Some Contributions to Optimization Theory of Nonequilibrium Diagonal MHD Generator Duct

By

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For the purpose of contributing to the optimum design of the nonequilibrium diagonal MHD generator duct, the authors derive a new digital calculation from the basic quasi one-dimensional MHD equations of the diverging rectangular duct and the integrals which express the duct size or isentropic efficiency. The calculation is intended to minimize the duct size or maximize the efficiency for a given thermal input, when the applied magnetic flux density, the mass flow rate and the duct inlet or outlet stagnation temperature and pressure of the working gas are held constant.

1. Introduction

In designing an MHD generator, it is necessary that the generator duct be constructed in the optimum form. For example, when the thermal input, applied magnetic flux density, mass flow rate and inlet or outlet stagnation pressure and temperature are held constant, we should minimize the duct length, surface-area or volume, or maximize the isentropic efficiency of the duct for a given thermal input. According to such an idea, the authors and the others have already derived optimization theories of the equilibrium Faraday, Hall and diagonal MHD generator duct.[1~5]

For the optimization of the nonequilibrium Faraday generator duct, in which rare gas is used as the working fluid, some papers have been already published. However, as regards the optimization of the diagonal duct, no paper is presented. In the rare gas plasma, electrothermal instabilities occur when a high magnetic flux density is applied; and they prevent the effective Hall parameter from becoming high. On the other hand, respecting the diagonal duct we can expect a good performance in the low Hall parameter, and moreover we can use it for a single load. So in this

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paper, the authors investigate the optimization of the nonequilibrium diagonal MHD generator duct.

In the analysis of the performance of the MHD generator, we use the quasi one-dimensional MHD equations, in which the friction and heat transfer at the duct walls, the segmented electrode effect, leakage current effect and electrode drop are included. Though the conventional calculations of output and efficiency for a large MHD generator have been mainly based on averaged electron energy balance and electron density equations, where the fluctuations in the elastic collisions loss and in the electron temperature were neglected, in this paper more accurate calculation is made according to the Zampaglione theory.[6]

Moreover we derive the optimization theories of the duct of constant velocity, constant Mach or constant length.

2. Basic Equations

As is well known, the MHD flow in the diagonal generator duct (Fig. 1) is described by the following set of the quasi one-dimensional basic equations:

$$\rho u A = \rho_0 u_0 A_0 = m_0 \quad : \text{ continuity equation,} \tag{1}$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = J_y B - dF \quad : \text{ momentum equation,} \tag{2}$$

$$\rho u \frac{d}{dx}\left(C_p T + \frac{u^2}{2}\right) = E_x J_x + E_y J_y - dQ \quad : \text{ energy equation,} \tag{3}$$

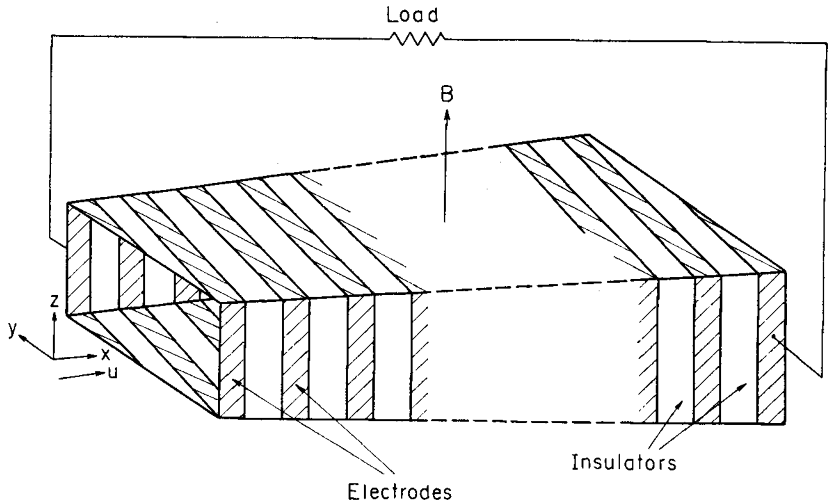

Fig. 1. Sketch of segmented electrode diagonal generator duct.

