# Some Properties of Multicolored-Branch Graphs

By

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**Abstract**

A multicolored-branch graph is such a linear graph that the branches of the graph are partitioned into several sets, and a certain color is assigned to the branches belonging to each of the sets. The assignment is called a coloring. The degree of interference of loops or cutsets in such a graph is deffned to be the minimum number of indenpedent loops or cutsets respectively containing all the colors. The maximum of the degree of interference taken over all the possible colorings is studied. Theorems concerning the colorings to give the maximum in a two-colored-branch graph are derived. Moreover, the maximum of the degree of interference is shown to be equal to the topological degree of freedom and to the maximum distance between a pair of trees in the graph. The degree of interference is also related to the rank of a certain submatrix of the fundamental loop or cutset matrix. An upper bound and a lower bound on the degree of interference in a three-colored-branch graph are given.

## 1. Introduction

A multicolored-branch graph is such a linear graph that the branches of the graph are partitioned into several sets, and a certain color is assigned to the branches belonging to each of the sets to signify the nature of the branches. For instance, if the graph represents an electrical network, the sets of branches may correspond to resistors, capacitors and inductors, or may correspond to passive elements and active elements. Two-colored-branch graphs were introduced by Reza[1] to investigate the order of complexity of electrical networks. Hattori[2][3] presented a theory of multicolored-branch graphs and used it in the derivation of the state equations of electrical networks. He introduced the concept of the degree of interference of loops or cutsets in a multi-colored-branch graph, as is defined below. The order of complexity of a linear passive or active network was given, using the degree of interference of loops or cutsets.

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Let us first give some notations concerning multicolored-branch graphs. Given a graph $G$, we partition the branches of $G$ into sets $\alpha_1, \alpha_2, \cdots, \alpha_p$. A coloring of the branches of $G$ or simply a coloring is an assignment of colors to the branches of $G$, or in other words, a specification of the partition of branches into sets $\alpha_1$, $\alpha_2$, $\cdots$, $\alpha_p$. With a particular coloring, the graph is denoted by $G$ $(\alpha_1, \alpha_2, \cdots, \alpha_p)$. We regard $\alpha_j$ $(j=1, 2, \cdots, p)$as variables taking one of values $a_j$, 0 and 1. By $\alpha_j = a_j$, 0 or 1, we mean all the branches in set $\alpha_j$ are assigned color $a_j$, open-circuited or short-circuited, respectively. The rank and the nullity of $G$ $(\alpha_1, \alpha_2, \cdots, \alpha_p)$ are denoted by $n$ $(\alpha_1, \alpha_2, \cdots, \alpha_p)$ and $m$ $(\alpha_1, \alpha_2, \cdots, \alpha_p)$ respectively. Now a loop in $G$ $(a_1, a_2, \cdots, a_p)$ is called $k$-chromatic if the total number of colors assigned to its branches is exactly $k$. A $k$-chromatic cutset is defined similarly. We have the following definition.

*Definition*: The degree of interference of loops in $G$ $(a_1, a_2, \cdots, a_p)$, denoted by $m$ $(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_p)$, is the minimum number of $p$-chromatic loop in a set of $m$ $(a_1, a_2, \cdots, a_p)$ independent loops, the minimum being taken over all possible sets of $m$ $(a_1, a_2, \cdots, a_p)$ independent loops. The degree of interference of cutsets in $G$ $(a_1, a_2, \cdots, a_p)$ is defined similarly and is denoted by $n$ $(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_p)$.

