Self-Excited Oscillations in a System with Two Degrees of Freedom

By

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Abstract

The behavior of a negative-resistance oscillator with two resonant circuits is studied. The amplitude and frequency characteristics of the self-excited oscillation are calculated and shown graphically for some combinations of the system parameters. The stability of the oscillation is discussed. Particular attention is directed toward the internal resonance of the two resonant circuits which occurs when a certain relationship exists between their natural frequencies.

1. Introduction

This paper deals with self-excited oscillations in a negative-resistance oscillator having two resonant circuits. These circuits are inductively coupled, and one of them is connected to a negative-resistance element. The behavior of such systems has been studied by I. G. Malkin [1], N. Minorsky [2], N. V. Butenin [3], A. Tondl [4] and others. It has usually been assumed that the coupling between the two resonant circuits is large, and that there exist two distinct frequencies of the self-excited oscillation.

This paper is intended to discuss the oscillations of the system, not only for the large coupling between the two resonant circuits but also for the small coupling between them. In the latter case the entrainment of the two frequencies occurs if they are close each other. More generally, the internal resonance of the two circuits occurs when their natural frequencies have a certain relationship.

2. Fundamental equations and their transformation

The schematic diagram of Fig. 1 shows a vacuum-tube oscillator which contains two resonant circuits $L_1R_1C_1$ and $L_2R_2C_2$. We assume that the inductive coupling exists between L_1 and L_2, between L_1 and L_3, but no coupling between L_2 and L_3.

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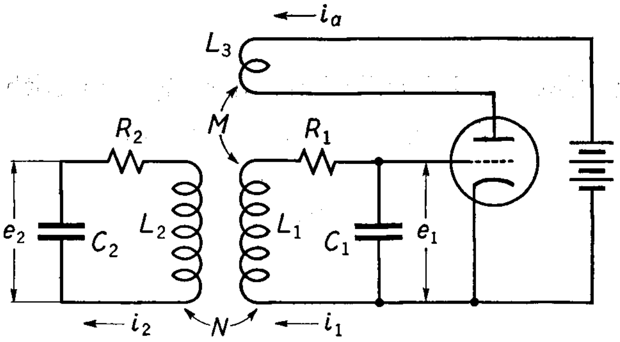

Fig. 1. Vacuum-tube oscillator with two resonant circuits.

Using Kirchhoff's laws and neglecting the grid current of the vacuum tube, we write the equations of the circuit as

$$\begin{aligned} L_1\frac{di_1}{dt}+R_1i_1+e_1 &= M\frac{di_a}{dt}+N\frac{di_2}{dt} \\ L_2\frac{di_2}{dt}+R_2i_2+e_2 &= N\frac{di_1}{dt} \\ e_1 = \frac{1}{C_1}\int i_1dt, &\qquad e_2 = \frac{1}{C_2}\int i_2dt \end{aligned} \tag{2.1}$$

Eliminating i_1 and i_2 in Eqs. (2.1), we obtain

$$\begin{aligned} L_1C_1\frac{d^2e_1}{dt^2}+C_1R_1\frac{de_1}{dt}+e_1 &= M\frac{di_a}{dt}+NC_2\frac{d^2e_2}{dt^2} \\ L_2C_2\frac{d^2e_2}{dt^2}+C_2R_2\frac{de_2}{dt}+e_2 &= NC_1\frac{d^2e_1}{dt^2} \end{aligned} \tag{2.2}$$

If we neglect the plate reaction and assume that the plate current - grid voltage characteristic is a cubic, then

$$i_a = S_1e_1-\frac{1}{3}S_3e_1^3 \tag{2.3}$$

where S_1 and S_3 are certain constants specific to the vacuum tube. Introducing the dimensionless variables u and v defined by

$$u = e_1\sqrt{\frac{MS_3}{MS_1-C_1R_1}}, \qquad v = e_2\sqrt{\frac{MS_3}{MS_1-C_1R_1}} \tag{2.4}$$

