Applications of Fourier Series Technique to Inverse Laplace Transform (Part II)

By

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(Received June 29, 1973)

Abstract

This paper describes some basic properties of the Fourier series, the finite Fourier approximation and the method of applying them to the numerical analysis of the Laplace transform. Some considerations of error analysis in numerical treaties and some numerical examples are given.

1. Introduction

In the previous paper[1], we reported the method of determining the time domain solutions numerically from the operational functions (Laplace transform) by using the Fourier series technique. The method is very useful to engineering applications because of the following reasons.

1) In the analysis, we can get numerical solutions easily when their analytical forms are unknown or very complicated.

2) The method can be applied to matrix functions easily.

3) By applying the Fast Fourier Transform method, we can get numerical solutions with desired accuracy in a little computational time.

In the previous paper, the principle of the method and some numerical examples were shown. Some considerations of the error analysis were also shown, but sufficient discussions were not given.

Therefore, the main purpose of this paper is to further develop the discussions which were not sufficient in the previous paper. For this purpose, we shall show some basic properties of the Fourier series and the finite Fourier approximation. Also, we shall give some numerical examples and some considerations of error analysis. For simplicity, we shall treat of scaler functions here. The same discussions are also applied to matrix functions.

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2. Fourier series and finite Fourier approximation[2),3)]

Here we shall show some basic properties of Fourier series and finite Fourier approximation. First we shall show following theorems.

Theorem 1: Consider a periodic function $x(t)$ of period $2T$ which is summable in the sense of Lebesgue, and has a bounded variation in the neighborhood of $t=t$. Then, for the partial sum of Fourier series,

$$S_n(t) = \sum_{k=-n}^{n} X_k e^{ik(\pi/T)t} \tag{2.1}$$

where

$$X_k = \frac{1}{2T}\int_0^{2T} x(t)e^{-ik(\pi/T)t}dt \tag{2.2}$$

we have

$$S_n(t) = \frac{x(t-0)+x(t+0)}{2} \qquad n\rightarrow\infty . \tag{2.3}$$

We can write

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik(\pi/T)t} \tag{2.4}$$

Then, by the above thorem:

$$x(t) = \begin{cases} x(t) & t\text{: continuous point of } x(t) \\ \dfrac{x(t-0)+x(t+0)}{2} & t\text{: discontinuous point of } x(t). \end{cases}$$

Theorem 2: For (2.4) we have

$$||x||_{L^2} = ||X||_{l^2} \tag{2.5}$$

where

$$||x||_{L^2} = \left[\int_0^{2T}(x(t))^2 dt\right]^{1/2}$$

$$||X||_{l^2} = \left[\sum_{k=-\infty}^{\infty} |X_k|^2\right]^{1/2}.$$

By the above theorem, the Fourier series expansion of $x(t)$ defines an isometric operator from L^2-space to l^2-space and conserves all the properties of the topological space related to the algebraic operations and the norm.

For the finite Fourier approximation of $2N$ frequency spectra X_k's ($k=0\sim 2N-1$), from (2.4), we have

$$x(t) = \sum_{k=-(2N-1)}^{2N-1} X_k e^{ik(\pi/T)t} = 2\left[R_e\sum_{k=0}^{2N-1} X_k e^{ik(\pi/T)t} - \frac{1}{2}X_0\right] \tag{2.6}$$

and if X_k's are real numbers, from the periodicity of complex exponential function,

$$x(2T-t) = x(t) \tag{2.7}$$

Therefore, we can get the time function $x(t)$ only in the interval $0 \leq t < T$. In (2.2), if integral is approximated by finite series of sample points $2N$

$$X_{2N-k} = X_k^* \tag{2.8}$$

and we can get only N frequency spectra.
This effect is called aliase and for band limited signals this phenomenon is famous as the sampling theory.

