# An Algorithm of Interaction Coordination in Multilevel Control of Nonlinear Systems

By

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**Abstract**

This paper proposes a coordination algorithm for a multilevel control of a large-scale dynamical system. The system considered consists of weakly interconnected nonlinear subsystems and the performance index is quadratic in states and controls.

According to the variational principle, the optimal control is given by solving a nonlinear two-point boundary-value problem, of which analytical solution is generally impossible. The present technique is to solve the overall problem, first by solving decomposed problems of the subsystems, and secondly by coordinating interactions among the subsystems. Since each subsystem problem is a linear two-point boundary-value problem, it is relatively easy to solve. The present idea of coordination is to adjust directly the interaction variables by an iteration without using the conventional Lagrange multiplier. A sufficient condition for convergence of the iteration algorithm is presented in the paper.

The algorithm is computationally simple and the convergence is quite rapid for the problem of weakly coupled systems with small nonlinearities. The effectiveness of the method is illustrated in two examples.

## 1. Introduction

Recently, several attempts have appeared in the optimal design and control of large, complex systems not only of an industrial but also of a social or an ecological nature. A basic idea for the study of large-sclae systems is to decompose the original problem into a number of smaller and simpler subproblems which can be dealt with by some conventional mathematical or computational tools. The subsystems have to be later coordinated to reconstruct the original system, then the decomposition-coordination algorithm inevitably involves a multilevel structure.

Usually a difficulty arises in deciding interconnection variables and a coordination

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policy. A common technique is to introduce the Lagrange multipliers corresponding to interconnection constraints[1-4]. However, the Lagrange multiplier method does not offer an efficient means for nonlinear dynamical systems. The so-called feasible method[4] always guarantees satisfaction of the interconnection constraint even when the iterative procedure does not converge. But, unfortunately, the method is applicable only to output-controllable systems.

The purpose of this paper is to present another coordination algorithm for the multilevel control of a large-scale dynamical system. The system considered here consists of weakly interconnected nonlinear subsystems. The algorithm is an extension of the one previously proposed for a linear dynamical system[5]. The performance index is taken to be quadratic in states and controls.

According to the variational principle, a nonlinear two-point boundary-value problem is derived, the solution of which gives the optimal control. Since the overall problem can not be solved analytically, it is decomposed into subsystem problems. Each subsystem problem is a linear two-point boundary-value problem of relatively low dimension. The present idea of the coordination is to adjust directly the interconnection variables by iterations without using the Lagrange multiplier. A sufficient condition for convergence of the iteration procedure is presented in the paper.

The algorithm is computationally simple and the convergence is quite rapid for the problem of weakly coupled systems with small nonlinearities. The effectiveness of the method is illustrated in two examples.

## 2. Notation

$I_n$ : the identity matrix of dimension $n$

$x'(t)$: vector transpose

$\|x(t)\| \triangleq \max_{t\in[t_0, t_f]} (x'x)^{1/2}$

$A'(t, \tau)$: matrix transpose

$\|A(t, \tau)\| \triangleq \max_{t, \tau\in[t_0, t_f]} (\text{trace } AA')^{1/2}$

$\text{diag}(A, B) \triangleq \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

${}^k z$ : the value of $z$ at the $k$-th iteration step

## 3. Problem Statement

Consider dynamical systems governed by the differential equation

$$\dot{x} = A(t;\varepsilon)x + B(t)u + \lambda f(t, x; \varepsilon), \quad x(t_0) = x_0 \tag{1}$$

with the associated performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} [x'Q(t;\varepsilon)x + u'R(t)u]\, dt \tag{2}$$

where $x$ is the $n$-dimensional state vector, $u$ the $m$-dimensional control vector; $A$ and $B$ are $n\times n$- and $n\times m$-matrices, respectively, continuous in time $t$. $f$ is a nonlinear vector function of the class $C^2$. The matrices $Q$ and $R$, both continuous in $t$, are positive semidefinite and positive definite, respectively. The initial time $t_0$ and the final time $t_f$ are assumed to be fixed.

The scalar parameter $\lambda$ is associated with the system nonlinearities. The parameter $\varepsilon$ in $f$, $A$, and $Q$ represents interconnection among the subsystems; that is, when $\varepsilon=0$ the problem of (1) and (2) is decomposed into several independent subproblems. The partitioned form of a system consisting of two subsystems is

$$\left.\begin{aligned} & x = (x_1', x_2')', \quad u = (u_1', u_2')', \qquad A(t;\varepsilon) = \begin{pmatrix} A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix} \\ & f(t, x; \varepsilon) = (f_1'(t, x_1, x_2; \varepsilon), f_2'(t, x_1, x_2; \varepsilon))' \\ & B(t) = \mathrm{diag}(B_{11}, B_{22}),\ R(t) = \mathrm{diag}(R_{11}, R_{22}),\ Q(t;\varepsilon) = \begin{pmatrix} Q_{11} & \varepsilon Q_{12} \\ \varepsilon Q_{12}' & Q_{22} \end{pmatrix} \end{aligned}\right\} \tag{3}$$

where $x_1$ and $x_2$ are $n_1$- and $n_2$-dimensional substates, $n_1+n_2=n$, and $u_1$ and $u_2$ are $m_1$- and $m_2$-dimensional subcontrols, $m_1+m_2=m$. In the following, two subsystems are considered for simplicity. Generalization to an arbitrary number of subsystems is straightforward.