$$p=\rho RT \quad : \text{ state equation,} \tag{4}$$

$$(J_x+aJ_y)A=I \quad : \text{ current continuity equation,} \tag{5}$$

$$\left.\begin{aligned} J_x &= \frac{\sigma_{eff}}{1+\beta_{eff}^2}\left[E_x+\beta_{eff}\{uB(1-\varDelta)-E_y\}\right] \\ J_y &= \frac{\sigma_{eff}}{1+\beta_{eff}^2}\left[\{E_y-uB(1-\varDelta)\}+\beta_{eff}E_x\right] \end{aligned}\right\} \quad : \text{ Ohm's low,} \tag{6}$$

$$E_x=aE_y. \tag{7}$$

In these equations and Fig. 1

A: duct cross-sectional area,
$a=\tan\theta$: electrode inclination parameter*,
B: magnetic flux density*,
$C_p=\alpha R$: specific heat* at constant pressure,
C_v: specific heat* at constant volume,
dF: friction loss on the duct wall,
dQ: heat transfer loss on the duct wall,
E_x, E_y: x- and y- components of internal electric field intensity respectively,
I: total current*,
J_x, J_y: x- and y- components of current density respectively,
M_0: mass flow rate*,
p: gas pressure,
R: gas constant*,
T: gas temperature,
u: gas velocity,
$\alpha=\gamma/(\gamma-1)$*,
β_{eff}: effective Hall parameter,
σ_{eff}: effective electrical conductivity,
$\varDelta$: dimensionless voltage drop*,
θ: electrode angle*,
suffix 0 and 1: show the quantities in duct inlet and outlet respectively,
*: shows the quantities which are assumed constant in analysis. (8)

Next, as the estimation functions we adopt the integrals which represent the duct size and the isentropic efficiency, namely

$$I_N = \int_0^l A^N dx = \begin{cases} l & \text{for} \quad N=0, \\ V & \text{for} \quad N=1, \end{cases} \tag{9}$$

$$S = I_{1/2} = 4\int_0^l \left\{A + \left(\frac{1}{4}\frac{dA}{dx}\right)^2\right\}^{1/2} dx, \tag{10}$$

$$\eta_i = \frac{\int_0^l |E_x J_x + E_y J_y| A\,dx}{m_0 C_p T_{s0}\{1-(p_{s1}/p_{s0})^{1/\alpha}\}}. \tag{11}$$

where

$$\left.\begin{array}{l} l\text{: duct length,} \\ V\text{: duct volume,} \\ S\text{: duct surface area,} \\ \eta_i\text{: isentropic efficiency,} \\ P_{s0} \text{ and } P_{s1}\text{: stagnation pressure at duct inlet and outlet,} \\ T_{s0}\text{: stagnation temperature at inlet.} \end{array}\right\} \tag{12}$$

3. Electrical Conductivity, Hall Parameter and Losses

3.1 Electrical conductivity and Hall parameter

In this paper we assume that electrons lose their energy only by collisions and we neglect the radiation loss. Up to now, the power output and efficiency calculations for a large MHD generator have been mainly based on averaged electron energy balance and electron density equations, where fluctuations in the elastic collisions loss and in the electron temperature were neglected. However Zampaglione more accurately estimated the electron temperature for the Faraday generator; and his calculation results agreed more with experimental ones than the calculation by the old theory.[6]

And so we apply his theory to the diagonal generator. Then, the electron temperature T_e in the duct becomes

$$T_e = T + \frac{2}{3}\frac{m}{k\delta}\frac{u^2B^2\beta^2}{(1+a^2)(1+\beta_e^2)}\{(1+a^2)+\kappa(\beta_e^2-a^2) -(\beta_e-a)^2(1-\kappa)\kappa\}\left(\frac{\sigma_e}{\sigma}\right)^2, \tag{13}$$

where the effective electrical conductivity σ_e of the plasma, effective Hall parameter β_e and loading parameter κ are defined by the following relations:

$$\langle J_x\rangle = \frac{\sigma_e}{1+\beta_e^2}\{\langle E_x\rangle + \beta_e(uB-\langle E_y\rangle)\}, \tag{14}$$

$$\langle J_y\rangle = \frac{\sigma_e}{1+\beta_e{}^2}\{\langle E_y\rangle - uB + \beta_e\langle E_x\rangle\}, \tag{15}$$

$$\kappa = -\frac{1+a^2}{(\beta_e - a)uB}\langle E_x\rangle, \tag{16}$$

where $\langle \quad \rangle$ denotes the mean value in time and in the main stream within the cross section, and m and δ are defined in Relations (19).

In this connection, σ in Eq. (13) is the laminar conductivity of a plasma bearing an electron current density $\langle J\rangle = (\langle J_x\rangle^2 + \langle J_y\rangle^2)^{1/2}$, which is the actual current density flowing through the generator, and β is the Hall parameter corresponding to σ.

