A Finite Element Formulation of The Coupled Thermoelastic-Plastic Problems

By

Tatsuo INOUE* and Kikuaki TANAKA*

(Received March 29, 1973)

Abstract

A finite element formulation for the thermoelastic-plastic material is presented in the incremental form with thermomechanical considerations. Emphasis is focused upon the constitutive relation by using the generalized yield function. The governing equations for finite deformation containing the increments of displacement and temperature as unknowns are deduced for the material.

1. Introduction

This paper is concerned with the finite element formulation of coupled thermoelastic-plastic materials, which is a special case of the various kinds of coupled problems in the field of natural science. The thermodynamic consideration in addition to the kinetic treatment is required to formulate such problems in continuum mechanics.[1),2)]

Many attempts have been made to deal with the coupled problems of thermoelastic or thermoviscoelastic materials[3)~5)], and the fundamental equations have been established.[6)~9)] In particular, Biot's variational formulation[10)] is one of the brilliant investigations, where he deduced a variational principle similar to Hamilton's principle in dynamics.

Recently, the finite element method was introduced to analyze the coupled problems for elastic, viscoelastic or general simple materials.[11)~14)] On the other hand, the finite element analyses for elastic-plastic materials were made, for the uncoupled problems, such as isothermal deformations with infinitesimal[15),16)] or finite strain[16)~21)] and thermal stress problems.[22)~24)]

As for the coupled thermoelastic-plastic materials, however, a difficulty is raised

* Department of Mechanical Engineering

about the treatment of the thermomechanical constitutive relations, especially, of the internal dissipation function. The emphasis focused upon in the present paper is the formulations of stress-strain relation in terms of yield function and the incremental equations of motion and heat conduction for finite deformation with the internal dissipation function.

2. Preliminaries

Consider the continuous body $\mathcal{B}$ under the action of the general system of external forces and heat condition. The body is under a natural unstrained state with a uniform temperature T_0 at the time $t=0$, and the body is referred to occupy the configuration C_0.

In order to trace the motion of the body and the time variation of the temperature, we select the other state in the configuration C_t. The material coordinates $x_i (i=1, 2, 3)$, which coincide with the spatial rectangular cartesian coordinates $X_i (i=1, 2, 3)$ in the reference configuration C_o, are introduced.

The motion of the body can be defined by the equation of the form

$$X_i = X_i(x_1, x_2, x_3, t), \qquad \left|\frac{\partial X_i}{\partial x_j}\right| > 0. \tag{1}$$

Let the base vactor referred to X_i be denoted $\boldsymbol{i}_i$ and the base vectors referred to x_i, $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$. The material point P_o in the body $\mathcal{B}$ in C_o transforms to the point P in C_t. If the points P_o and P are identified by the position vectors $\boldsymbol{r}_0$ and $\boldsymbol{r}$, respectively, the displacement vector is defined by

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{r}_0 = (X_i - x_i)\boldsymbol{i}_i = u_i \boldsymbol{i}_i, \tag{2}$$

where u_i is the cartesian component of the displacement relative to C_o. It is easy to show the relation between $\boldsymbol{G}_i$ and $\boldsymbol{i}_i$:

$$\boldsymbol{G}_i = X_{i,m} \boldsymbol{i}_m = (\delta_{im} + u_{i,m}) \boldsymbol{i}_m, \tag{3}$$

where the comma denotes the partial derivatives with respect to x_i.

For later convenience we introduce the interpolating function $\psi(\boldsymbol{x})$ to approximate the displacement and the temperature field in the finite element. The components of the displacement vector can be described by making use of this function as

$$u_i(x_1, x_2, x_3, t) = \psi_N(\boldsymbol{x}) u_i^N(t), \tag{4}$$

where the repeated nodal index N is to be summed up from 1 to N_e, while N_e is the total number of the nodes composing each element.

The components of the Green strain tensor and the strain rate tensor can be also written as

$$\gamma_{ij}=\frac{1}{2}(\psi_{N,j}u_i^N+\psi_{N,i}u_j^N+\psi_{N,i}\psi_{M,j}u_m^N u_m^M), \tag{5}$$

and

$$\dot{\gamma}_{ij}=\frac{1}{2}[\psi_{N,j}\dot{u}_i^N+\psi_{N,i}\dot{u}_j^N+\psi_{N,i}\psi_{M,j}(u_m^N\dot{u}_m^M+u_m^M\dot{u}_m^N)] \tag{6}$$

where the superimposed dot denotes the time derivative.