## 4. Necessary Condition for the Optimality

A necessary condition for optimality of the problem is derived by using the variational principle. Define the Hamiltonian:

$$H = \frac{1}{2}(x'Qx + u'Ru) + p'(Ax + Bu + \lambda f) \tag{4}$$

where $p$ is the $n$-dimensional costate vector satisfying the differential equation

$$\dot{p} = -Qx - \left[A' + \lambda\left(\frac{\partial f}{\partial x}\right)'\right]p, \qquad p(t_f) = 0 \tag{5}$$

The optimal control which minimizes the Hamiltonian of (4) is given by

$$u = -R^{-1}B'p \tag{6}$$

Substituting (6) into (1) and thereupon combining (1) and (5) yields the following nonlinear two-point boundary-value problem:

$$\dot{z}_i = D_{ii}(t)z_i + \varepsilon D_{ij}(t)z_j + \lambda h_i(t, z_1, z_2; \varepsilon) \tag{7}$$

$$L_{i1}z_i(t_0) + L_{i2}z_i(t_f) = l_i \qquad (i, j = 1, 2; i \neq j) \tag{8}$$

where

$$\left.\begin{aligned} &z_i \triangleq (x_i', p_i')', \quad h_i \triangleq (f_i', -p_i'\psi_{ii} - p_j'\psi_{ji})', \quad \psi_{ji} \triangleq \frac{\partial f_j}{\partial x_i} \\ &D_{ij} \triangleq \begin{pmatrix} A_{ij} & -E_{ij} \\ -Q_{ij} & -A_{ji}' \end{pmatrix}, \qquad E \triangleq BR^{-1}B' = \mathrm{diag}(E_{11}, E_{22}) \\ &L_{i1} \triangleq \mathrm{diag}(I_{n_i}, 0), \; L_{i2} \triangleq \mathrm{diag}(0, I_{n_i}), \; l_i \triangleq (x_{i0}', 0')' \quad (i, j = 1, 2) \end{aligned}\right\} \tag{9}$$

By putting $\lambda = 0$ in (7), the problem is reduced to the linear two-point boundary-value problem. Further, when $\varepsilon = 0$ the problem is decomposed into two individual sub-problems. The system of (7) with $\varepsilon = \lambda = 0$ is called the unperturbed system.

## 5. Multilevel Solution Procedure

A multilevel technique is used to solve the problem of (7) and (8). Our procedure is essentially to solve linear two-point boundary-value problems of lower dimensions iteratively until the coordination of interconnections among the subsystems is achieved. An iteration algorithm for the coordination is proposed.

### 5.1 The First-Level Calculation

Rewriting $z_i$ and $z_j$ in the terms containing $\varepsilon$ and $\lambda$ in (7) into $w_i$ and $w_j$, respectively, leads to

$$\dot{z}_i = D_{ii}(t)z_i + \varepsilon D_{ij}(t)w_j + \lambda h_i(t, w_1, w_2) \tag{10}$$

Henceforth $\varepsilon$ in $h$ and $f$ will be omitted. The vector $w_i \triangleq (y_i', q_i')'$ $(i = 1, 2)$, called the interaction vector, is to be given from the second level. At the first level, equation (10) is solved regarding $w_i$ as pseudoinputs, under the boundary condition (8). Since the problem is linear and of lower dimension, it is relatively easy to solve. We have the following lemma.

*Lemma* 1

Let $\Phi_i(t)$ be the fundamental matrix of the unperturbed system

$$\dot{z}_i(t) = D_{ii}(t)\, z_i(t) \tag{11}$$

satisfying the initial condition $\Phi_i(t_0) = I_{2n_i}$. If the matrix

$$L_i \triangleq L_{i1} + L_{i2}\Phi_i(t_f) \tag{12}$$

is nonsingular, then equation (10) has the unique solution satisfying the boundary condition (8) for arbitrary vectors $w_i(t)$ and $w_j(t)$. The solution can be written in terms of the fundamental matrix as

$$z_i(t) = \Phi_i(t) L_i^{-1} l_i + \int_{t_0}^{t_f} \Gamma_i(t, \tau) \{\varepsilon D_{ij}(\tau) w_j(\tau) + \lambda h_i[\tau, w_1(\tau), w_2(\tau)]\} d\tau \quad (13)$$
$$(i, j = 1, 2;\ i \neq j)$$

where $\Gamma_i(t, \tau)$ is the Green function given by

$$\Gamma_i(t, \tau) = \begin{cases} \Phi_i(t) [I_{2n_i} - L_i^{-1} L_{i2} \Phi_i(t_f)] \Phi_i^{-1}(\tau), & t_0 \leq \tau \leq t \\ -\Phi_i(t) L_i^{-1} L_{i2} \Phi_i(t_f) \Phi_i^{-1}(\tau), & t \leq \tau \leq t_f \end{cases} \quad (14)$$

As to the proof of Lemma 1, refer to 6).

