004	2768	742	+0.0001	6216	582	−0.0000	5444	9041
0.05	−0.0001	1577	040	−0.0000	3711	1942	+0.0000	4791	0057
6	+0.0002	6069	137	−0.0001	7633	924	0.0000	8175	2871
7	0.0004	2978	725	−0.0001	6805	637	+0.0000	3987	9462
8	0.0003	7986	959	−0.0000	6652	1448	−0.0000	2418	2612
9	+0.0001	9795	182	+0.0000	5381	6419	−0.0000	6966	3330
0.10	−0.0000	2371	4187	+0.0001	4434	254	−0.0000	8241	2650
20	+0.0001	6035	129	−0.0001	8913	331	+0.0000	3884	8722
30	+0.0002	9351	275	+0.0001	1053	739	−0.0000	7703	0240
40	−0.0002	7930	614	+0.0001	5284	926	+0.0000	4127	8065
0.50	−0.0004	2411	715	−0.0000	5318	6727	0.0000	7933	3641
60	−0.0001	8137	466	−0.0001	8188	778	+0.0000	1596	9912
70	+0.0001	4423	746	−0.0001	6155	767	−0.0000	5497	6371
80	0.0003	6766	281	−0.0000	5426	3180	−0.0000	8283	8092
90	0.0004	3861	397	+0.0000	6545	9538	−0.0000	6612	8250
1.00	+0.0003	8046	183	0.0001	5196	750	−0.0000	2431	8212
1.50	−0.0003	5406	011	+0.0000	0868	33457	+0.0000	7304	0311
2.00	−0.0003	4633	788	−0.0001	9447	243	−0.0000	4777	9430
2.50	+0.0000	6248	7561	−0.0001	0573	604	−0.0000	8327	1902
3.00	0.0003	7941	675	+0.0000	6435	4426	−0.0000	2792	1892
3.50	0.0005	0287	216	0.0001	8077	122	+0.0000	4074	8091
4.00	0.0004	8966	527	0.0002	2291	385	0.0000	8421	7378
4.50	0.0004	1217	797	0.0002	1394	038	0.0000	9881	0962
5.00	0.0003	1859	103	0.0001	8051	008	0.0000	9383	3139