3. Inverse Laplace transform

When a Laplace transform $X(s)$ is given, its time function $x(t)$ is given numerically

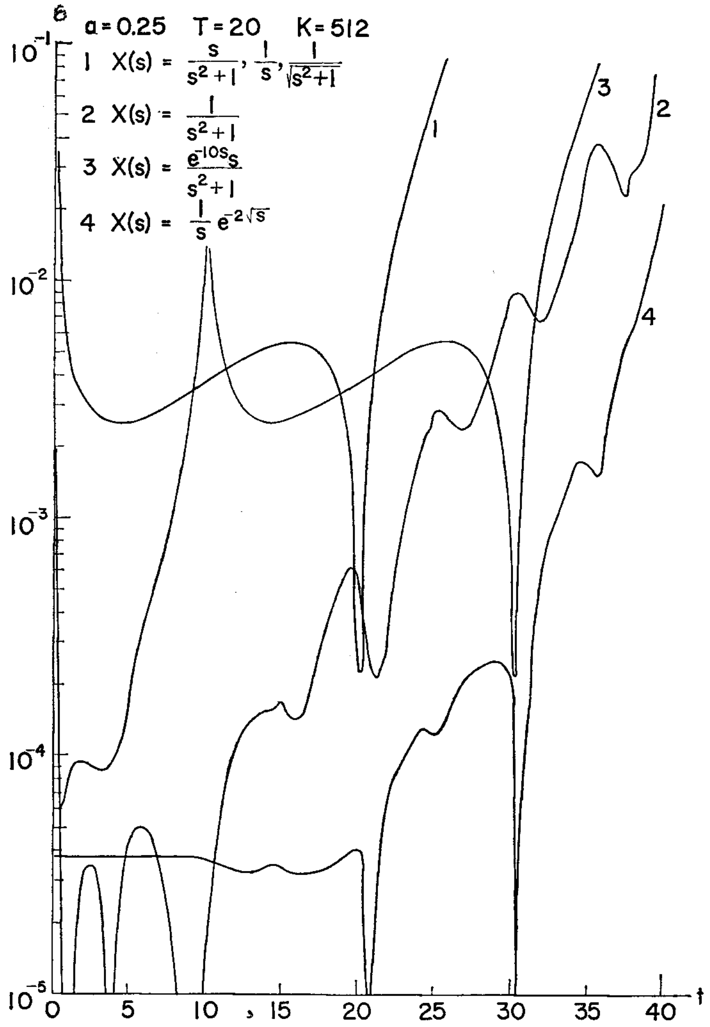

Fig. 1. Error for various functions

as follows[1].

$$\left.\begin{aligned} x\left(n\frac{2T}{K}\right) &= \frac{e^{2anT/K}}{T}R_e\left[\sum_{k=0}^{K-1}X\left(a+ik\frac{\pi}{T}\right)e^{i2\pi nk/K}-\frac{1}{2}X(a)\right] \\ n &= 0,\,1,\,2,\,\cdots,\,K-1\,. \end{aligned}\right\} \tag{3.1}$$

This is of the same form as (2.6), but here $X(a+ik\pi/T)$ is a complex number and an aliase effect does not appear. Therefore, we can get a numerical solution in the interval $0 \leq t < 2T$.

When the parameters a and T are given, the error caused by (3.1) depends heavily on the value of K and function $X(s)$. Fig. 1 shows the error for various