and using Eq. (2.3), we rewrite Eqs. (2.2) in the form

$$\begin{aligned} \ddot{u}-\chi_1\ddot{v}+n_1^2u &= \mu n_1(1-u^2)\dot{u} \\ \ddot{v}-\chi_2\ddot{u}+n_2^2v &= -\mu\frac{n_2^2}{n_1}\delta\dot{v} \end{aligned} \tag{2.5}$$

where

$$
\begin{aligned}
n_1^2 &= \frac{1}{L_1C_1}, & n_2^2 &= \frac{1}{L_2C_2} \\
\chi_1 &= \frac{NC_2}{L_1C_1}, & \chi_2 &= \frac{NC_1}{L_2C_2} \\
\mu &= (MS_1-C_1R_1)n_1, & \delta &= \frac{C_2R_2}{MS_1-C_1R_1}
\end{aligned}
\tag{2.6}
$$

The dots over u and v refer to differentiation with respect to t. Equations (2.5) are the fundamental equations that describe the behavior of the system of Fig. 1. When $N=0$, i.e., $\chi_1=\chi_2=0$, the first equation of (2.5) is reduced to van der Pol's equation and the second equation describes a damped oscillation in the $L_2R_2C_2$-circuit. Since the system is self-oscillatory, μ must be positive. We assume hereafter that μ is a small positive quantity.

Transformation of the fundamental equations to the standard form

The solution of the linear system obtained by putting $\mu=0$ in Eqs. (2.5) is given by

$$
\begin{aligned}
u(t) &= r_1 \cos(\omega_1 t+\theta_1)+r_2 \cos(\omega_2 t+\theta_2) \\
v(t) &= k_1 r_1 \cos(\omega_1 t+\theta_1)+k_2 r_2 \cos(\omega_2 t+\theta_2)
\end{aligned}
\tag{2.7}
$$

where r_i, $\theta_i (i=1, 2)$ are integration constants, ω_i are the roots of the following equation

$$(1-\chi_1\chi_2)\omega^4-(n_1^2+n_2^2)\omega^2+n_1^2n_2^2 = 0 \tag{2.8}$$

and k_i are given by

$$k_i = \frac{\omega_i^2-n_1^2}{\chi_1\omega_i^2} = \frac{\chi_2\omega_i^2}{\omega_i^2-n_2^2} \tag{2.9}$$

The linear system [$\mu=0$ in Eqs. (2.5)] has two natural frequencies ω_1 and ω_2.[1] Hence, for a small value of μ, we may assume that the solution of Eqs. (2.5) takes a form similar to Eqs. (2.7). Therefore we write

$$
\begin{aligned}
u &= x+y \\
v &= k_1x+k_2y
\end{aligned}
\tag{2.10}
$$

Substituting Eqs. (2.10) into Eqs. (2.5) leads to the standard form of equations

[1] Without loss of generality we may assume that $\omega_1<\omega_2$. Then we obtain the relations: $\omega_1<n_1<\omega_2$, $\omega_1<n_2<\omega_2$, $k_1<0$ and $k_2>0$.

$$\ddot{x}+\omega_1^2 x = \mu f(x, y, \dot{x}, \dot{y})$$
$$\ddot{y}+\omega_2^2 y = \mu g(x, y, \dot{x}, \dot{y}) \tag{2.11}$$

where

$$f(x, y, \dot{x}, \dot{y}) = \frac{\omega_1^2}{n_1}\frac{1}{k_2-k_1}\{k_2[1-(x+y)^2](\dot{x}+\dot{y})+\delta(k_1\dot{x}+k_2\dot{y})\}$$
$$g(x, y, \dot{x}, \dot{y}) = \frac{\omega_2^2}{n_1}\frac{1}{k_1-k_2}\{k_1[1-(x+y)^2](\dot{x}+\dot{y})+\delta(k_1\dot{x}+k_2\dot{y})\} \tag{2.12}$$

It is to be noted that, if ω_1 and ω_2 are sufficiently close each other, one may expect the internal resonance, i.e., the entrainment between these frequencies owing to the nonlinearity of the system. In this case, the present analysis ceases to be meaningful. Since the discriminant of Eq. (2.8) is given by $(n_1^2-n_2^2)^2+4\chi_1\chi_2 n_1^2 n_2^2$, this type of internal resonance occurs when $n_1 \simeq n_2$ and $N \simeq 0$ [see Eqs. (2.6)]. This case will be discussed in Section 4.2.