When it is assumed that the electron density n_e is given by Saha's equation and the mean electron collision frequency is given only by the collisions between the electrons and the seed atoms and between the former and the parent atoms, then σ and β are given as follows:

$$\sigma = \left(\frac{2\pi m_e k}{h^2}\right)^{3/4}\left(\frac{\pi\varepsilon_s}{8m_e}\right)^{1/2} e^2 \frac{p^{-1/2}T^{1/2}T_e^{1/4}}{(1-\varepsilon_s)Q_0 + \varepsilon_s Q_s}\exp\left(-\frac{eV_i}{2kT_e}\right), \tag{17}$$

$$\beta = \left(\frac{k\pi}{8m_e}\right)^{1/2} \frac{eBp^{-1}TT_e^{-1/2}}{(1-\varepsilon_s)Q_0 + \varepsilon_s Q_s}, \tag{18}$$

where

e: electrical charge of electron,
h: Plank's constant,
k: Boltzman's constant,
m: equivalent mass,
m_e: electron mass,
Q_0: collision cross-section between electron and parent gas atom,
Q_s: collision cross-section between electron and seed atom,
T_e: electron temperature,
V_i: ionized voltage of seed atom,
ε_s: seed fraction,
δ: collision loss factor. (19)

On the other hand σ_e and β_e are investigated through the experiment and the theory by Velikhof,[7] Dykhne,[8] Zampaglione,[6] Evans[9] and others. According to their discussions, we assume the following σ_e and β_e,

(1) When $\beta \leqq \beta_{crit}$,

$$\left.\begin{aligned}\sigma_e &= \sigma,\\ \beta_e &= \beta.\end{aligned}\right\} \tag{20}$$

(2) When $\beta_{crit} \leqq \beta \leqq 10$,

$$\left.\begin{aligned} \frac{\sigma_e}{\sigma} &= A_e \frac{\beta_e}{\beta}, \\ \beta_e &= \beta_{crit}, \end{aligned}\right\} \tag{21}$$

where A_e is a constant given by the experiment or the theory and β_{crit} is the critical Hall parameter.

(3) When $\beta \geqq 10$,

$$\left.\begin{aligned} \frac{\sigma_e}{\sigma} &= \frac{A_e \beta_{crit}}{\sqrt{10}} \frac{1}{\sqrt{\beta}}, \\ \beta_e &= \beta_{crit}. \end{aligned}\right\} \tag{22}$$

As the losses due to the leakage current in the generator duct and the finite segmented electrodes make the effective electrical conductivity low, we put the effective electrical conductivity σ_{eff} as follows:

$$\sigma_{eff} = \lambda \sigma_e \tag{23}$$

And we assume the effective Hall parameter β_{eff} as follows:

$$\beta_{eff} = \beta_e \tag{24}$$

Here for the Faraday duct, the expression of λ is derived as follows[10,11)]

$$\lambda = \frac{1}{1+\frac{h}{s}(\beta_e - 0.4)} \frac{1+\dfrac{\beta_e^2}{1+(1+\beta_e^2)\sum w/\sigma_e}}{1+\beta_e^2}, \tag{25}$$

where h, s and $\sum w$ are defined in Relations (27). However, for the diagonal duct the expression has not yet been obtained, and so for that expression, we assume λ constant, considering Eq. (25).

3.2 Friction and heat losses

It is assumed that the friction loss dF and heat loss dQ are given as follows:

$$\left.\begin{aligned} dF &= \frac{2}{D} \rho u^2 C_f, \\ dQ &= \frac{4}{D} \rho u C_p \frac{C_f/2}{1+\sqrt{C_f/2}\,\beta_N} (T_s - T_W), \\ C_f &= \frac{\{2.87+0.65 \log(x/R_s)\}^{-2.5}}{1+r(\gamma-1)M_0^2/2}, \\ r &= \sqrt[3]{P_r}, \\ \beta_N &= 0.52\left(\frac{\rho u \sqrt{C_f/2}\, k_s}{\mu}\right)^{0.45} P_r^{0.8}, \end{aligned}\right\} \tag{26}$$

where

$$\left.\begin{array}{l}
C_f:\ \text{friction factor,}\\
D:\ \text{hydraulic diameter,}\\
h:\ \text{electrode height,}\\
k_s:\ \text{equivalent sand roughness (representing the roughness}\\
\quad \text{level of the plate),}\\
M_0:\ \text{Mach number in the main stream,}\\
P_r:\ \text{Prandtl number,}\\
r:\ \text{recovery factor,}\\
s:\ \text{electrode pitch,}\\
T_s:\ \text{local stagnation temperature in the main stream,}\\
T_W:\ \text{wall temperature,}\\
X:\ \text{distance from the duct inlet,}\\
\mu:\ \text{viscosity of the gas,}\\
\Sigma_W:\ \text{electrical conductivity of duct wall.}
\end{array}\right\} \quad (27)$$