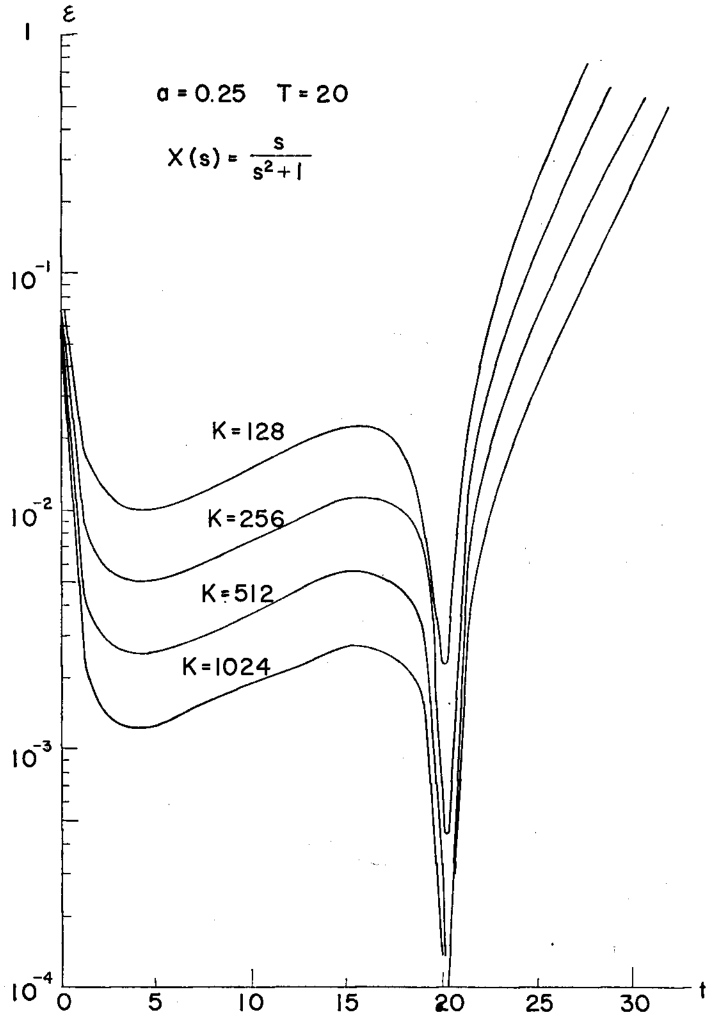

Fig. 2. Error for various K

functions $X(s)$ at constant K and Fig. 2 for same function at various value of K, computed by (3.1) using a Fast Fourier transform.

Truncation error at $t=t_n$ caused by (3.1) is given as

$$\varepsilon(t_n) = \frac{e^{at_n}}{T}\left| R_e \sum_{k=K}^{\infty} X(a+ik\pi/T)e^{ik(\pi/T)t_n} \right| \tag{3.2}$$

and when $t_n = T$ we have

$$\varepsilon(T) = \frac{e^{aT}}{T}\left| R_e \sum_{k=K}^{\infty} (-1)^k X(a+ik\pi/T) \right|. \tag{3.3}$$

In lamped constants systems, operational function of desired variables are given by

$$\left.\begin{aligned} &X(s) = N(s)/M(s) \\ &M(s) \text{ polynomial of } s \text{ with degree } m \\ &N(s) \text{ polynomial of } s \text{ with degree } n \\ &m \geq n+1 \end{aligned}\right\} \tag{3.4}$$

Therefore, if K is large we have

$$\left.\begin{aligned} &R_e X(s) \propto \frac{1}{K^2} \qquad I_m X(s) \propto \frac{1}{K} \\ &s = a+iK\pi/T \end{aligned}\right\} \tag{3.5}$$

and

$$\left.\begin{aligned} &\varepsilon(T) \simeq \varepsilon_K\left[1-\frac{K^2}{(K+1)^2}+\frac{K^2}{(K+2)^2}-\frac{K^2}{(K+3)}+\cdots\right] < \frac{K}{K-1}\varepsilon_K \\ &\varepsilon_K = \frac{e^{aT}}{T} R_e|X(a+iK\pi/T)|\,. \end{aligned}\right\} \tag{3.6}$$

At any t_n, imaginary part of $X(s)$ affects $\varepsilon(t_n)$. The evaluation of $\varepsilon(t_n)$ is very complicated and especially in the interval $T<t<2T$ $\varepsilon(t_n)$ increases more rapidly than in $0 \leq t \leq T$ by the factor e^{at}.