The absolute temperature θ at the point P of the body is the sum of the uniform temperature T_0 and the change in temperature T, that is

$$\theta=T_0+T=T_0+\psi_N T^N. \tag{7}$$

The temperature increment $\Delta\theta$ is then given by

$$\Delta\theta=\Delta T=\psi_N\Delta T^N. \tag{8}$$

3. Thermomechanical Constitutive Relations

It has already been mentioned that, to investigate the coupled problems in the continuum mechanics, the thermomechanical consideration should be introduced. At the first step, the discussion of the constitutive relations for the thermoelastic-plastic material is made in this section before the formulation of the coupled thermoelastic-plastic problem. Here, we deduce the incremental stress-strain relation following the constitutive relations proposed by Perzyna and Wojno.[25,26]

The constitutive relations for the thermoelastic-plastic body are generally given by

$$\begin{aligned}
&\phi=\phi(\gamma_{ij}^e, \gamma_{ij}^p, \theta)\\
&\eta=-\frac{\partial\phi}{\partial\theta}\\
&K^{ij}=\rho_0\frac{\partial\phi}{\partial\gamma_{ij}^e}\\
&\boldsymbol{q}=\boldsymbol{q}(\gamma_{ij}^e, \gamma_{ij}^p, \theta, \mathrm{grad}\,\theta)\\
&\dot{\gamma}_{ij}^p=\Lambda n_{ij}
\end{aligned} \tag{9}$$

where γ_{ij}^e and γ_{ij}^p denote the elastic and plastic components of the strain tensor γ_{ij}, respectively. The functions ϕ and η are the specific Helmholtz free energy and the specific entropy, K^{ij} the Kirchhoff stress tensor, ρ_0 the density in C_0, $\boldsymbol{q}$ the heat flux vector per unit area in C_0, while $\mathrm{grad}\,\theta$ is the gradient of the temperature with respect to x_i. In the above relation, Λ is the constant which should be determined in terms of the yield function and n_{ij} is the second order symmetric tensor function.

Assume now the yield function in the form

$$F=f(K^{ij}, \gamma_{ij}^p, \theta)-k(\kappa, \theta)=0 \tag{10}$$

where k is the function of the isotropic hardening parameter κ and the absolute temperature θ.[27]

The constant Λ in Eq.(9)$_5$ can be determined by the condition $\dot{F}=0$ as

$$\Lambda=\hat{G}\left[\frac{\partial F}{\partial K^{ij}}\dot{K}^{ij}+\frac{\partial F}{\partial \theta}\dot{\theta}\right], \tag{11}$$

with the non-negative function

$$\hat{G}=\left[-\frac{\partial F}{\partial \gamma_{ij}^p}n_{ij}\right]^{-1}. \tag{12}$$

Hence, the plastic strain rate $\dot{\gamma}_{ij}^p$ is deduced for each case as

$$\dot{\gamma}_{ij}^p=\begin{cases} \hat{G}\left(\dfrac{\partial F}{\partial K^{mn}}\dot{K}^{mn}+\dfrac{\partial F}{\partial \theta}\dot{\theta}\right)n_{ij}; & F=0 \text{ and } \dfrac{\partial F}{\partial K^{mn}}\dot{K}^{mn}+\dfrac{\partial F}{\partial \theta}\dot{\theta}>0 \\ 0; \quad F=0 \text{ and } \dfrac{\partial F}{\partial K^{mn}}\dot{K}^{mn}+\dfrac{\partial F}{\partial \theta}\dot{\theta}\leq 0 \text{ or } F<0. & \end{cases} \tag{13}$$