By use of these notations and definitions, the order of complexity $\sigma$ of a passive LCR network represented by $G$ $(l, c, r)$ where $l$, $c$ and $r$ correspond to the inductors, capacitors and resistors in the network respectively, is given by[2)3)]

$$\sigma = m\ (l, o, o) + m\ (\bar{l}, \overline{c+r}) + n\ (1, 1, c) + n\ (\overline{l+r}, \bar{c}). \qquad (1)$$

Here $c+r$ and $l+r$ correspond to the joint set of capacitor branches and resistor branches, and to the joint set of inductor branches and resistor branches, respectively. The sum of the second and fourth terms in (1) gives the number of non-zero natural frequencies of the network. This number is equal to the number of state variables, if the variables corresponding to the zero natural frequencies are omitted from the state equations. In many state variable approaches, the variables corresponding to the zero natural frequencies are treated separately from those corresponding to the non-zero natural frequencies. Therefore, (1) is a convenient form giving the order of complexity of a network. From (1) we also see that the number of non-zero natural frequencies of an LR or CR network is exactly the degree of interference of loops or cutsets respectively in the corresponding two-colored-branch graph.

In this paper the maximum of the degree of interference in a multicolored-branch graph, the maximum being taken over all possible colorings of the graph, is studied. In Section II several fundamental theorems are given concerning the colorings for the case of two-colored-branch graghs. As is discussed in Section

III, the maximum is closely related to the topological degree of freedom[4)5)] and also to the maximally distant trees of the graph[6)]. The theorems in Section II are, in a way, extensions of theorems in references[4),5)] derived from a different point of view. In section IV, an upper bound and a lower bound on the degree of interference of loops or cutsets in a three-colored-bracnh graph are derived.

## 2. Degree of Interference of Loops or Cutsets in a Two-Colored-Branch Graph

We consider a two-colored-branch graph in this section. For a given graph $G$, colors $a$ and $b$ are assigned to sets $\alpha$ and $\beta$ of branches respectively. With a particular coloring the graph is denoted by $G(\alpha, \beta)$. The degree of interference of loops in $G(a, b)$, denoted by $m(\bar{a}, \bar{b})$, is the minimum number of bichromatic loops in a set of $m(a, b)$ independent loops. We can easily obtain that

$$m(\bar{a}, \bar{b}) = m(a, b) - m(a, 0) - m(0, b), \tag{2}$$

if we observe $m(a, 0) + m(0, b)$ is the maximum number of independent monochromatic loops. Dually, the degree of interference of cutsets is given by

$$n(\bar{a}, \bar{b}) = n(a, b) - n(a, 1) - n(1, b). \tag{3}$$

Using the formulas (a brief proof for which is given in Section III)

$$m(a, b) = m(a, 0) + m(1, b) = m(a, 1) + m(0, b) \tag{4}$$

$$n(a, b) = n(a, 0) + n(1, b) = n(a, 1) + n(0, b) \tag{5}$$

and a relation such as

$$n(a, 0) + m(a, 0) + n(0, b) + m(0, b) = n(a, b) + m(a, b),$$

we obtain that the degree of interference of loops in $G(a, b)$ is equal to the degree of interference of cutsets in $G(a, b)$. Hence, we simply call them the degree of interference in $G(a, b)$ and denote it by $\nu(\bar{a}, \bar{b})$, that is,

$$m(\bar{a}, \bar{b}) = n(\bar{a}, \bar{b}) = \nu(\bar{a}, \bar{b}) \, . \tag{6}$$

Now for a given graph $G$, the degree of interference varies depending on the coloring of branches of $G$. There must be a maximum of the degree taken over all possible colorings. We denote the maximum by $\nu_m$ and now investigate colorings to give $\nu_m$. From (2) and (3) we get the following theorem.

*Theorem* 1

Given a coloring $P$ of the branches of $G$. $P$ is a coloring to give $\nu_m$, if there exists no monochromatic loop in $G(a, b)$, or if there exists no monochromatic cutset in $G(a, b)$.

Proof: If there exists no monochromatic loop, $m(a, 0)=m(0, a)=0$ and thus $m(\bar{a}, \bar{b})=m(a, b)$, the maximum number of independent loops in $G(a, b)$. A dual proof to the above can be shown if the condition is given in terms of cutsets.

Moreover we have

*Theorem* 2

If coloring $P$ gives $\nu_m$, there is no branch belonging to a monochromatic loop and to a monochromatic cutset at the same time.