For distributed constants systems, operational functions of desired variables are given by

$$X(s) = e^{-Q(s)}N(s)/M(s) \tag{3.7}$$

and $X(s)$ decreases more rapidly by the factor $e^{-Q(s)}$ than in the case of lamped constants systems.

From Fig. 1, we know that for a various function, graph of $\varepsilon(t)$ to t is of a like shape, from Fig. 2, the truncation error did not improve very much in the interval $T<t<2T$ by increasing the value of K. Therefore, in applications we do not need a detailed discussion to determine the value of $\varepsilon(t)$; and if the parameters are selected as $aT\simeq 5$ and $K\simeq 256$, we can get a sufficient numerical value of $x(t)$ in the interval $0\leq t\leq T$.

4. Laplace transform

When a function $x(t)$ is given, its Laplace transform is given as

$$\left.\begin{aligned} X(a+ik\pi/T) &= \int_0^{2T} x_p(t)e^{-at}e^{-ik(\pi/T)t}dt \\ x_p(t) &= x(t)+\sum_{n=1}^{\infty} e^{-2anT}x(t+2nT)\,. \end{aligned}\right\} \tag{4.1}$$

For the inverse Laplace transform, we need a complex frequency spectra $X(a+ik\pi/T)$ $(k=0, 1, \cdots, K-1)$ and we can get them numerically by the same method as in the case of the inverse Laplace transform. However, here, the effects of aliase and discontinuity of $x(t)$ must be considered. In 3.3 of the previous paper[1], these effects have not been stated and discussions stated there must be corrected as follows.

Theoretically, $X(a+ik\pi/T)$ is given by integral transform, and therefore the discontinuous points of $x(t)$ do not cause any influence. However, numerically, the relation between $x(t)$ and $X(a+ik\pi/T)$ is given by Theorem 1. In the Laplace transform, function $x(t)$ to be considered is equal to 0 in $t<0$,
Therefore, approximately, we have

$$X(a+ik\pi/T) = \frac{2T}{N}\left[\sum_{n=0}^{N-1} x\left(n\cdot\frac{2T}{N}\right)e^{-a\cdot n(2T/N)}\cdot e^{-i2\pi nk/N}-\frac{1}{2}x(0)\right] \tag{4.2}$$

and if we need K frequency spectra, N must be determined as $N\geq 2K$ by the aliase effect. In (4.2) if $x(t)$ is discontinous at $t=t_n$, then we must correct $x(t_n)$ by

$$\left.\begin{aligned} x(t_n) &= x(t_n-0)+\frac{1}{2}\delta_n = x(t_n+0)-\frac{1}{2}\delta_n \\ \delta_n &= x(t_n+0)-x(t_n-0)\,. \end{aligned}\right\} \tag{4.3}$$

In using (4.2) two kind of errors are caused, which are

ε_1 in approximating $x_p(t)$ by $x(t)$

ε_2 in approximating integral by the finite series.

For ε_1 we have

$$\left.\begin{aligned} \varepsilon_1 &= \left|\int_{2T}^{\infty} x(t)e^{-at}e^{-i\omega t}dt\right| \leq \int_{2T}^{\infty}|x(t)|e^{-at}dt\leq\frac{M}{a}e^{-2aT} \\ M &= \sup_{t\in(2T,\infty)}|x(t)|\,. \end{aligned}\right\} \tag{4.4}$$

Generally, Fourier coefficients are proportional to $1/K^p (p \geq 1)$ as K increases and they are very small. Therefore, the effects of ε_1 and ε_2 cannot be ignored. As with the case of analysing linear systems with arbitrary input functions, we must get a numerical solution by (3.1) using a frequency spectra given by (4.2). In that case, ε_1 and ε_2 must as little as possible, for by inversion formula they are magnified at $t=t_n$

$$\varepsilon'(t_n) \simeq \frac{e^{an(2T/K)}}{T} (\varepsilon_1 + \varepsilon_2) \tag{4.5}$$