The internal dissipation function[2] which will be clarified in the next section is expressed in the form,[26]

$$\begin{aligned}\sigma&=\sigma(\gamma_{ij}^e, \gamma_{ij}^p, \theta)\\ &=\hat{G}\left(\frac{\partial F}{\partial K^{mn}}\dot{K}^{mn}+\frac{\partial F}{\partial \theta}\dot{\theta}\right)\left(K^{ij}-\rho_0\frac{\partial \phi}{\partial \gamma_{ij}^p}\right)n_{ij}.\end{aligned} \tag{14}$$

Of course, these quantities must be constrained by the general dissipation inequality

$$\sigma+\frac{1}{\theta}\boldsymbol{q}\,\mathrm{grad}\,\theta\geq 0. \tag{15}$$

Assumption is made on the specific Helmholtz free energy ϕ,[28] which is given as the summation of eleastic and plastic parts as

$$\phi=\phi^e(\gamma_{ij}^e, \theta)+\phi^p(\gamma_{ij}^p), \tag{16}$$

where ϕ^e is equal to the free energy adopted in thermoelastic problems and given by

$$\rho_0\phi^e=\frac{1}{2}E^{ijkl}\gamma_{ij}^e\gamma_{kl}^e+B^{ij}\gamma_{ij}^e T+\frac{1}{2}\frac{c}{T_0}T^2, \tag{17}$$

with the elastic constants

$$E^{ijkl}=E^{jikl}=E^{ijlk}=E^{klij}, \quad B^{ij}=B^{ji}$$

and the specific heat c. The plastic part of the free energy ϕ^p may have the form

$$\rho_0\phi^p=\frac{1}{2}\Phi^{ijkl}\gamma_{ij}^p\gamma_{kl}^p \tag{18}$$

where the tensor coefficients Φ^{ijkl} satisfy the same symmetry relations as E^{jikl}.

We now try to deduce the incremental stress-strain relation for the thermoelastic-plastic body from the constitutive equation (9) under the assumptions already mentioned. The free energy ϕ^* in the configuration $C_{t+\Delta t}$ at $t=t+\Delta t$ after an infinitesimal change from C_t, is obtained by expanding it in the Taylor series at C_t and neglecting the higher order terms;

$$\begin{aligned}\phi^*(\gamma_{ij}{}^e+\Delta\gamma_{ij}{}^e,\ \gamma_{ij}{}^p+\Delta\gamma_{ij}{}^p,\ \theta+\Delta\theta) \\ =\phi(\gamma_{ij}{}^e, \gamma_{ij}{}^p, \theta)+\frac{\partial\phi}{\partial\gamma_{ij}{}^e}\Delta\gamma_{ij}{}^e+\frac{\partial\phi}{\partial\gamma_{ij}{}^p}\Delta\gamma_{ij}{}^p+\frac{\partial\phi}{\partial\theta}\Delta\theta.\end{aligned} \tag{19}$$

Hence, by making use of Eqs.(9), (18) and (19), the stress tensor $\overset{*}{K}{}^{ij}$ in $C_{t+\Delta t}$ is

$$\begin{aligned}\overset{*}{K}{}^{ij}&=\rho_0\frac{\partial\phi^*}{\partial\gamma_{ij}{}^e} \\ &=K^{ij}+E^{ijkl}(\Delta\gamma_{ij}-\Delta\gamma_{kl}{}^p)+B^{ij}\Delta T,\end{aligned} \tag{20}$$

where the condition $\partial^2\phi/\partial\gamma_{ij}{}^e\partial\gamma_{ij}{}^p=0$ is used, which is deduced from Eq.(16) and $\Delta\gamma_{ij}{}^e$ is replaced by $\Delta\gamma_{ij}-\Delta\gamma_{ij}{}^p$. We can have the stress increment ΔK^{ij} corresponding to the change of the configuration $C_{t+\Delta t}$ from C_t by substituting Eq.(13) into Eq.(20),

$$\Delta K^{ij}=E^{ijkl}\Delta\gamma_{kl}-E^{ijkl}\hat{G}\left(\frac{\partial F}{\partial K^{mn}}\Delta K^{mn}+\frac{\partial F}{\partial\theta}\Delta T\right)n_{kl}+B^{ij}\Delta T. \tag{21}$$