## 5.2 The Second-Level Calculation

After the first-level calculation, the subproblem solutions do not necessarily satisfy the interconnection constraints:

$$r_i(t) \triangleq z_i(t) - w_i(t) = 0 \qquad (i = 1, 2) \quad (15)$$

The task of the second level is to correct the interaction vector $w_i(t)$ so as to satisfy the constraint (15). The coordination method which uses the Lagrange multiplier as an adjustable variable is well known[1~4]. However, the Lagrange multiplier method does not offer an efficient means for coordinating interconnected nonlinear subsystems.

The present idea is to correct directly the interaction vector by an iteration. To this end, corresponding to (15), the cost function at the second level is introduced:

$$G \triangleq \left[ \sum_{i=1}^{2} \int_{t_0}^{t_f} r_i'(t) r_i(t) dt \right]^{1/2} \quad (16)$$

The goal is to adjust the interaction vector $w_i(t)$ to reduce (16) to zero. A nearly steepest-descent algorithm is adopted to obtain ${}^{k+1}w_i(t)$, the $(k+1)$-th iterate, as follows:

$$\left. \begin{array}{ll} {}^{k+1}w_i(t) = {}^{k}w_i(t) + {}^{k+1}\alpha\, {}^{k}r_i(t) & \\ |{}^{k+1}\alpha| \leq \bar{\alpha} & (i = 1, 2;\ k = 0, 1, 2, \cdots) \end{array} \right\} \quad (17)$$

where ${}^{k+1}\alpha$ is the step size along the search direction ${}^{k}r_i$. ${}^{k+1}\alpha$ is determined by a one-dimensional search so as to minimize the coordination error (16). $\bar{\alpha}$ is a constant such that $\bar{\alpha} \geq 1$. The initial guess ${}^{0}w_i(t)$ of $w_i(t)$ is chosen to be the solution of the unperturbed system, i.e.,

$$\bar{z}_i(t) \triangleq \Phi_i(t) L_i^{-1} l_i \quad (18)$$

## 5.3 Convergence Proof of the Iteration Algorithm

In this section the convergency of the proposed iteration is examined. Define the closed region of $w_1(t)$ and $w_2(t)$ as follows:

$$\Omega \triangleq \left\{(w_1, w_2) \,\Big|\, \sum_{i=1}^{2} \| w_i(t) - \bar{z}_i(t) \| \leq 2\delta \right\}, \qquad \delta > 0 \tag{19}$$

Under the assumption on the matrices $A$, $B$, $Q$, $R$ and the function $f$ in (1) and (2), the following quantities are introduced:

$$\left.\begin{aligned}
& a_1 \triangleq \max_{i,j;\, i\neq j} \| D_{ij}(t) \|, \qquad a_2 \triangleq (t_f - t_0) \max_i \| \Gamma_i(t, \tau) \| \\
& a_3 \triangleq \max_i \| \bar{z}_i(t) \|, \qquad b_1 \triangleq \max_i \| f_i[t, \bar{z}_1(t), \bar{z}_2(t)] \| \\
& b_2 \triangleq \max_{i,j} \| \psi_{ij}[t, w_1(t), w_2(t)] \| \qquad \text{for}(w_1, w_2) \in \Omega \\
& \| f_i(t, w_1^1, w_2^1) - f_i(t, w_1^2, w_2^2) \| \leq b_{i1} \| w_1^1 - w_1^2 \| + b_{i2} \| w_2^1 - w_2^2 \| \\
& \qquad \text{for}(w_1^l, w_2^l) \in \Omega \\
& \| \psi_{ij}(t, w_1^1, w_2^1) - \psi_{ij}(t, w_1^2, w_2^2) \| \leq b_{ij1} \| w_1^1 - w_1^2 \| \\
& \qquad + b_{ij2} \| w_2^1 - w_2^2 \| \qquad \text{for}(w_1^l, w_2^l) \in \Omega \\
& b_3 \triangleq \max_i \,[b_{1i} + b_{2i} + 2(a_3 + \delta)(b_{11i} + b_{12i} + b_{21i} + b_{22i})] \\
& \mu \triangleq a_2[|\varepsilon| a_1 + |\lambda| (2b_2 + b_3)] \\
& \nu \triangleq a_2[|\varepsilon| a_1 a_3 + |\lambda| (2a_3 b_2 + b_1)] \qquad (i, j, l = 1, 2)
\end{aligned}\right\} \tag{20}$$