*Proof:* If there were such a branch, the degree of interference could be increased by changing its color.

If there are monochromatic loops for a coloring, let us consider a series of sets of branches determined by the following procedure.

*Procedure* 1

$L_0=$ {all the branches belonging to all the monochromatic loops}

$L_j=$ {all the branches belonging to all the loops which consist of one of the branches in $L_{j-1}$ and of the branches of the color different from that of the branch} $\cup L_{j-1}$ $(j=1, 2, \cdots)$,

$$\Delta L_0 = L_0\,, \quad \Delta L_j = L_j - L_{j-1} \quad (j=1, 2, \cdots)\,.$$

There must be an integer $l$ such that $L_0 \subset L_1 \subset \cdots \subset L_l = L_{l+1} = \cdots$ or $\Delta L_j \neq \phi$ for $j \leq l$, and $\Delta L_j = \phi$ for $j > l$. Sets $L_0$, $L_1$, $\cdots$ and $L_l$ are called the series of sets of branches associated with monochromatic loops. For examples of the sets, see Fig. 1 shown later in this section. Suppose $l>0$ for coloring $P$, and there is a monochromatic cutset S which intersects $\bigcup_{j=1}^{l} \Delta L_j$ but not $L_0$. Then, choose a branch belonbing to S and at the same time to $\Delta L_k$, $k$ being as small as possible $(k \geq 1)$. The branch is denoted by $e_k$. Change the color of $e_k$, and obtain a new coloring $P'$. Furthermore, construct the new series of sets of branches associated with monochromatic loops $L_j'(j=0, 1, 2, \cdots)$. We have the following lemma concerning $P$ and $P'$.

*Lemma* 1

(i) $L_i'=L_i$ and $\Delta L_i' = \Delta L_i$ $(i=0, 1, \cdots, k-1)$.

(ii) There is a monochromatic cutset containing $e_k$ and $e_{k-1}$, where $e_{k-1}$ is a branch contained in $\Delta L_{k-1}$ and thus in $\Delta L'_{k-1}$.

*Proof:* (i) We prove by induction with respect to $i$. For $P'$ there is no monochromatic loop containing $e_k$, since if there were such a loop, it could have only one branch $e_k$ in common with cutset S, which is impossible for a loop. Thus $L_0'=L_0$. Assume $L'_{i-1}=L_{i-1}$ $(i<k)$. $L_i'$ $(i<k)$ is determined by finding a loop

which consists of one of the branches in $L'_{i-1}$ and branches of the color different from the branch. Since only the color of $e_k$ is changed to get $P'$ from $P$, we investigate the existence of a loop which contains $e_k$ and which also meets the above condition. Before changing the color of $e_k$, such a loop consists of $e_k$, one branch of the same color as $e_k$ and contained in $L_{i-1}$, and branches of the color different from $e_k$. But there is no such a loop for $P$, since it would have $e_k$ and the branch of the same color as $e_k$ and contained in $L_{i-1}$ in common with $S$, contradicting the choice of $e_k$. Thus $L_i'=L_i\,(i<k)$.

(ii) For $P$ there is no monochromatic loop containing $e_k$. Therefore, every loop containing $e_k$ contains at least one branch of the color different from $e_k$, and is dissected by removing the branch and $e_k$. Thus, there is a cutset consisting of $e_k$ and the branches of the color different from $e_k$. Besides, since $e_k, \in \Delta L_k$ there is a loop consisting of $e_k$, one branch of the color different from $e_k$, say $e_{k-1}\in \Delta L_{k-1}$, and branches of the same color as $e_k$. Thus, the cutset should contain $e_{k-1}$. By changing the color of $e_k$ we get a monochromatic cutset containing $e_k$ and $e_{k-1}$.

Now let $G_s$ be a subgraph of $G$ obtained from $G$ by open-circuitring one or more branches. For a coloring of $G$, let the color of each remaining branch in $G_s$ be the same as that of the corresponding branch in $G$. Denote the nullity of $G_s\,(\alpha, \beta)$ by $m_s\,(\alpha, \beta)$. Then the following lemma is obvious.