The incremental stress-strain relation is now obtained by solving Eq.(21) on ΔK^{ij}, i.e.,

$$\begin{aligned}\Delta K^{ij}&=\left(E^{ijkl}-\frac{\hat{G}}{S}E^{ijrs}n_{rs}E^{pqkl}\frac{\partial F}{\partial K^{pq}}\right)\Delta\gamma_{kl} \\ &\quad+\left(B^{ij}-\frac{\hat{G}}{S}E^{ijkl}n_{kl}B^{mn}\frac{\partial F}{\partial K^{mn}}-\frac{\hat{G}}{S}E^{ijkl}n_{kl}\frac{\partial F}{\partial\theta}\right)\Delta T \\ &\equiv A^{ijkl}\Delta\gamma_{kl}+D^{ij}\Delta T,\end{aligned} \tag{22}$$

with

$$S=1+\hat{G}E^{ijkl}n_{kl}\frac{\partial F}{\partial K^{ij}}. \tag{23}$$

It is not difficult to show that in the case of an elastic deformation, the material constants are reduced to

$$A^{ijkl}=E^{ijkl},\quad D^{ij}=B^{ij}\quad\text{and}\quad S=1.$$

This means that A^{ijkl} and D^{ij} have the same symmetry law as E^{ijkl} and B^{ij}, respectively. Kawahara and Horii[20] introduced a constitutive equation similar to Eq. (22) under the condition that the yield function was not affected by the temperature. In the isothermal deformation, the relation similar to Eq.(22) was also presented under different conditions.[21),29)]

The increment of the plastic strain $\Delta\gamma_{ij}^p$ is obtained by substituting Eq.(22) into Eq.(13);

$$\Delta\gamma_{ij}^p=\hat{G}\frac{\partial F}{\partial K^{mn}}A^{mnkl}n_{ij}\Delta\gamma_{kl}+\hat{G}\left(\frac{\partial F}{\partial K^{mn}}D^{mn}+\frac{\partial F}{\partial\theta}\right)n_{ij}\Delta T$$
$$\equiv a_{ij}^{kl}\Delta\gamma_{kl}+\beta_{ij}\Delta T, \tag{24}$$

with

$$a_{ij}^{kl}=a_{ij}^{lk}=a_{ji}^{kl} \quad \text{and} \quad \beta_{ij}=\beta_{ji}.$$

For later convenience, we try to evaluate the explicit expressions of $\rho_0\dot{\eta}$, $\rho_0\Delta\dot{\eta}$ and $\Delta\sigma$. First of all, $\rho_0\dot{\eta}$ can be expressed by substituting Eq.(16) into Eq.(9)

$$\rho_0\dot{\eta}=-\rho_0\frac{\partial\dot{\phi}}{\partial\theta}=-\left(B^{ij}\dot{\gamma}_{ij}^e+\frac{c}{T_0}\dot{\theta}\right), \tag{25}$$

where we have used Eq.(17).

A combination of Eqs.(16) and (17) leads to the form as

$$\rho_0\Delta\dot{\eta}=(B^{kl}a_{kl}^{ij}-B^{ij})\Delta\dot{\gamma}_{ij}+\left(\beta_{ij}B^{ij}-\frac{c}{T_0}\right)\Delta\dot{T}$$
$$+\dot{B}^{ij}a_{ij}^{kl}\Delta\gamma_{kl}+B^{ij}\dot{\beta}_{ij}\Delta T$$
$$\equiv{}_{\dot{\gamma}}S^{ij}\Delta\dot{\gamma}_{ij}+{}_{\dot{T}}S\Delta\dot{T}+{}_{\gamma}S^{ij}\Delta\gamma_{ij}+{}_{T}S\Delta T. \tag{26}$$