Eq.s (26) are the ones which Schlichting has derived for the compressible turbulent boundary layer on the rough plate.[12]

4. Optimization Theory of Diverging Rectangular Duct

4.1 Constant velocity duct

In this article, let us discuss the optimization in the case of constant velocity, viz. $u=u_0$.

Now putting

$$\left.\begin{array}{l}
\xi=\log(p_0/p),\\
\zeta=\log(T_0/T),
\end{array}\right\} \quad (28)$$

since $du/dx=0$ in this case, Eq.s (2) and (3) are transformed as follows:

$$\frac{d\xi}{dx}=\frac{\sigma_{eff}u_0B^2\{(1+a^2)+(\beta_{eff}{}^2-a^2)\kappa\}(1-\Delta)}{(1+\beta_{eff}{}^2)(1+a^2)p}+\frac{dF}{p}, \quad (29)$$

$$\frac{dS}{dx}=\frac{I(\beta_{eff}-a)B\kappa(1-\Delta)}{(1+a^2)Aap}+\frac{dQ}{apu_0}, \quad (30)$$

where

$$\left.\begin{aligned}
p &= p_0 \exp(-\xi),\\
T &= T_0 \exp(-\zeta),\\
A &= \frac{m_0 R T_0}{u_0 p_0} \exp(\xi-\zeta),\\
\kappa &= 1-\frac{u_0 p_0 I \exp(\xi-\zeta)}{m_0 R T_0 u_0 B(1-\Delta)} \frac{1+\beta_{eff}{}^2}{\sigma_{eff}(\beta_{eff}-a)}\,.
\end{aligned}\right\} \tag{31}$$

From Eq. (13) the differential equations for the electron temperature T_e are obtained as follows:

(1) When $\beta \leqq \beta_{crit}$,

$$\frac{dT_e}{dx} = C_{1\xi}\frac{d\xi}{dx}+C_{1\zeta}\frac{d\zeta}{dx}\,. \tag{32}$$

(2) When $\beta_{crit} \leqq \beta \leqq 10$,

$$\frac{dT_e}{dx} = C_{2\xi}\frac{d\xi}{dx}+C_{2\zeta}\frac{d\zeta}{dx}\,. \tag{33}$$

(3) When $\beta \geqq 10$,

$$\frac{dT_e}{dx} = C_{3\xi}\frac{d\xi}{dx}+C_{3\zeta}\frac{d\zeta}{dx}\,. \tag{34}$$

In these equations

$$\left.\begin{aligned}
C_{1\xi} &= \{1-(f_{11}f_{12e}+f_{12}f_{11e})\}^{-1}(f_{11}f_{12p}+f_{12}f_{11p}),\\
C_{1\zeta} &= \{1-(f_{11}f_{12e}+f_{12}f_{11e})\}^{-1}(-T+f_{11}f_{12T}+f_{12}f_{11T}),\\
C_{2\xi} &= \left\{1-f_{22e}\frac{2}{T_e}\left(\frac{3}{4}+\frac{T_i}{T_e}\right)\right\}^{-1} f_{22e},\\
C_{2\zeta} &= -\left\{1-f_{22e}\frac{2}{T_e}\left(\frac{3}{4}+\frac{T_i}{T_e}\right)\right\}^{-1}(T+f_{22e}),\\
C_{3\xi} &= \{1-(f_{31}f_{32e}+f_{32}f_{31e})\}^{-1}(f_{31}f_{32p}+f_{32}f_{31p}),\\
C_{3\zeta} &= \{1-(f_{31}f_{32e}+f_{32}f_{31e})\}^{-1}(-T+f_{31}f_{32T}+f_{32}f_{31T}),\\
f_{11} &= \frac{2u_0{}^2}{3\delta R(1+a^2)}\frac{\beta^2}{1+\beta^2}\,,\\
f_{12} &= (1+a^2)+\kappa(\beta^2-a^2)-(\beta-a)^2(1-\kappa)\kappa,\\
f_{21} &= \frac{2u_0{}^2 A_e{}^2\beta_e{}^2}{3\delta R(1+a^2)(1+\beta_e{}^2)}\,,\\
f_{31} &= \frac{u_0{}^2 A_e{}^2\beta_e{}^2\beta}{15\delta R(1+a^2)(1+\beta_e{}^2)}\,,\\
f_{32} &= (1+a^2)+\kappa(\beta_e{}^2-a^2)-(\beta_e-a)(1-\kappa)\kappa,\\
f_{11p} &= \frac{2f_{11}}{1+\beta^2}\,,
\end{aligned}\right\} \tag{35}$$