Owing to (13), the difference between ${}^k z_i$ and ${}^{k+1} z_i$ is given by

$$\begin{aligned}
{}^{k+1}z_i(t) - {}^k z_i(t) = \int_{t_0}^{t_f} \Gamma_i(t, \tau) \Big\{ & \varepsilon D_{ij}(\tau)[{}^{k+1}w_j(\tau) - {}^k w_j(\tau)] \\
& + \lambda [h_i(\tau, {}^{k+1}w_1, {}^{k+1}w_2) - h_i(\tau, {}^k w_1, {}^k w_2)] \Big\} d\tau \\
& \qquad (i, j = 1, 2;\ i \neq j)
\end{aligned} \tag{21}$$

On taking (9) into account, we have

$$\begin{aligned}
& \| h_i(t, {}^{k+1}w_1, {}^{k+1}w_2) - h_i(t, {}^k w_1, {}^k w_2) \| \leq \| f_i(t, {}^{k+1}w_1, {}^{k+1}w_2) \\
& \quad - f_i(t, {}^k w_1, {}^k w_2) \| + \| \mathrm{diag}[0, -\psi_{ii}'(t, {}^{k+1}w_1, {}^{k+1}w_2)]^{k+1} w_i \\
& \quad - \mathrm{diag}[0, -\psi_{ii}'(t, {}^k w_1, {}^k w_2)]^k w_i \| + \| \mathrm{diag}[0, -\psi_{ji}'(t, {}^{k+1}w_1, \\
& \quad {}^{k+1}w_2)]^{k+1} w_j - \mathrm{diag}[0, -\psi_{ji}'(t, {}^k w_1, {}^k w_2)]^k w_j \|
\end{aligned} \tag{22}$$

For $({}^k w_1, {}^k w_2) \in \Omega$ and $({}^{k+1}w_1, {}^{k+1}w_2) \in \Omega$, the second term in the right side of (22) is bounded by

$$\begin{aligned}
& \| \mathrm{diag}[0, -\psi_{ii}'(t, {}^{k+1}w_1, {}^{k+1}w_2)]^{k+1} w_i - \mathrm{diag}[0, -\psi_{ii}'(t, {}^k w_1, {}^k w_2)]^k w_i \| \\
& = \| \mathrm{diag}[0, \psi_{ii}'(t, {}^k w_1, {}^k w_2) - \psi_{ii}'(t, {}^{k+1}w_1, {}^{k+1}w_2)]^k w_i \\
& \quad + {}^{k+1}\alpha\, \mathrm{diag}[0, -\psi_{ii}'(t, {}^{k+1}w_1, {}^{k+1}w_2)]^k r_i \| \\
& \leq |{}^{k+1}\alpha| [b_2 \| {}^k r_i \| + 2(a_3 + \delta)(b_{ii1} \| {}^k r_1 \| + b_{ii2} \| {}^k r_2 \|)]
\end{aligned} \tag{23}$$

Substituting (17), (22) and (23) into (21) yields

$$\begin{aligned}\| {}^{k+1}z_i - {}^{k}z_i \| &\leq \int_{t_0}^{t_f} \| \Gamma_i \| \Big\{ |\varepsilon| \, \| D_{ij} \| \, \| {}^{k+1}w_j - {}^{k}w_j \| \\ &+ |\lambda| \, \| h_i(\tau, {}^{k+1}w_1, {}^{k+1}w_2) - h_i(\tau, {}^{k}w_1, {}^{k}w_2) \| \Big\} \, d\tau \\ &\leq |{}^{k+1}\alpha| \, a_2 \Big\{ |\lambda| \, b_2 \| {}^{k}r_i \| + (|\varepsilon| \, a_1 + |\lambda| \, b_2) \| {}^{k}r_j \| \\ &+ |\lambda| \, [b_{i1} + 2(b_{ii1} + b_{ji1})(a_3 + \delta)] \| {}^{k}r_1 \| \\ &+ |\lambda| \, [b_{i2} + 2(b_{ii2} + b_{ji2})(a_3 + \delta)] \| {}^{k}r_2 \| \Big\} \end{aligned} \tag{24}$$

Hence we obtain

$$\sum_{i=1}^{2} \| {}^{k+1}z_i(t) - {}^{k}z_i(t) \| \leq |{}^{k+1}\alpha| \, \mu \sum_{i=1}^{2} \| {}^{k}r_i(t) \| \tag{25}$$