Since the increment of the internal dissipation function $\Delta\sigma$ is expressed from Eq.(14) as

$$\Delta\sigma=\frac{\partial\sigma}{\partial\gamma_{ij}^e}\Delta\gamma_{ij}+\left(\frac{\partial\sigma}{\partial\gamma_{mn}^p}-\frac{\partial\sigma}{\partial\gamma_{mn}^e}\right)\Delta\gamma_{ij}^p+\frac{\partial\sigma}{\partial\theta}\Delta T,$$

the explicit from of $\Delta\sigma$ is given as follows:

$$\Delta\sigma=\left[\frac{\partial\gamma_{ij}^e}{\partial\sigma}+\left(\frac{\partial\sigma}{\partial\gamma_{mn}^p}-\frac{\partial\sigma}{\partial\gamma_{mn}^e}\right)a_{mn}^{ij}\right]\Delta\gamma_{ij}$$
$$+\left[\beta_{mn}\left(\frac{\partial\sigma}{\partial\gamma_{mn}^p}-\frac{\partial\sigma}{\partial\gamma_{mn}^e}\right)+\frac{\partial\sigma}{\partial\theta}\right]\Delta T$$
$$\equiv{}_{\gamma}\sigma^{ij}\Delta\gamma_{ij}+{}_{T}\sigma\Delta T, \tag{27}$$

with

$${}_{\gamma}\sigma^{ij}={}_{\gamma}\sigma^{ji}.$$

4. Governing Equations for a Finite Element

A typical finite element with the volume v_0 and the surface area A_0 at the reference configuration C_0 is considered to discuss the energy balance of the continuum under the action of external forces and heat supply. As will be discussed at the end of this chapter, it is easy to obtain the equations governing the entire system.

The law of conservation of energy for the finite element, considering only thermo-mechanical behavior, is

$$\dot{K}+\dot{U}=\Omega+Q, \tag{28}$$

where K and U are the kinetic energy and internal energy of the element, and Ω and Q are the mechanical power and heat supply, respectively. If all quantities are refered to C_o, these are expressed as follows,

$$\begin{aligned}
\dot{K}&=\int_{v_0}\rho_0\ddot{u}_i\dot{u}_i dv\\
\dot{U}&=\int_{v_0}\rho_0\dot{\varepsilon}dv\\
\Omega&=\int_{v_0}\rho_0F_i\dot{u}_i dv+\int_{A_0}P_i\dot{u}_i dA\\
Q&=\int_{A_0}q_i\nu_{0i}dA+\int_{v_0}\rho_0 h dv.
\end{aligned} \tag{29}$$

In Eq.(29) ε and h are the internal energy and the heat supply per unit mass, respectively, F_i is the body force per unit mass, P_i and q_i are the surface force and the heat flux per unit area referred to x_i, respectively; while ν_{oi} is the unit normal to A_o.

Substitution of Eq.(29) into Eq.(28) gives the global form of energy balance for the element as

$$\begin{aligned}
&\int_{v_0}\rho_0\ddot{u}_i\dot{u}_i dv+\int_{v_0}\rho_0\dot{\varepsilon}dv\\
&\quad=\int_{v_0}\rho_0F_i\dot{u}_i dv+\int_{A_0}P_i\dot{u}_i dA+\int_{A_0}q_i\nu_{0i}dA+\int_{v_0}\rho_0 h dv.
\end{aligned} \tag{30}$$

Bearing in mind the following expression for the surface force,

$$P_m=K^{ij}\nu_{0i}(\delta_{jm}+u_{m,j}), \tag{31}$$

and assuming that the principle of balance of linear momentum hold, we have the local form of energy balance from Eq.(30) under the certain continuity requirement,

$$\rho_0\dot{\varepsilon}=(K^{mj}X_{i,j})\dot{u}_{i,m}+q_{i,i}+\rho_0 h. \tag{32}$$

The equation of motion for the element is obtained by substituting Eq.(32) into Eq.(30) and using Eq.(4),

$$\begin{aligned}
&\int_{v_0}\rho_0\psi_N\psi_M dv\ddot{u}_i{}^M+\int_{v_0}K^{mj}(\delta_{ij}+\psi_{M,j}u_i{}^M)\psi_{N,m}dv\\
&\quad=\int_{v_0}\rho_0\psi_N F_i dv+\int_{A_0}\psi_N P_i dA.
\end{aligned} \tag{33}$$