Derivation of an autonomous system by using the averaging method

When $\mu=0$, the solution of Eqs. (2.5) is given by Eqs. (2.7). Bearing this in mind, we write for x and y in Eqs. (2.10) as

$$\begin{aligned} x(t) &= r_1(t) \cos [\omega_1 t+\theta_1(t)] \\ y(t) &= r_2(t) \cos [\omega_2 t+\theta_2(t)] \\ \dot{x}(t) &= -\omega_1 r_1(t) \sin [\omega_1 t+\theta_1(t)] \\ \dot{y}(t) &= -\omega_2 r_2(t) \sin [\omega_2 t+\theta_2(t)] \end{aligned} \tag{2.13}$$

We assume that, for small values of μ, both the amplitudes $r_1(t)$, $r_2(t)$ and the phase angles $\theta_1(t)$, $\theta_2(t)$ are slowly varying functions of t. Substituting Eqs. (2.13) into Eqs. (2.11) gives

$$\begin{aligned} \dot{r}_1 \sin (\omega_1 t+\theta_1)+r_1\dot{\theta}_1 \cos (\omega_1 t+\theta_1) &= -\frac{\mu}{\omega_1} f(r_1, r_2, \theta_1, \theta_2) \\ \dot{r}_2 \sin (\omega_2 t+\theta_2)+r_2\dot{\theta}_2 \cos (\omega_2 t+\theta_2) &= -\frac{\mu}{\omega_2} g(r_1, r_2, \theta_1, \theta_2) \end{aligned} \tag{2.14}$$

where the functions f and g on the right-hand sides of Eqs. (2.14) are obtained by the substitution of Eqs. (2.13) into Eqs. (2.12). We obtain from Eqs. (2.13) that

$$\begin{aligned} \dot{r}_1 \cos (\omega_1 t+\theta_1)-r_1\dot{\theta}_1 \sin (\omega_1 t+\theta_1) &= 0 \\ \dot{r}_2 \cos (\omega_2 t+\theta_2)-r_2\dot{\theta}_2 \sin (\omega_2 t+\theta_2) &= 0 \end{aligned} \tag{2.15}$$

Solving Eqs. (2.14) and Eqs. (2.15) for $\dot{r}_1$, $\dot{r}_2$, $\dot{\theta}_1$, and $\dot{\theta}_2$ gives

$$\begin{aligned}
\dot{r}_1 &= -\frac{\mu}{\omega_1} f(r_1, r_2, \theta_1, \theta_2) \sin(\omega_1 t+\theta_1) \\
\dot{r}_2 &= -\frac{\mu}{\omega_2} g(r_1, r_2, \theta_1, \theta_2) \sin(\omega_2 t+\theta_2) \\
r_1\dot{\theta}_1 &= -\frac{\mu}{\omega_1} f(r_1, r_2, \theta_1, \theta_2) \cos(\omega_1 t+\theta_1) \\
r_2\dot{\theta}_2 &= -\frac{\mu}{\omega_2} g(r_1, r_2, \theta_1, \theta_2) \cos(\omega_2 t+\theta_2)
\end{aligned} \tag{2.16}$$

Equations (2.16) show that $\dot{r}_i$ and $\dot{\theta}_i (i=1, 2)$ are both of the order of μ. Therefore, as expected, r_i and θ_i are slowly varying functions of t. Hence, upon application of the averaging method, Eqs. (2.16) can be transformed into an autonomous system, i.e.,

$$\begin{aligned}
\dot{r}_1 &= -\frac{\mu}{\omega_1} \lim_{T\to\infty} \frac{1}{T}\int_0^T f(r_1, r_2, \theta_1, \theta_2) \sin(\omega_1 t+\theta_1)dt \\
&= \frac{\mu\omega_1^2}{8n_1} \frac{k_2}{k_2-k_1} (\rho_1-r_1^2-2r_2^2)r_1
\end{aligned}$$