*Lemma* 2

$$m\,(a, 0)+m\,(0, b)\geq m_s\,(a, 0)+m_s(0, a) \tag{7}$$

and if all the monochromatic loops in $G\,(a, b)$ are also included in $G_s\,(a, b)$, the equality in (7) holds.

With the preparation of Lemma 1 and 2 we have Theorem 3 which is the main result concerning colorings to give $\nu_m$.

*Theorem* 3

If $L_l$ ($l\geq 0$) exists for coloring $P$ of $G$, a necessary and sufficient condition for $P$ to give $\nu_m$ is that there is no monochromatic cutset intersecting $L_l$ in $G\,(a, b)$.

*Proof:* Necessity: If $l=0$ this condition is the same as that in Theorem 2. For $l\geq 1$, assume that there were such a cutset. From Theorem 2, the cutset cannot intersect $L_0$. Choosing a branch which belongs to the cutset and also to $\Delta L_k$, $k$ being as small as possible, and changing the color of the branch, we get a monochromatic cutset intersecting $\Delta L'_{k-1}$ $(=\Delta L_{k-1})$ by Lemma 1. Repeating the process we get a monochromatic cutset intersecting $L_0$, which contradicts Theorem 2.

Sufficiency: Consider the subgraph of $G$ consisting of all the branches in $L_l$. Since there is no monochromatic cutset in the subgraph, the degree of interference

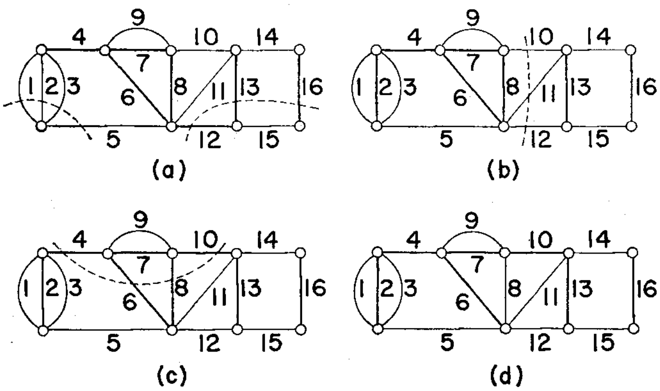


Fig. 1. Procedure to get a coloring to give $\nu_m$.
(a) $L_0=\{1, 2, 3, 6, 7, 8\}$, $\Delta L_1=\{9, 10, 11\}$, $\Delta L_2=\{12, 13\}$
(b) $L_0=\{1, 3, 6, 7, 8\}$, $\Delta L_1=\{2, 9, 10, 11\}$
(c) $L_0=\{1, 3, 6, 7, 8\}$, $\Delta L_1=\{2, 9\}$
(d) A coloring to give $\nu_m$.

of loops in the subgraph takes the maximum by Theorem 1. Thus, from Lemma 2 we see $m(a, 0)+m(0, b)$ takes the minimum value over all possible colorings.

From Theorem 2, Lemma 1 and Theorem 3, we can get an algorithm to get a coloring giving $\nu_m$. Since its presentation is almost same as repeating the proofs of the theorems and lemma, we here omit writing it down, but give an example as illustrated in Fig. 1. The colors of the branches are indicated by the thickness of the lines. The monochromatic cutsets intersecting $L_l$ are shown by the dotted lines.

Interesting examples of colorings to give the maximum of the degree of interference are the realizations of LC, CR or LR networks in Foster's or Cauer's form, as are discussed later. If a coloring gives $\nu_m$, the subgraph of $G$ corresponding to set $L_l$ obtained from the coloring is called the principal subgraph associated with monochromatic loops, and is denoted $G_l$.

With the discussions dual to the above we can get a procedure, a lemma and a theorem concerning cutsets corresponding to Procedure 1, Lemma 1 and Theorem 3 respectively. The principal subgraph associated with monochromatic cutsets can also be defined, and is denoted $G_c$.