Since the increment of the surface force ΔP_i can be denoted as

$$\begin{aligned}\Delta P_i &= \Delta K^{mj}(\delta_{ij}+\psi_{M,j}u_i{}^M)\nu_{0m}+K^{mj}\psi_{M,j}\nu_{0m}\Delta u_i{}^M \\ &\equiv \Delta P_0{}^j(\delta_{ij}+\psi_{M,j}u_i{}^M)+P_0{}^j\psi_{M,j}\Delta u_i{}^M,\end{aligned} \tag{34}$$

the incremental form of Eq.(33) is obtained by adopting the constitutive equation (22) in the following

$$\begin{aligned}&\int_{v_0}\rho_0\psi_M\psi_N dv\Delta\ddot{u}_i{}^M \\ &+\Big[\int_{v_0}A^{mjkl}\psi_{M,k}\psi_{N,m}(\delta_{lr}+\psi_{K,l}u_r{}^K)(\delta_{ij}+\psi_{L,j}u_i{}^L)dv \\ &\qquad+\int_{v_0}K^{mj}\psi_{M,j}\psi_{N,m}dv\delta_{ir}-\int_{A_0}P_0{}^j\psi_N\psi_{M,j}dA\delta_{ir}\Big]\Delta u_r{}^M \\ &+\int_{v_0}D^{mj}\psi_{N,m}\psi_M(\delta_{ij}+\psi_{K,j}u_i{}^K)dv\Delta T^M \\ &=\int_{v_0}\rho_0\psi_N\Delta F_i d\tau+\int_{A_0}\Delta P_0{}^j\psi_N(\delta_{ij}+\psi_{M,j}u_i{}^M)dA,\end{aligned} \tag{35}$$

and is written briefly as

$$M_{MN}\Delta\ddot{u}_i{}^M+K_{MNri}\Delta u_r{}^M+\Theta_{MNi}\Delta T^M=\Delta\Pi_{Ni}. \tag{36}$$

We now introduce the free energy ϕ and internal dissipation function σ in order to deduce the equation of heat conduction,

$$\phi=\varepsilon-\eta\theta, \tag{37}$$

$$\sigma=K^{ij}\dot{\gamma}_{ij}-\rho_0(\dot{\phi}+\eta\dot{\theta}). \tag{38}$$

The concrete expression for these functions has already been given in Sec. 3. Since the relation

$$(K^{ij}X_{m,j})\dot{u}_{m,i}=\sigma+\rho_0(\dot{\varepsilon}-\dot{\eta}\theta)$$

is obtained from Eqs.(37) and (38), it can be shown that the following equation of the local form of energy balance is rewritten as

$$\rho_0\dot{\eta}\theta=q_{i,i}+\rho_0 h+\sigma. \tag{39}$$

Upon multiplying both sides of Eq.(39) by T and making use of the Green-Gauss theorem, we obtain the following general equation of heat conduction

$$\begin{aligned}\int_{v_0}\rho_0 T\theta\dot{\eta}dv&=\int_{A_0}Tq_i\nu_{0i}dA \\ &\quad-\int_{v_0}(q_iT_i-\rho_0Th)dv+\int_{v_0}\sigma Tdv.\end{aligned} \tag{40}$$

The incremental form of Eq.(40) is obtained by operating Δ after substituting Eqs.

(4), (7) and (8) into Eq.(40) and simplifying it, i.e.,

$$\int_{v_0} \rho_0 \psi_N (\eta \Delta\theta + \theta\Delta\eta) dv = \int_{A_0} \psi_N \Delta q_i \nu_{0i} dA$$
$$- \int_{v_0} (\psi_{N,i} \Delta q_i - \rho_0 \psi_N \Delta h) dv + \int_{v_0} \psi_N \Delta\sigma dv. \tag{41}$$

We now require the Fourier's law of the heat conduction which can be denoted as

$$q_i = -\lambda^{mn} (\delta_{im} + u_{i,m})(\delta_{jn} + u_{j,n}) T_{,j}, \tag{42}$$

where λ^{ij} is the thermal conductivity tensor in the configuration C_t. By making use of Eqs.(25), (26) and (42), the incremental form of the general equation of heat conduction is obtained from Eq.(41),