$$
\left.
\begin{aligned}
f_{11T} &= -\frac{2f_{11}}{1+\beta^2},\\
f_{11e} &= -\frac{f_{11}}{1+\beta^2}\frac{1}{T_e},\\
f_{12p} &= 2\{a+\kappa(\beta-a)\}\Big\{\frac{1}{2}+(1-\kappa)(\beta-a)\\
&\qquad +\beta\frac{2\kappa\beta(\beta-a)+(\beta^2-2a\beta-1)}{1+\beta^2}\Big\},\\
f_{12T} &= 2\{a+\kappa(\beta-a)\}\Big\{-\frac{1}{2}+(1-\kappa)(\beta-a)\\
&\qquad -\beta\frac{2\kappa\beta(\beta-a)+(\beta^2-2a\beta-1)}{1+\beta^2}\Big\},\\
f_{12e} &= 2\{a+\kappa(\beta-a)\}\Big\{-\frac{\beta}{2}\frac{2\kappa\beta(\beta-a)+(\beta^2-2a\beta-1)}{1+\beta^2}\\
&\qquad +\left(\frac{1}{4}+\frac{T_i}{T_e}\right)\Big\}\frac{1}{T_e},\\
f_{22e} &= f_{21}(\beta_e-a)(1-\kappa)\{a+\kappa(\beta_e-a)\},\\
f_{31p} &= f_{31},\\
f_{31T} &= -f_{31},\\
f_{31e} &= -\frac{f_{31}}{2T_e},\\
f_{32p} &= 2(\beta_e-a)(1-\kappa)\{a+\kappa(\beta_e-a)\},\\
f_{32T} &= -2(\beta_e-a)(1-\kappa)\{a+\kappa(\beta_e-a)\},\\
f_{32e} &= 2(\beta_e-a)(1-\kappa)\{a+\kappa(\beta_e-a)\}\left(\frac{1}{2}+\frac{T_i}{T_e}\right)\frac{1}{T_e}.
\end{aligned}
\right\}
$$

And Eq.s (5), (6) and (7) are transformed as follows:

$$J_x = \frac{\sigma_{eff}u_0B(1-\Delta)}{1+\beta_{eff}^2}\left\{\beta_{eff}-\frac{(1-a\beta_{eff})(\beta_{eff}-a)\kappa}{1+a^2}\right\}, \tag{36}$$

$$J_y = -\frac{\sigma_{eff}u_0B(1-\Delta)}{1+\beta_{eff}^2}\left\{1+\frac{(\beta_{eff}^2-a^2)\kappa}{1+a^2}\right\}, \tag{37}$$

$$E_x = -\frac{(\beta_{eff}-a)u_0B(1-\Delta)\kappa}{1+a^2}. \tag{38}$$

Now when the values of T_{s0}, T_{s1}, P_{s0}, u_0, κ_0 etc. are given, the values of the duct inlet and outlet temperature T_0 and T_1, the duct inlet pressure P_0 and ξ_1 can be evaluated by the following relations:

$$T_0 = T_{s0}-u_0^2/2C_p, \tag{39}$$

$$T_1 = T_{s1}-u_0^2/2C_p, \tag{40}$$

$$p_0 = p_{s0}\left(1+\frac{u_0^2}{2C_pT_0}\right)^{-\alpha}, \tag{41}$$

$$\zeta_1 = \log(T_0/T_1). \tag{42}$$

If the numerical values of T_0, T_1, P_0 and ξ_1 are obtained with these equations, we can find the inlet value T_{e0} of the electron temperature by solving numerically Eq. (13). When the numerical values of T_0, T_1, T_{e0} and etc. are decided, we can digitally solve Eq.s (29) to (38), for example, by the Runge-Kutta-Gill method, where in analysis we use ζ as the independent variable. Then, applying the obtained numerical solution to Eq.s (9) to (11), we can get the values of I_N and η_i. When we give the various values to u_0 and κ_0 as the parameters, we can find numerically the values of the optimum velocity $u_0 = u_{opt}$ and the optimum inlet loading factor $\kappa_0 = \kappa_{opt}$, which make the duct size minimum or the isentropic efficiency maximum.