On the other hand, the difference between ${}^{k}z_i$ and ${}^{k}w_i$ is

$$\begin{aligned} {}^{k}r_i(t) &= {}^{k}z_i(t) - {}^{k}w_i(t) \\ &= \bar{z}_i(t) + \int_{t_0}^{t_f} \Gamma_i(t, \tau) \Big\{ \varepsilon D_{ij}(\tau) {}^{k}w_j(\tau) + \lambda h_i[\tau, {}^{k}w_1(\tau), {}^{k}w_2(\tau)] \Big\} d\tau - {}^{k}w_i(t) \\ &= (1 - {}^{k}\alpha)^{k-1} r_i(t) + \int_{t_0}^{t_f} \Gamma_i(t, \tau) \Big\langle \varepsilon D_{ij}(\tau) [{}^{k}w_j(\tau) - {}^{k-1}w_j(\tau)] \\ &\quad + \lambda \Big\{ h_i[\tau, {}^{k}w_1(\tau), {}^{k}w_2(\tau)] - h_i[\tau, {}^{k-1}w_1(\tau), {}^{k-1}w_2(\tau)] \Big\} \Big\rangle \, d\tau \end{aligned} \tag{26}$$

Therefore

$$\sum_{i=1}^{2} \| {}^{k}r_i(t) \| \leq \varphi({}^{k}\alpha) \sum_{i=1}^{2} \| {}^{k-1}r_i(t) \| \tag{27}$$

where

$$\varphi({}^{k}\alpha) \triangleq |1 - {}^{k}\alpha| + \mu |{}^{k}\alpha| \tag{28}$$

Now the following lemma is established:

*Lemma 2*

Assume that $\mu < 1$ and $\nu \leq \delta(1-\mu)/\bar{\alpha}$. Further choose the step size ${}^{k}\alpha$ of each iteration so as to minimize the norm

$$\sum_{i=1}^{2} \| {}^{k}r_i(t) \|$$

or equivalently ${}^{k}G$. Then the following relations are obtained:

$$\sum_{i=1}^{2} \| {}^{k}r_i(t) \| \leq 2 \nu \mu^{k} \tag{29}$$

$$\sum_{i=1}^{2} \| {}^{k+1}z_i(t) - {}^{k}z_i(t) \| \leq 2\bar{a}\,\nu\mu^{k+1} \tag{30}$$

$$\sum_{i=1}^{2} \| {}^{k+1}w_i(t) - \bar{z}_i(t) \| \leq 2\delta \qquad (k=0, 1, 2, \cdots) \tag{31}$$

*Proof*

The lemma is proved inductively. First, since ${}^0w_i(t) = \bar{z}_i(t)$, evidently $({}^0w_1, {}^0w_2) \in \Omega$. Since (13) with $k=0$ reads

$${}^0z_i(t) = \bar{z}_i(t) + \int_{t_0}^{t_f} \Gamma_i(t, \tau)\Big\{ \varepsilon D_{ij}(\tau){}^0w_j(\tau) + \lambda h_i[\tau, {}^0w_1(\tau), {}^0w_2(\tau)]\Big\}\, d\tau \tag{32}$$

we have

$$\| {}^0r_i(t) \| = \| {}^0z_i(t) - \bar{z}_i(t) \| \leq a_2[|\varepsilon| a_1a_3 + |\lambda| (2a_3b_2 + b_1)] = \nu \tag{33}$$

which, upon substitution into (17) with $k=0$, gives

$$\| {}^1w_i - \bar{z}_i \| = |{}^1\alpha|\, \| {}^0r_i(t) \| \leq \bar{a}\nu \leq \delta \tag{34}$$

Thereupon using (25) with $k=0$ gives

$$\sum_{i=1}^{2} \| {}^1z_i(t) - {}^0z_i(t) \| \leq |{}^1\alpha|\, \mu \sum_{i=1}^{2} \| {}^0r_i(t) \| \leq 2\bar{a}\nu\mu \tag{35}$$

Thus the relations (29)～(31) are proved for $k=0$.

Secondly, we show that the relations (29)～(31) hold for $k$ if they hold up to $k-1$. Since $\min_{|\alpha| \leq \bar{a}} \varphi(\alpha) = \varphi(1) = \mu$, ${}^k\alpha$ can be chosen to derive

$$\sum_{i=1}^{2} \| {}^kr_i(t) \| \leq \mu \sum_{i=1}^{2} \| {}^{k-1}r_i(t) \| \tag{36}$$

from (27). In fact, ${}^k\alpha$ minimizing ${}^kG$ suffices to give (36). This implies that (29) holds for $k$. We also obtain, from (17) and (29), the relation

$$\sum_{i=1}^{2} \| {}^{k+1}w_i(t) - \bar{z}_i(t) \| = \sum_{i=1}^{2} \| {}^{k+1}w_i(t) - {}^0w_i(t) \| \leq \sum_{i=1}^{2} \sum_{j=0}^{k} \| {}^{j+1}w_i(t) - {}^jw_i(t) \|$$

$$= \sum_{i=1}^{2} \sum_{j=0}^{k} |{}^{j+1}\alpha|\, \| {}^jr_i(t) \| \leq 2\nu\bar{a} \sum_{j=0}^{k} \mu^j \leq \frac{2\nu\bar{a}}{1-\mu} \leq 2\delta \tag{37}$$

Inequality (37) implies that $({}^{k+1}w_1, {}^{k+1}w_2) \in \Omega$. Then substituting (29) into (25) gives

$$\sum_{i=1}^{2} \|{}^{k+1}z_i(t) - {}^{k}z_i(t)\| \leq 2\bar{\bar{a}}\nu\mu^{k+1} \tag{38}$$

Q.E.D.