To reduce ε_1 we must use $x(t) + \sum_{n=1}^{\infty} e^{-2anT} x(t+2nT)$ instead of $x(t)$, but it is

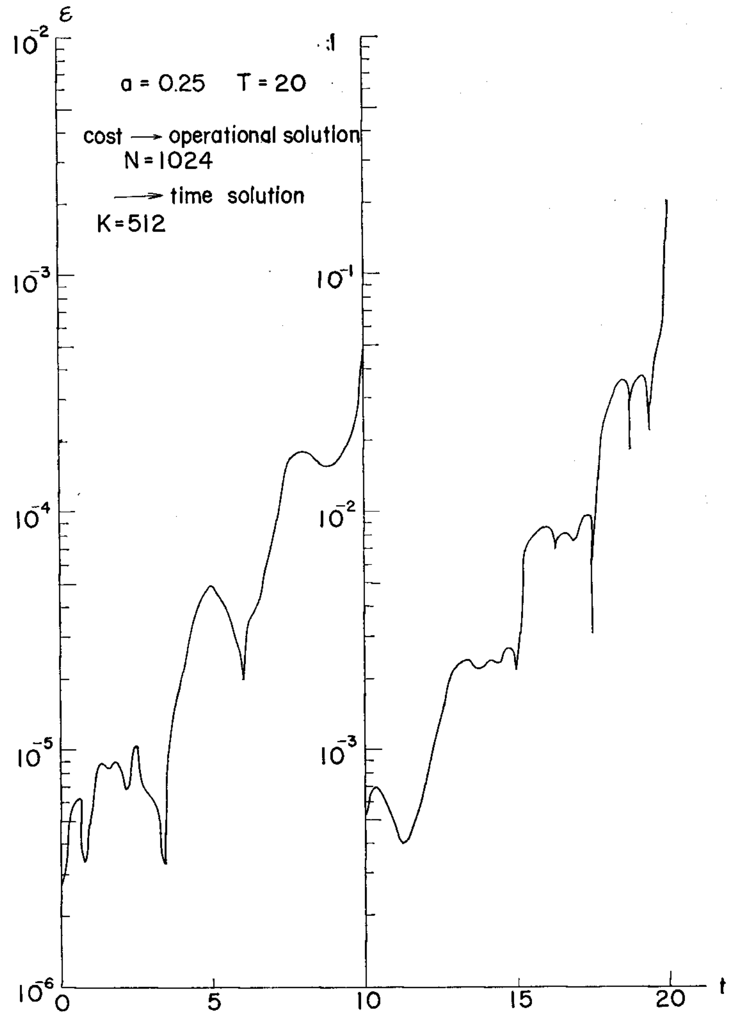

Fig. 3. Error in using (4.2) and (3.2)

pretty difficult in a general case because we cannot know the value of $x(t)$ at large t. To reduce ε_2 we must increase the number of sample points N and pay attention to the discontinuous points of $x(t)$ if they exist.

In Fig. 3, a numerical example of error analysis is shown. First, for $x(t)=\cos t$ ($a=0.25$, $T=20$ and $N=1024$) by (4.2) 512 spectra have been computed and then, using them by (3.1) numerical solutions have been computed. This provided a fairly good approximation in $0 \leq t < T$.

5. Conclusion

As mentioned above, some basic properties of the Fourier series and the finite Fourier approximation are given; and are applied to the numerical analysis of Laplace transform. Some considerations of error analysis and numerical examples are also shown. The method, along with that in the previous paper should be a very helpful tool in engineering applications.

Here digital computations were carried out by FACOM 230–60 in the Data Processing Center of Kyoto University.

6. Acknowledgement

The authors wish to thank Assistant Prof. T. Ozawa of Kyoto University for his useful suggestions.

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