The values of the other quantities, for example, σ, β and κ can be digitally calculated by using the values of ζ, ξ, T_e etc.

Next, the power output P_W, the total heat loss P_Q and the total friction loss P_f are evaluated by the following equations:

$$p_W = \frac{Iu_0B(1-\Delta)}{1+a^2}\int_0^{\zeta_1}(\beta_{eff}-a)\frac{dx}{d\zeta}d\zeta, \tag{43}$$

$$p_Q = \int_0^{\zeta_1} dQA\frac{dx}{d\zeta}d\zeta, \tag{44}$$

$$p_f = \int_0^{\zeta_1} dFA\frac{dx}{d\zeta}d\zeta, \tag{45}$$

where

$$A = \frac{m_0RT_0}{u_0p_0}\exp(\xi-\zeta). \tag{46}$$

4.2 Constant Mach number duct

Here we shall derive an optimization theory for a constant Mach number duct, i.e in the case of Mach number $M = M_0$. We can rewrite the basic flow equations (2) and (3) in terms of the stagnation values and by use of the adiabatic law

$$\log(p_s/p) = \alpha\log(T_s/T) = \alpha\log(1+X_0). \tag{47}$$

Namely putting

$$\left.\begin{aligned}\xi_s &= \log(p_{s0}/p_s),\\ \zeta_s &= \log(T_{s0}/T_s),\end{aligned}\right\} \tag{48}$$

Eq.s (3) and (2) are transformed as follows:

$$\frac{d\xi_s}{dx} = \frac{(1+\beta_{eff}^2)p}{\sigma_{eff}uB^2(1-\Delta)}\left\{1+\frac{(\beta_{eff}^2-a)^2\kappa}{1+a^2}\right\}-\alpha X_0\frac{d\zeta_s}{dx}+\frac{dF}{p}, \tag{49}$$

$$\frac{d\zeta_s}{dx}=\frac{I(\beta_{eff}-a)B\kappa(1-\Delta)}{\alpha(1+X_0)pA(1+a^2)}+\frac{dQ}{\alpha(1+X_0)up}, \tag{50}$$

where

$$\left.\begin{aligned}
X_0&=(\gamma-1)M_0{}^2/2,\\
p&=p_{s0}(1+X)^{-\alpha}\exp(-\xi_s),\\
u&=\{2C_pT_{s0}X_0/(1+X_0)\}^{1/2}\exp(-\zeta_s/2),\\
A&=\frac{m_0R}{p_{s0}}\left(\frac{T_{s0}}{2C_p}\right)^{1/2}(1+X_0)^{\alpha-1/2}X_0{}^{-1/2}\exp(\xi_s-\zeta_s/2).
\end{aligned}\right\} \tag{51}$$

Also the differential equations for the electron temperature are given by equations similar to Eq.s (32) to (34), where

$$\left.\begin{aligned}
C_{1\xi}&=\{1-T(f_{11}f_{12e}+f_{12}f_{11})\}^{-1}T(f_{11}f_{12p}+f_{12}f_{11p}),\\
C_{1\zeta}&=\{1-T(f_{11}f_{12e}+f_{12}f_{11})\}^{-1}\\
&\qquad\times T(-T_e/T+f_{11}f_{12T}+f_{12}f_{11T}),\\
C_{2\xi}&=\left\{1-\frac{2f_{22e}}{T_e}\left(\frac{3}{4}+\frac{T_i}{T_e}\right)\right\}^{-1}f_{22e},\\
C_{2\zeta}&=-\left\{1-\frac{2f_{22e}}{T_e}\left(\frac{3}{4}+\frac{T_i}{T_e}\right)\right\}^{-1}(T_e+f_{22e}),\\
C_{3\xi}&=\{1-T(f_{31}f_{32e}+f_{32}f_{31e})\}^{-1}T(f_{32}f_{32p}+f_{32}f_{31p}),\\
C_{3\zeta}&=\{1-T(f_{31}f_{32e}+f_{32}f_{31e})\}^{-1}\\
&\qquad\times T(f_{32}f_{32T}+f_{31}f_{32T}-T_e/T),\\
f_{11}&=\frac{2\gamma M_0{}^2}{3\delta(1+a^2)}\,\frac{\beta^2}{1+\beta^2},\\
f_{21}&=\frac{2\gamma M_0{}^2A_e{}^2\beta_e{}^2}{3\delta(1+a^2)(1+\beta_e{}^2)},\\
f_{31}&=\frac{\gamma M_0{}^2A_e{}^2\beta_e{}^2\beta}{15\delta(1+a^2)(1+\beta_e{}^2)},\\
f_{32}&=(1+a^2)+\beta_e{}^2-a^2)\kappa-(\beta_e-a)^2(1-\kappa)\kappa,\\
f_{22e}&=Tf_{21}(1-\kappa)(\beta_e-a)\{a+\kappa(\beta_e-a)\}.
\end{aligned}\right\} \tag{52}$$