Lemma 2 is sufficient to establish the following theorem.

*Theorem*

If all the conditions of Lemmas 1 and 2 are satisfied, the sequences $\{{}^{k}z_i(t)\}$ and $\{{}^{k}w_i(t)\}$ $(i=1, 2)$ converge uniformly to a same limit function $z_i^*(t)$ as $k \to \infty$. The limit function $z_i^*(t)$ is the unique solution to the problem of (7) and (8).

*Proof*

Since $0<\mu<1$ in Lemma 2, it follows that

$$\lim_{k\to\infty} \|{}^{k}r_i(t)\| = \lim_{k\to\infty} \|{}^{k+1}z_i(t) - {}^{k}z_i(t)\| = 0 \qquad (i = 1, 2) \tag{39}$$

This implies that, as $k \to \infty$, ${}^{k}z_i(t)$ coincides with ${}^{k}w_i(t)$ for all $t \in [t_0, t_f]$. Moreover, the solution $z_i(t)$ to the problem (7) and (8) is continuous in $t \in [t_0, t_f]$. Then the space of the functions $z_i(t)$ with the definition of the norm $\|z_i\|$ is complete. Therefore the sequences $\{{}^{k}z_i(t)\}$ and $\{{}^{k}w_i(t)\}$ have $z_i^*(t)$ as a limit. From (31) evidently $(z_1^*, z_2^*) \in \Omega$. By substituting the limit $z_i^*(t)$ into (13), it is readily observed that $z_i^*(t)$ is a solution to the problem of (7) and (8). The uniqueness of the solution is obvious due to the principle of contraction mapping. This completes the convergence proof.

Q.E.D.

*Remark* 1

The theorem guarantees the convergence of the iteration algorithm for the problem of weakly interconnected systems with small nonlinearities.

*Remark* 2

In this paper our consideration is confined to the problem of large systems composed of nonlinear subsystems interconnected through state vectors. By adding appropriate conditions, the algorithm applies also to systems with linear and nonlinear interconnections through state and control vectors, i.e.,

$$B = B(t; \varepsilon), \quad R = R(t; \varepsilon)$$
$$f = (f_1'(t, x_1, x_2, u_1, u_2; \varepsilon), \ f_2'(t, x_1, x_2, u_1, u_2; \varepsilon))' \tag{40}$$

In Section 7 an example of such a system will be examined.

## 6. Computational Algorithm

The computational procedure of the proposed algorithm is summarized as follows:

*Step* 0 (Initial guess of $w_i(t)$)

Set $k=0$. Find the solution $\bar{z}_i(t)$ to the problem (7) and (8) with $\varepsilon=\lambda=0$. Then choose $\bar{z}_i(t)$ as an initial estimate ${}^0w_i(t)$ of the interaction vector.

*Step* 1 (Level 1)

Given the interaction vector ${}^kw_i(t)$, find the solution ${}^kz_i(t)$ to the problem (8) and (10).

*Step* 2 (Level 2)

Given ${}^kz_i(t)$ from Level 1, calculate the coordination error ${}^kG$ defined by (16). If ${}^kG \leq \sigma$ ($\sigma$: small positive number prescribed), compute the optimal control $u$ and the associated performance index $J$ from (6) and (2), respectively. Then the calculation is terminated. If ${}^kG > \sigma$, proceed to Step 3.

*Step* 3 (Level 2)

Correct the interaction vector ${}^kw_i(t)$ to ${}^{k+1}w_i(t)$ by (17). The step size ${}^{k+1}\alpha$ is determined by a one-dimensional search to minimize the coordination error ${}^{k+1}G$. Replace $k$ by $k+1$ and return to Step 1.

*Remark* 3

If the step size ${}^k\alpha$ is always chosen as unity without the one-dimensional search, the present algorithm coincides with that proposed by Mesarović et al.[1]. However, the one-dimensional search is often efficient for accelerating the convergence rate.