$$\begin{aligned}
&\int_{v_0} {}_{\dot\gamma}S^{ij} \psi_N \psi_{M,j} (T_0 + \psi_K T^K)(\delta_{ir} + \psi_{L,j} u_r{}^L) dv \Delta \dot{u}_r{}^M \\
&+ \int_{v_0} {}_{\dot T}S \psi_N \psi_M (T_0 + \psi_K T^K) dv \Delta \dot{T}^M \\
&+ \Big\{ \int_{v_0} \psi_N \psi_{M,i} (T_0 + \psi_K T^K)[{}_{\dot\gamma}S^{ij} \psi_{L,j} \dot{u}_r{}^L + {}_{\gamma}S^{ij} (\delta_{jr} + u_r{}^K \psi_{K,j})] dv \\
&\quad - \int_{v_0} \lambda^{mj} \psi_{N,i} \psi_{M,j} [(\delta_{lm} + u_{l,m}) \psi_{K,l} \delta_{ir} + (\delta_{im} + u_{i,m}) \psi_{K,r}] T^K dv \\
&\quad - \int_{v_0} {}_{\gamma}\sigma^{ij} \psi_N \psi_{M,i} (\delta_{jr} + \psi_{K,j} u_r{}^K) dv \Big\} \Delta u_r{}^M \\
&+ \Big\{ \int_{v_0} \psi_N \psi_M \left({}_{\dot\gamma}S_{ij} \gamma_{ij} + {}_{\dot T}S \dot{T} \right) dv + \int_{v_0} {}_T S \psi_N \psi_M (T_0 + \psi_K T^K) dv \\
&\quad - \int_{v_0} \lambda^{mn} \psi_{N,i} \psi_{M,j} (\delta_{im} + u_{i,m})(\delta_{jn} + u_{j,n}) dv \\
&\quad - \int_{v_0} {}_T\sigma \psi_N \psi_M dv \Big\} \Delta T^M \\
&= \int_{A_0} \psi_N \Delta q_i \nu_{0i} dA + \int_{v_0} \psi_N \Delta h dv,
\end{aligned} \tag{43}$$

or briefly

$$a_{NMr} \Delta \dot{u}_r{}^M + b_{NM} \Delta \dot{T}^M + k_{NMr} \Delta u_r{}^M + \vartheta_{NM} \Delta T^M = \Delta Z_N. \tag{44}$$

Now we have the fundamental equations for the coupled thermoelastic-plastic problems for an element in the form of Eqs.(36) and (44). The final equations of the whole system are deduced by superposing these equations of an element over the entire domain, and taking into account the mechanical and the thermal boundary conditions of the system considered. They can be expressed in the matrix form as

$$\boldsymbol{M} \Delta \ddot{\boldsymbol{u}} + \boldsymbol{K} \Delta \boldsymbol{u} + \boldsymbol{\Theta} \Delta \boldsymbol{T} = \Delta \boldsymbol{\Pi}, \tag{45}$$
$$\boldsymbol{a} \Delta \dot{\boldsymbol{u}} + \boldsymbol{b} \Delta \dot{\boldsymbol{T}} + \boldsymbol{k} \Delta \boldsymbol{u} + \boldsymbol{\vartheta} \Delta \boldsymbol{T} = \Delta \boldsymbol{Z}. \tag{46}$$

where $\Delta \boldsymbol{u}$ and $\Delta \boldsymbol{T}$ are the nodal displacement increment and nodal temperature increment vectors of the whole system, and the coefficient matrices in the above equations are obtained by superposing the element coefficient matrices in Eqs.(36) and (44).

As the time derivatives of $\Delta \boldsymbol{u}$ and $\Delta \boldsymbol{T}$ in the equations such as $\Delta \dot{\boldsymbol{u}}$, $\Delta \ddot{\boldsymbol{u}}$ and $\Delta \dot{\boldsymbol{T}}$ are expressed approximately by their current and preceding values,[30] we can obtain the incremental displacement $\Delta \boldsymbol{u}$ and temperature $\Delta \boldsymbol{T}$ by solving the linear simultaneous equations. It is easy to calculate the strain increment $\Delta \gamma_{ij}$, the stress increment ΔK^{ij} and also the current values of γ_{ij} and K^{ij} of the elements once $\Delta \boldsymbol{u}$ and $\Delta \boldsymbol{T}$ are determined.

6. Concluding Remarks

In order to formulate the coupled thermoelastic-plastic problem, the thermomechanical constitutive equation for the thermoelastic-plastic material was deduced with the assumption that the free energy of the material is given as the summation of two parts, i.e., elastic and plastic parts.