Similarly

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1} \frac{k_1}{k_1-k_2} (\rho_2-2r_1^2-r_2^2)r_2 \tag{2.17}$$

and

$$\dot{\theta}_1 = 0$$
$$\dot{\theta}_2 = 0$$

where

$$\begin{aligned}
\rho_1 &= 4\left(1+\frac{k_1}{k_2}\delta\right) = 4\left(1-\frac{\omega_1^2-n_1^2}{\omega_1^2-n_2^2}\frac{n_2^2}{n_1^2}\delta\right) \\
\rho_2 &= 4\left(1+\frac{k_2}{k_1}\delta\right) = 4\left(1-\frac{\omega_2^2-n_1^2}{\omega_2^2-n_2^2}\frac{n_2^2}{n_1^2}\delta\right)
\end{aligned} \tag{2.18}$$

If we write $r_i^2=R_i$ $(i=1, 2)$, the first two equations of (2.17) can be rewritten as

$$\begin{aligned}
\dot{R}_1 &= \frac{\mu\omega_1^2}{4n_1} \frac{k_2}{k_2-k_1} (\rho_1-R_1-2R_2)R_1 \equiv F(R_1, R_2) \\
\dot{R}_2 &= \frac{\mu\omega_2^2}{4n_1} \frac{k_1}{k_1-k_2} (\rho_2-2R_1-R_2)R_2 \equiv G(R_1, R_2)
\end{aligned} \tag{2.19}$$

It is to be noted that, if there exists a certain relationship between the two frequencies ω_1 and ω_2, additional terms may appear on the right-hand sides of Eqs.

(2.17). The functions f and g in Eqs. (2.16) contain terms of several frequencies, and some of them which apparently look different from ω_1 or ω_2 may coincide with these frequencies. For instance, if $3\omega_1=\omega_2$, the frequency components $\omega_2-2\omega_1$ and $3\omega_1$ which are contained in f and g coincide with ω_1 and ω_2, respectively. Therefore these terms do not vanish upon integration in Eqs. (2.17). Similarly to the foregoing case where $\omega_1 \simeq \omega_2$, the internal resonance also occurs when $3\omega_1$ is in the neighborhood of ω_2. This type of internal resonance will be discussed in Section 4.1.

3. Steady-state solutions and their stability

Let us consider the steady-state solutions of Eqs. (2.17) in which r_i and θ_i are constant. The last two equations of (2.17) show that the phase angles, θ_1 and θ_2, are constant in the steady state. From Eqs. (2.19), we have

$$\begin{aligned} (\rho_1-R_{10}-2R_{20})R_{10} &= 0 \\ (\rho_2-2R_{10}-R_{20})R_{20} &= 0 \end{aligned} \tag{3.1}$$

where R_{10} and R_{20} denote the steady-state values of R_1 and R_2, respectively. We see, from Eqs. (3.1), that there are four different states of equilibrium, i.e.,

$$\begin{aligned} &\text{(a)}\ R_{10} = 0, \qquad R_{20} = 0 \\ &\text{(b)}\ R_{10} = \rho_1, \qquad R_{20} = 0 \\ &\text{(c)}\ R_{10} = 0, \qquad R_{20} = \rho_2 \\ &\text{(d)}\ R_{10} = \frac{1}{3}(2\rho_2-\rho_1), \qquad R_{20} = \frac{1}{3}(2\rho_1-\rho_2) \end{aligned} \tag{3.2}$$

No self-excited oscillation exists in (a); while, in (b) and (c), the solutions are periodic of frequencies ω_1 and ω_2, respectively. Since ω_1 and ω_2 are generally incommensurable, the solution in (d) is almost periodic.