## 3. Degree of Interference Distance between a Pair of Trees and Topological Degree of Freedom

A principal partition of a linear graph introduced by Kishi and Kajitani[6] in connection with maximally distant trees is a partition of a graph into three principal subgraphs, $G_1$, $G_2$ and $G_0$, called the principal subgraph with respect to common chords, with respect to common tree-branches and of disjoint trees respectively. If the branches belonging to the subgraphs are painted with color $x, y$ and $z$ respectively, graphs $G_1'=G(x, 0, 0)$ $G_2'=G(1, y, 1)$ and $G_0'=(G(1, 0, z)$ are uniquely determined regardless of a pair of maximally distant trees used to obtain the subgraphs. It has been shown[7] that $G_0'$ can be further partitioned into a certain partially ordered set of subgraphs. The partial ordering of the subgraphs together with the subgraphs of $G_1'$ and $G_2'$ is called the structure of the graph. The structure of a graph is useful for the mixed analysis of electrical networks. In connection with the number of equilibrium equations necessary in the mixed analysis, the topological degree of freedom was defined[4)5)]. It can be written as, in our terms,

$$d = \min_{\text{all colorings of } G} \{n(u, 0) + m(1, v)\}$$

where $u$ and $v$ indicate the colors of branches, different notations being used to distinguish them from colors $a$ and $b$. Among the colorings to give $d$, the colorings with a minimum number of branches of color $u$ and those of color $v$, respectively, define subgraphs $G_n^* = G(u, 0)$ and $G_m^* = G(1, v)$. It can be shown that $G_n^* = G_1'$ and $G_m^* = G_2'$ from references[4)5)6)].

In order to derive the relation between the degree of interference and the distance between a pair of trees, we consider the following pair of trees of $G(a, b)$. Let $T_a(T_b)$ be a tree of $G(a, b)$ containing a maximum of branches of color $a$ ($b$) and a minimum of branches of color $b$ ($a$). The number of branches of color $a$ and $b$ in $T_a$ is $n(a, 0)$ and $n(1, b)$ respectively. From this fact the first half of (5) follows immediately. The other equations in (4) and (5) can be proved similarly. In general, there are many choices for $T_a$ and $T_a$ of $G(a, b)$, but we choose a special pair $T_a^*$ and $T_b^*$, which has as many common branches as possible. Such a pair can be obtained as follows. Let a tree of $G(1, b)$ and $G(a, 1)$ be $T_{1b}$ and $T_{a1}$ respectively. Then choose a tree of $G(a, 0)$ which contains $T_{a1}$, and denote it $T_{a0}$. Likewise let $T_{0b}$ be a tree of $G(0, b)$ containing $T_{1b}$. Construct $T_a^* = T_{1b} \cup T_{a0}$ and $T_b^* = T_{a1} \cup T_{0b}$. Since $T_a^*$ and $T_b^*$ have $T_{a1}$ and $T_{1b}$ in common, the distance between $T_a^*$ and $T_b^*$ is

$$D(T_a^*, T_b^*) = n(a, 0) - n(a, 1) = n(0, b) - n(1, b) = n(\bar{a}, \bar{b}) = \nu(\bar{a}, \bar{b}) . \tag{8}$$

In general

$$D(T_a, T_b) \geq \nu(\bar{a}, \bar{b}) \tag{9}$$

Considering all the colorings of branches, we see the maximum of the degree of interference is not more than the maximum of the distance between a pair of trees in $G$. In fact, if a pair $T_A$ and $T_B$, of maximally distant trees is given, a coloring to give $\nu_m$ can be obtained by assigning color $a$ to the branches of $T_A - T_B$, and color $b$ to those of $T_B - T_A$. and $\nu_m$ is equal to the maximum distance, denoted by $D_m$, between the pair of trees. The coloring of the common tree-branches of $T_A$ and $T_B$ and that of the common chords are arbitrary, but the coloring of the common tree-branches determines the monochromatic cutsets; and the coloring of the common chords determines the monochromatic loops in $G(a, b)$. All the monochromatic loops and cutsets are included in $G_1$ and $G_2$ respectively. $G_0$ consists of a pair of trees of different colors. Conversely, if a coloring to give $\nu_m$ is given, a pair of maximally distant trees can be obtained by a procedure to get $T_a$ and $T_b$.