## 7. Illustrative Examples

*Example* 1 (Three-axis attitude control)

The following equations approximately describe a three-axis attitude control system of an orbiting space vehicle[7]:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \varepsilon x_4 + \varepsilon x_4 x_6 + \varepsilon x_3 u_3 + u_1 \end{aligned}\right\} \text{Subsystem 1}$$
$$\left.\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\varepsilon x_2 - \varepsilon x_2 x_6 - \varepsilon x_1 u_3 + u_2 \end{aligned}\right\} \text{Subsystem 2} \qquad (41)$$
$$\left.\begin{aligned} \dot{x}_5 &= x_6 \\ \dot{x}_6 &= \varepsilon x_2 x_4 + \varepsilon x_1 u_2 + u_3 \end{aligned}\right\} \text{Subsystem 3}$$

The quantities $x_1$, $x_3$ and $x_5$ represent the roll, yaw and pitch motion, respectively, of the body about its principal axes. For convenience, the parameter $\lambda$ is set equal to $\varepsilon$ in (41). The index of performance is

$$J = \frac{1}{2}\int_0^{t_f}\left(\sum_{i=1}^{6} x_i^2 + \sum_{i=1}^{3} u_i^2\right)dt \qquad (42)$$

Corresponding to (8) and (10), the following two-point boundary-value problem is obtained for Subsystem 1:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -p_2+\varepsilon[y_4(1+y_6)-y_3(\varepsilon y_3 q_2-\varepsilon y_1 y_4+q_6)] \\ \dot{p}_1 &= -x_1+\varepsilon^2[-y_3 q_2 q_4+y_1(q_4{}^2+q_6{}^2)] \\ \dot{p}_2 &= -x_2-p_1+\varepsilon[q_4(1+y_6)-y_4 q_6] \\ x_1(0) &= x_{10}, \quad x_2(0) = x_{20}, \quad p_1(t_f) = p_2(t_f) = 0 \end{aligned}\right\} \tag{43}$$

$y_i$ $(i=1, 3, 4, 6)$ and $q_i$ $(i=2, 4, 6)$ are interaction variables corresponding to $x_i$ and $p_i$, respectively. Similar problem obtained for Subsystem 2 and 3 are omitted here.

As an example, let $t_f=10$ and the initial state $x_1(0)=x_3(0)=x_5(0)=1$, $x_2(0)=x_4(0)=x_6(0)=0$. Figure 1 shows the convergence rates of the performance index $J$ and the coordination error $G$ for various values of $\varepsilon$. A rapid convergence is observed for small $\varepsilon$. Figure 2 shows the step size obtained by the one-dimensional search at each iteration. Figure 3 illustrates the optimal trajectories of Subsystem 1 for various $\varepsilon$.

*Example* 2 (Minimum-fuel orbit transfer)

The second example deals with the minimum-fuel transfer of a low-thrust propulsion system between circular orbits[8]. The system dynamics is described by

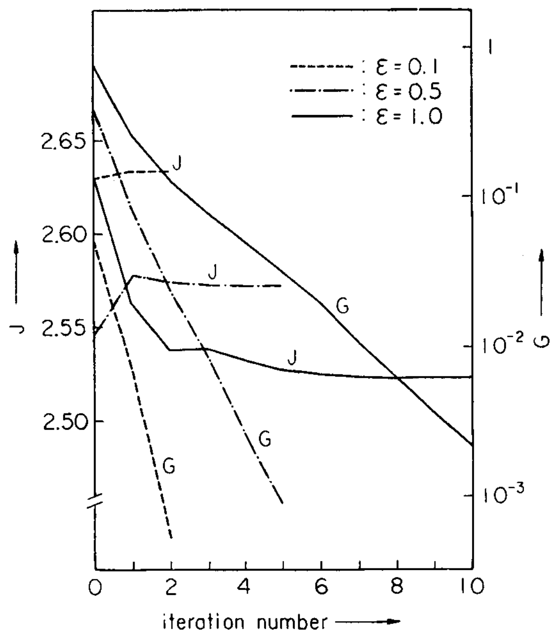


Fig. 1. Convergence rates of $J$ and $G$.

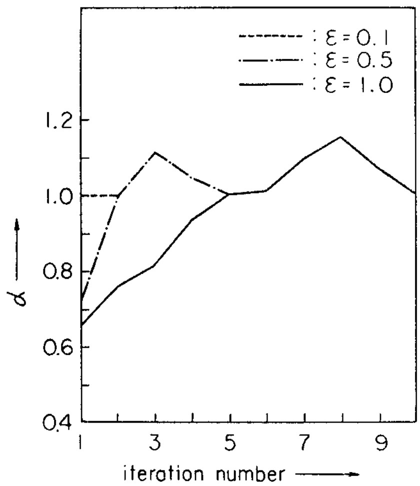


Fig. 2. Variations of the step size $\alpha$.