and f_{12}, f_{11p}, f_{11e}, f_{11T}, f_{12p}, f_{12T}, f_{12e}, f_{31p}, f_{31T}, f_{31e}, f_{32p}, f_{32T} and f_{32e} are given by Eq.s (35).

J_x, J_y and E_x are rewritten as follows:

$$\left.\begin{aligned}
J_x&=\frac{\sigma_{eff}uB(1-\Delta)}{1+\beta_{eff}{}^2}\left\{\beta_{eff}-\frac{(1-a\beta_{eff})(\beta_{eff}-a)\kappa}{1+a^2}\right\},\\
J_y&=-\frac{\sigma_{eff}uB(1-\Delta)}{1+\beta_{eff}{}^2}\left\{1+\frac{(1+a^2)\kappa}{\beta_{eff}{}^2-a^2}\right\},\\
E_x&=-\frac{(\beta_{eff}-a)uB(1-\Delta)\kappa}{1+a^2}.
\end{aligned}\right\} \tag{53}$$

Next, the values of T_0, T_1, P_0 and ζ_{s1} are determined by the following relations:

$$\left.\begin{aligned} T_0 &= T_{s0}(1+X_0)^{-1}, \\ T_1 &= T_{s1}(1+X_0)^{-1}, \\ p_0 &= p_{s0}(1+X_0)^{-\alpha}, \\ \zeta_{s1} &= \log(T_{s0}/T_{s1}). \end{aligned}\right\} \qquad (54)$$

When they are calculated by these equations, the value of T_{e0} can be evaluated by solving Eq. (13). Then, when we give M_0 and κ_0 the various values, solve numerically Eq.s (13), (49) and (50) for those values and apply the results to Eq.s (9) to (11), we can obtain the values of M_{opt} and κ_{opt}, which make the duct size minimum or the isentropic efficiency maximum.

4.3 Constant length duct

In this article we discuss the optimization in the case of the constant length.

Now using Eq.s (48), Eq.s (2) and (3) are transformed to the following equations:

$$\frac{d\xi_s}{dx} = \frac{\sigma_{eff} u B^2(1-\Delta)}{(1+\beta_{eff}{}^2)p}\left\{1+\frac{(\beta_{eff}{}^2-a^2)\kappa}{1+a^2}\right\}-aX\frac{d\zeta_s}{dx}+\frac{dF}{p}, \qquad (55)$$

$$\frac{d\zeta_s}{dx} = \frac{I(\beta_{eff}-a)B\kappa(1-\Delta)}{\alpha(1+X)pA(1+a^2)}+\frac{dQ}{\alpha(1+X)up}, \qquad (56)$$

where

$$\left.\begin{aligned} X &= (\gamma-1)M^2/2, \\ p &= p_{s0}(1+X)^{-\alpha}\exp(-\xi_s), \\ u &= \{2C_pT_{s0}X/(1+X)\}^{1/2}\exp(-\zeta_s/2), \\ A &= \frac{m_0R}{p_{s0}}\left(\frac{T_{s0}}{2C_p}\right)^{1/2}(1+X)^{\alpha-1/2}X^{-1/2}\exp(\xi_s-\zeta_s/2). \end{aligned}\right\} \qquad (57)$$

Now let us investigate the case where Eq. (11) is used, which expresses the isentropic efficiency η_i, as the estimation function. Eq. (11) becomes as follows:

$$\eta_i = \frac{IB}{m_0C_pT_{s0}(1+a^2)}\,\frac{\int_0^l(\beta_{eff}-a)u\kappa dx}{1-(p_{s1}/p_{s0})^{1/\alpha}}. \qquad (58)$$

Here, if we define $\chi(x)$ as follows:

$$\chi(x) = \int_0^x(\beta_{eff}-a)u\kappa dx, \qquad (59)$$

Eq. (58) is expressed by

$$\eta_i = \frac{IB}{m_0C_pT_{s0}(1+a^2)}\,\frac{\chi(l)}{1-(p_{s1}/p_{s0})^{1/\alpha}}. \qquad (60)$$

Then our problem becomes the so-called terminal control one, and so we can solve this problem by means of Pontryagin's principle.