Now Theorem 5 of references[4)5)] states that

$$d = r(G) - r_s^* \tag{10}$$

where $r(G)$ is the rank of $G$ and $r_s^*$ is equal to the number of common tree-branches of a pair of maximally distant trees. If a coloring to give $\nu_m$ is given, $T_a^*$ and $T_b^*$ obtained from the coloring are maximally distant trees, and the number of common tree-branches of the pair is $n(a, 1) + n(1, b)$. Thus we have

*Theorem* 4

$$\nu_m = D_m = d. \tag{11}$$

Examining Lemma 1 and Theorem 3 for the special cases where the branches of color $a$ form a tree and the branches of color $b$, its co-tree, we also get

$$G_l = G_1' = G_n^* \quad \text{and} \quad G_c = G_2' = G_m^* \tag{12}$$

Thus $G_l$ and $G_c$ also have the known properties of $G_1'$ and $G_2'$ respectively.

As an application of the above statements, let us consider an LC network. From (1) we see that the number of non-zero natural frequencies is twice the degree of interference. With the given network topology the maximum of the number of non-zero natural frequencies can be obtained by assigning inductors and capacitors according to a coloring to give $\nu_m$. Moreover, after open-circuiting or short-circuiting the branches corresponding to the zero natural frequency, the number of branches of the graph of the network is equal to the number of non-zero natural frequencies if, and only if, the graph consists of a pair of trees of an different color, that is, trees of inductors and of capacitors. This can be applied to the network topology to realize a rational reactance function with the minimum number of elements for the given degree. The realization in Foster's or Cauer's form satisfies the above condition, if the two terminals are properly open- or short-circuited. Similar discussion can be given for LR or CR networks.

Next, let us consider the fundamental cutset matrix $Q_f$ defined by $T_a$. We can write $Q_f$ in the form

$$Q_f = [U\, Q_c] = \begin{bmatrix} U_a & O & Q_{ab} & Q_{aa} \\ O & U_b & Q_{bb} & O \end{bmatrix} \tag{13}$$

where $U$ is a unit matrix and $Q_c$ is called the characteristic part of $Q_f$. $U_a$ and $U_b$ are unit matrices corresponding to the tree-branches of color $a$ and $b$ respectively. $Q_{aa}$, $Q_{ab}$ and $Q_{bb}$ are the submatrices of $Q_c$ corresponding to the tree-branches of color specified by the first subscript, and to the chords of color specified by the second subscript. The lower rows of (13) correspond to the monochromatic cutsets of color $b$, and thus we have a zero-submatrix at the right-lower corner.

*Theorem* 5

The rank of $Q_{ab}$ is equal to the degree of interference of cutsets in $G(a, b)$.

*Proof:* Consider a graph, $G_a$, obtained from $G$ by short-circuiting all the tree-branches of color $b$. The fundamental cutset matrix of $G_a$ corresponds to the upper rows of (13). Since there are $n(a, 1)$ independent monochromatic cutsets of color $a$ in $G_a$, the upper rows of (13) can be converted to the form

$$\begin{bmatrix} Q_a' & O & \begin{pmatrix} O \\ Q'_{ab} \end{pmatrix} & Q'_{aa} \end{bmatrix}$$

by proper additions or subtractions among the rows. The zero-submatrix above $Q'_{ab}$ has $n(a, 1)$ rows, and thus the rank of $Q_{ab}$ is not more than $n(a, 0) - n(1, b) = n(\bar{a}, \bar{b})$. Now open-circuiting the branches of color $a$ in $G_a$ leaves a graph consisting of the chords of color $b$ of $T_a$. A tree of such a graph contains $n(0, b) - n(1, b) = n(\bar{a}, \bar{b})$ branches. As the columns of $Q_{ab}$ contain the columns corresponding to the branches of such a tree, the rank of $Q_{ab}$ is no less than $n(\bar{a}, \bar{b})$.