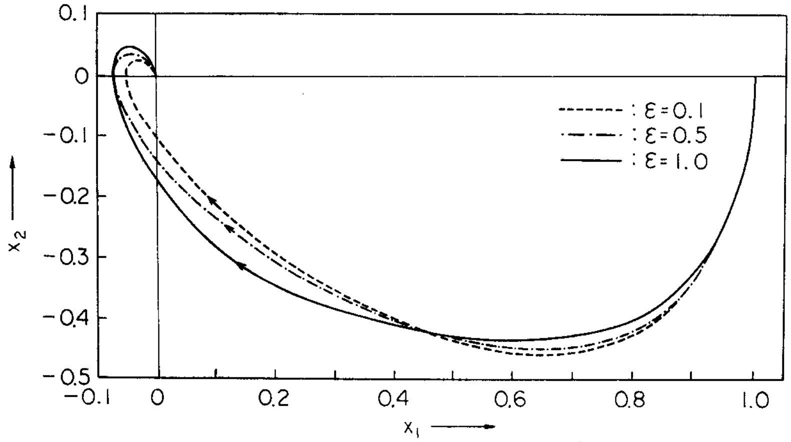


Fig. 3. Trajectories on the $x_1x_2$ plane.

$$\left.\begin{array}{ll}\left.\begin{array}{l}\dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - \varepsilon\left\{2x_4 + \dfrac{x_1}{(x_1^2+x_3^2+x_5^2)^{3/2}}\right\} + u_1\end{array}\right) & \text{Subsystem 1} \\ \left.\begin{array}{l}\dot{x}_3 = x_4 \\ \dot{x}_4 = x_3 + \varepsilon\left\{2x_2 - \dfrac{x_3}{(x_1^2+x_3^2+x_5^2)^{3/2}}\right\} + u_2\end{array}\right) & \text{Subsystem 2} \\ \left.\begin{array}{l}\dot{x}_5 = x_6 \\ \dot{x}_6 = -\varepsilon\left\{\dfrac{x_5}{(x_1^2+x_3^2+x_5^2)^{3/2}}\right\} + u_3\end{array}\right) & \text{Subsystem 3}\end{array}\right\} \quad (44)$$

The quantities $x_1$, $x_3$ and $x_5$ represent three components of the vehicle displacement in a reference frame. The parameter $\varepsilon$ introduced for convenience is equal to unity. The performance index is

$$J=\frac{1}{2}\int_0^{t_f}(u_1^2+u_2^2+u_3^2)\,dt \tag{45}$$

By way of example, let $t_f=\pi$ and $x_1(0)=x_2(0)=x_4(0)=x_5(0)=x_6(0)=0$, $x_3(0)=1$, $x_2(\pi)=-0.75$, $x_3(\pi)=1.5$, $x_4(\pi)=x_5(\pi)=0$, $x_6(\pi)=-\pi/90$, and $x_1(\pi)$ be free. Figure 4 shows the variations of $J$, $G$ and $\alpha$ with computing time.

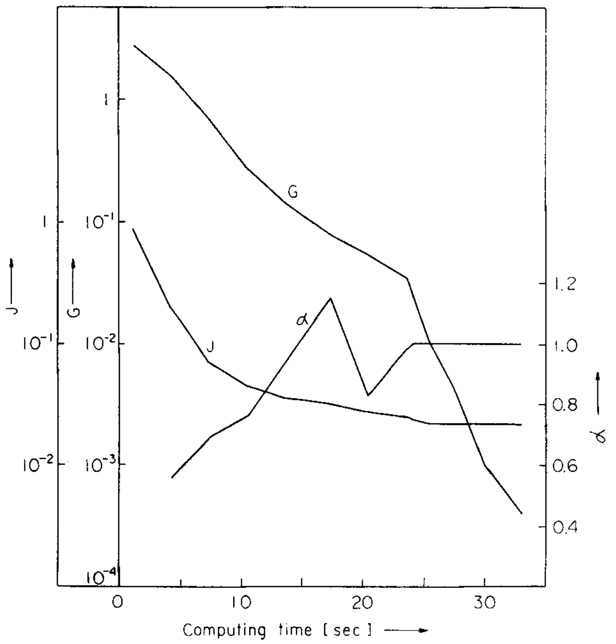


Fig. 4. Variations of $J$, $G$, and $\alpha$ with computing time.

## 8. Conclusion

A coordination algorithm is proposed for a multilevel control of nonlinear dynamical systems. The system considered consists of weakly interconnected nonlinear subsystems and the performance index is quadratic in states and controls.

Due to the variational principle, the optimal control is given by solving a nonlinear two-point boundary-value problem. The present technique is to solve the overall problem, first by solving linear problems of the subsystems, and secondly by coordinating interactions among the subsystems. The idea of coordination is to adjust directly the interaction vectors by an iteration without using the conventional Lagrange multiplier. The one-dimensional search to determine the step size $\alpha$ is often effective for accelerating the convergence rate.

The effectiveness of the method is illustrated in two examples. All the numerical computations were made by FACOM 230–75 at the Data Processing Center of Kyoto University.

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