The finite element formulation for the equation of motion and that of heat conduction during finite deformation was presented in the incremental form with the internal dissipation function proposed by Perzyna and Wojno.

The final governing equations containing nodal displacement increments and nodal temperature increments as unknowns were deduced.

References

1) C. Truesdell: "Elements of Continuum Mechanics", Springer-Verlag, Berlin (1966).
2) C. Truesdell: "Rational Thermodynamics", McGraw-Hill, New York (1969).
3) W. K. Nowacki ed.: "Progress in Thermoelasticity", Polish Scientific Publishers, Warszawa (1969).
4) A. C. Eringen: "Nonlinear Theory on Continuous Media", McGrow-Hill, New York (1962).
5) Y. C. Fung: "Foundation of Solid Mechanics", Prentice-Hall (1965).
6) B. A. Boley and J. H. Weiner: "Theory of Thermal Stresses", John Wiley & Sons, New York (1960).
7) W. K. Nowacki: "Thermoelasticity", Pergamon Press, Oxford (1962).
8) M. Sokolowski, H. Zorski and R. E. Czarnota-Bojarski ed.: "Trends in Elasticity and Thermoelasticity", Wolters-Noordhoff (1972).
9) Y. Takeuti and N. Noda: J. Soc. Materials Sci., Japan, **19**, 169 (1970).
10) M. Biot: J. Appl. Phys., **27**, 240 (1956).
11) J. T. Oden and D. A. Kross: Proc. Second Conf. on Matrix Methods, WPAFB, 1191 (1968).
12) J. T. Oden: Nuclear Engng. and Design, **10**, 465 (1969).
13) J. H. Argyris, H. Balmer, J. St. Doltsinis and K. J. Willam: Proc. Third Conf. on Matrix Methods in Structural Mechanics, WPAFB, (1971).
14) J. T. Oden: "Finite Elements on Nonlinear Continua", McGraw-Hill, New York (1972).
15) Y. Yamada, N. Yoshimura and T. Sakurai: Int. J. Mech. Sci., **10**, 343 (1968).

16) O. C. Zienkiewicz, S. Valliappan and I. P. King: Int. J. num. Meth. Engng., **1**, 75 (1969).
17) H. D. Hibbitt, P. V. Marcal and J. R. Rice: Int. J. Solids Structures, **6**, 1087 (1970).
18) J. T. Oden: Recent Advances in Matrix Methods of Structural Analysis and Design, The Univ. of Alabama Pr., 693 (1971).
19) J. T. Oden, T. J. Chung and J. E. Key; Proc. First Int. Conf. on Structural Mechanics in Reactor Technology, (1971).
20) M. Kawahara and K. Horii: Proc. of JSCE, 194, 163 (1971).
21) H. Kitagawa, Y. Seguchi and Y. Tomita: Ing.-Archiv, **40**, 213 (1971).
22) T. Fujino and K. Ohsaka: Proc. Second conf. on Matrix Methods in Structural Mechanics, WPAFB, 1181 (1968).
23) Y. Ueda and T. Yamakawa: Proc. Int. Conf. on Mechanical Behavior of Materials, Kyoto, **3**, 10 (1972).
24) T. Inoue, K. Tanaka and M. Aoki: Trans. JSME, **28**, 2490 (1972): T. Inoue and K. Tanaka; J. Soc. Materials Soc., Japan, **22**, 218 (1973).
25) P. Perzyna and W. Wojno: Arch. Mech. Stos., **20**, 499 (1968).
26) W. Wojno: Arch. Mech. Stos., **21**, 793 (1969).
27) W. Olszak and P. Perzyna: Proc. 11th Int. Congr Appl. Mech., 545 (1966).
28) A. E. Green and P. M. Naghdi: Arch. Rat. Mech. Anal., **18**, 251 (1965).
29) T. Tokuoka: Trans. Japan Soc. Aero. Space Sci., **15**, 22 (1972).
30) R. W. Colugh: Recent Advances in Matrix Methods of Structural Analysis and Design, The Univ. of Alabama Pr., 441 (1971).