Corollary: The rank of the characteristics part of a fundamental cutset matrix is equal to the rank of the graph obtained by open-circuiting all the branches of the corresponding tree, or is equal to the nullity of the graph obtained by short-circuiting all the chords.

The rank of $Q_{ab}$ is maximum if $T_a$ is an extremal tree, one of the maximally distant trees, with a coloring to give $\nu_m$.

Similar discussions concerning a fundamental loop matrix can be made.

## 3. Degree of Interference of Loops or Cutsets in a Multicolored-Branch Graph

The degree of interference of loops or cutsets in a multicolored-branch graph has not been sufficiently studied. We give only a few results for three-colored-branch graphs. In general, the degree of interference of loops in $G(a, b, c)$ may or may not be equal to the degree of interference of cutsets. Moreover, it can not be given merely in terms of the ranks and nullities of the graphs obtained by short-circuiting or open-circuiting all the branches belonging to some of the partitioned sets. The above statement can be verified by giving two graphs with different degrees of interference and showing all the ranks and nullities of the graphs as stated above are the same for the two graphs. We can, however, give an upper bound and a lower bound on the degree of interference of loops or cutsets in terms of these ranks and nullities as follows.

First we have

*Lemma* 3

There is an independent set of loops of $G(a, b, c)$ containing a minimum of trichromatic loops and a maximum of monochromatic loops, the minimum and the maximum being taken over all the independent sets of loops.

Proof: Suppose there is an independent set $S_1$ of loops containing a minimum of trichromatic loops but not a maximum of monochromatic loops, say, of color $a$. Then, there is a monochromatic loop in $G(a, 0, 0)$ which cannot be a linear combination of the monochromatic loops only in $S_1$. Let the loop be $l_i$. Thus, if loop $l_i$ is given as a linear combination of loops $l_1$, $l_2$, $\cdots$, and $l_r$ in $S_1$, at least one of $l_1, l_2, \cdots$, and $l_r$ is not monochromatic. Let the loop be $l_j$. If we form a new set $S_2$ from $S_1$ by omitting $l_i$ and adding $l_j$, $S_2$ is also an independent set of loops. Note that $l_j$ can not be trichromatic, since $S_1$ contains a minimum of trichromatic loops, and so does $S_2$. $S_2$ contains one more monochromatic loop than $S_1$. Repeating the process we can get a set as stated in the lemma.

By considering the independent sets of loops, each containing a minimum of trichromatic loops and a maximum of monochromatic loops, we get the following theorem.

*Theorem* 6

$$max\ (0,\ m_l) \leq m(\bar{a},\ \bar{b},\ \bar{c}) \leq min\ (m_{u1},\ m_{u2},\ m_{u3}) \tag{14}$$

where

$$\begin{aligned} m_l &= m\ (a,\ 1,\ 1) + m\ (a,\ 0,\ 0) - m\ (a,\ 0,\ 1) - m\ (a,\ 1,\ 0) \\ &= m\ (1,\ b,\ 1) + m\ (0,\ b,\ 0) - m\ (1,\ b,\ 0) - m\ (0,\ b,\ 1) \\ &= m\ (1,\ 1,\ c) + m\ (0,\ 0,\ c) - m\ (0,\ 1,\ c) - m\ (1,\ 0,\ c) \end{aligned} \tag{15}$$

$$m_{u1} = m\ (a,\ 1,\ 1) - m\ (a,\ 0,\ 1) = m\ (1,\ b,\ 1) - m\ (0,\ b,\ 1) \tag{16}$$

$$m_{u2} = m\ (1,\ b,\ 1) - m\ (1,\ b,\ 0) = m\ (1,\ 1,\ c) - m\ (1,\ 0,\ c) \tag{17}$$

$$m_{u3} = m\ (1,\ 1,\ c) - m\ (0,\ 1,\ c) = m\ (a,\ 1,\ 1) - m\ (a,\ 1,\ 0) \tag{18}$$

*Proof:* See Appendix.