When we put $\chi_s(x)$ and f_s

$$\chi_s(x) = \frac{\chi(x)}{1-\{p_s(x)/p_{s0}\}^{1/\alpha}}, \tag{61}$$

$$f_s = f_s(\xi_s, \zeta_s, X) = \frac{d\chi_s}{dx} \tag{62}$$

respectively and express the right sides of Eq.s (55) and (56) with $f_\xi = f_\xi(\xi_s, \zeta_s, X)$ and $f_\zeta = f_\zeta(\xi_s, \zeta_s, X)$ respectively, we can obtain a set of simultaneous differential equations to be solved under the independent variable of x, the state variables of ξ_s and ζ_s and the control variable of X as follows:

$$\frac{d\chi_s}{dx} = f_s(\xi_s, \zeta_s, X), \tag{63}$$

$$\frac{d\xi_s}{dx} = f_\xi(\xi_s, \zeta_s, X), \tag{64}$$

$$\frac{d\zeta_s}{dx} = f_\zeta(\xi_s, \zeta_s, X), \tag{65}$$

$$\frac{d\psi_\xi}{dx} = \frac{\partial f_s}{\partial \xi_s} - \psi_\xi \frac{\partial f_\xi}{\partial \xi_s} - \psi_\zeta \frac{\partial f_\zeta}{\partial \xi_s}, \tag{66}$$

$$\frac{d\psi_\zeta}{dx} = \frac{\partial f_s}{\partial \zeta_s} - \psi_\xi \frac{\partial f_\xi}{\partial \zeta_s} - \psi_\zeta \frac{\partial f_\zeta}{\partial \zeta_s}, \tag{67}$$

by Pontryagin's principle, where ψ_ξ and ψ_ζ are the adjoint variables pertaining to ξ_s and ζ_s respectively.

The boundary conditions are given by

$$\left.\begin{array}{l} \xi_{s0} = \zeta_{s0} = 0, \\ \zeta_{s1} = \log(T_{s0}/T_{s1}), \\ \psi_\xi|_{x=l} = 0. \end{array}\right\} \tag{68}$$

On the other hand, Hamiltonian H is given by

$$H = f_s + \psi_\xi f_\xi + \psi_\zeta f_\zeta. \tag{69}$$

We can obtain the value of the optimum efficiency $\eta_i = \eta_{i\,\max}$ and the optimum distributions of ξ_s, ζ_s, T_e, X etc. by numerically solving Eq.s (13) and (63) to (68) under the boundary condition (68) and the following condition

$$\min \mathrm{H} = \text{constant}$$

for the specific values of M_0 and κ_0.

Moreover, by using the calculation results of ξ_s, ζ_s, T_e and X for the various values of M_0 and κ_0, the various values of $\eta_{t\,\max}$ can be calculated. Therefore, we can find the values of M_{opt} and κ_{opt} which make $\eta_{t\,\max}$ maximum.

Also, the values of M_{opt} and κ_{opt}, which make the duct size minimum, can be obtained by a treatment similar to the above-mentioned.

5. Conclusion

In Articles 4.1 and 4.2, with respect to the constant velocity and constant Mach number duct, the numerically solvable differential equations for the pressure, temperature and electron temperature have been derived from the basic MHD equations. In these equations, we have considered the electrode voltage drop, leakage current, segmented electrode effect, friction and heat transfer at the channel walls, electrothermal instabilities of plasma and non-elastic collision effect.

When the differential equations can be solved in regard to the various values of u_0 or M_0 and κ_0, and the results are applied to the estimation functions, we can find the values of U_{opt} or M_{opt} and κ_{opt}, which make the estimation functions maximum or minimum, and give the optimum distribution of every quantity within the duct.

Also in Article 4.3, an optimization theory of the constant length duct has been derived, in which Pontryagin's principle was used.

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