A theorem giving bounds on $n(\bar{a}, \bar{b}, \bar{c})$ can be obtained by taking the dual of Theorem 6.

## 5. Conclusion

In this paper some properties of the degree of interference in multicolored-branch graphs are studied. Exposed are several theorems on the colorings to give its maximum for the two-colored case. An interesting fact revealed is that two kinds of degrees concerning electrical networks, namely the order of complexity and the topological degree of freedom, have intimate connections in the light of the two-colored-branch graph theory, although they were introduced from different points of view in the past literatures. Especially, the maximum of the degree of interference is equal to the topological degree of freedom, and the coolorings to give the maximum are closely related to the structure of the graph. It is hoped that new approaches in the network analysis or synthesis can be made using the relation.

A future interest in the multicolored case may be concerned with what kinds of measures should be considered besides ranks and nullities in order to obtain an exact and explicit formula of the degree of interference.

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## Appendix

Consider all the independent sets of loops containing a minimum of trichromatic loops and a maximum of monochromatic loops. Denote the number of monochromatic, bichromatic and trichromatic loops in one of such sets by $m_a$, $m_b$, $m_c$; $m_{ab}$, $m_{ac}$, $m_{bc}$ and $m_{abc}$ respectively, the colors of the branches contained in the loops being indicated by the subscripts. We have

$$m_a = m\ (a, 0, 0) \tag{A1}$$

$$m_b = m\ (0, b, 0) \tag{A2}$$

$$m_c = m\ (0, 0, c)\,. \tag{A3}$$

Let

$$m_{aa}{}^* = m\ (a, b, 0) - m\ (a, 0, 0) - m\ (0, b, 0) = \max\{m_{ab}\} \tag{A4}$$

$$m_{ac}{}^* = m\ (a, 0, c) - m\ (a, 0, 0) - m\ (0, 0, c) = \max\{m_{ac}\} \tag{A5}$$

$$m_{bc}{}^* = m\ (0, b, c) - m\ (0, b, 0) - m\ (0, 0, c) = \max\{m_{bc}\}\,, \tag{A6}$$

where each of the maximums is independently taken over all the sets considered. Since the loops corresponding to $m_{ab}{}^*$ and those to $m_{ac}{}^*$ are independent,

$$\max\ \{m_{ab} + m_{ac}\} = m_{ab}{}^* + m_{ac}{}^*\ , \tag{A7}$$

and similarly

$$\max\ \{m_{ac} + m_{bc}\} = m_{ac}{}^* + m_{bc}{}^* \tag{A8}$$

$$\max\ \{m_{ab} + m_{bc}\} = m_{ab}{}^* + m_{bc}{}^*\,. \tag{A9}$$

Obviously

$$\max\ \{m_{ab} + m_{ac} + m_{bc}\} \leq m_{ab}{}^* + m_{ac}{}^* + m_{bc}{}^*\,. \tag{A10}$$

From (A7), (A8) and (A9) we have

$$m_{ab}{}^* + m_{ac}{}^* + m_{bc}{}^* - \min(m_{ab}{}^*, m_{ac}{}^*, m_{bc}{}^*) \leq \max\{m_{ab} + m_{ac} + m_{bc}\} \tag{A11}$$

But

$$m_{abc} = m(a, b, c) - m_a - m_b - m_c - m_{ab} - m_{ac} - m_{bc}\,. \tag{A12}$$

From (A10) and (A11), therefore, we have an upper bound and a lower bound on $m_{abc}$. Into these bounds we substitute (A1), (A2),$\cdots$, and (A6), and then using the extended forms of (4) and (5) to three-colored-branch graphs, we have the theorem.