Stability analysis

The steady-state solutions given by Eqs. (3.2) are maintained only when they are stable. The stability of the solutions is tested by the behavior of small variations, ξ and η, from the steady-state values, R_{10} and R_{20}, respectively, i.e.,

$$R_1 = R_{10}+\xi, \qquad R_2 = R_{20}+\eta \tag{3.3}$$

Substituting Eqs. (3.3) into Eqs. (2.19) gives

$$\begin{aligned} \dot{\xi} &= a_{11}\xi+a_{12}\eta \\ \dot{\eta} &= a_{21}\xi+a_{22}\eta \end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
a_{11} &= \left(\frac{\partial F}{\partial R_1}\right)_0 = \frac{\mu\omega_1^2}{4n_1}\frac{k_2}{k_2-k_1}(\rho_1-2R_{10}-2R_{20}) \\
a_{12} &= \left(\frac{\partial F}{\partial R_2}\right)_0 = -\frac{\mu\omega_1^2}{2n_1}\frac{k_2}{k_2-k_1}R_{10} \\
a_{21} &= \left(\frac{\partial G}{\partial R_1}\right)_0 = -\frac{\mu\omega_2^2}{2n_1}\frac{k_1}{k_1-k_2}R_{20} \\
a_{22} &= \left(\frac{\partial G}{\partial R_2}\right)_0 = \frac{\mu\omega_2^2}{4n_1}\frac{k_1}{k_1-k_2}(\rho_2-2R_{10}-2R_{20})
\end{aligned}
\tag{3.5}
$$

The symbol $(\quad)_0$ denotes the insertion of the steady-state values, R_{10} and R_{20}, after differentiation. The characteristic equation of the system (3.4) is given by

$$\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}=0 \tag{3.6}$$

Hence we see that the solution is stable provided

$$a_{11}+a_{22}<0 \qquad \text{and} \qquad a_{11}a_{22}-a_{12}a_{21}>0 \tag{3.7}$$

Substituting Eqs. (3.5) into (3.7), we obtain the stability conditions for the equilibrium states represented by Eqs. (3.2), i.e.,

(a) For $R_{10}=R_{20}=0$, the conditions for stability are

$$\frac{\omega_1^2k_2}{k_2-k_1}\rho_1+\frac{\omega_2^2k_1}{k_1-k_2}\rho_2<0 \qquad \text{and} \qquad k_1k_2\rho_1\rho_2<0$$

Since $k_1<0$ and $k_2>0$, the above inequalities lead to

$$\rho_1<0 \qquad \text{and} \qquad \rho_2<0 \tag{3.8}$$

(b) For $R_{10}=\rho_1$, $R_{20}=0$, we obtain

$$\rho_1>\frac{1}{2}\rho_2 \qquad \text{and} \qquad \rho_1>0 \tag{3.9}$$

(c) For $R_{10}=0$, $R_{20}=\rho_2$, we obtain

$$\rho_2>\frac{1}{2}\rho_1 \qquad \text{and} \qquad \rho_2>0 \tag{3.10}$$

(d) For $R_{10}=\frac{1}{3}(2\rho_2-\rho_1)$, $R_{20}=\frac{1}{3}(2\rho_1-\rho_2)$, we obtain

$$R_{10}R_{20}<0 \tag{3.11}$$

There exists no oscillation in case (a). Since both R_{10} and R_{20} are positive, condition (3.11) is never fulfilled. Hence, in the steady state, the oscillation is periodic with frequency ω_1 [case (b)] or frequency ω_2 [case (c)], and no combination oscillation of the two frequencies is realized.

Numerical examples

The coupling k between the two resonant circuits of Fig. 1 is given by [see Eqs. (2.6)]

$$k = \sqrt{N^2/L_1L_2} = \sqrt{\chi_1\chi_2} \tag{3.12}$$

By solving Eqs. (2.8) for a given value of k, we obtain ω/n_1 as a function of n_2/n_1. An example of such a frequency characteristic is shown in Fig. 2 where $k=0.5$. If we furthermore assume δ or $(n_2/n_1)^2\delta$ in Eqs. (2.18), ρ_1 and ρ_2 are calculated.[2] Then, by using Eqs. (3.2), the amplitude characteristic of the self-excited oscillation is obtained. Calculated results are shown in Figs. 3 and 4. The stability of the oscillation is tested by using conditions (3.8) to (3.11), and the unstable portions of the characteristic curves are shown by broken lines. In Fig. 3, stable solutions with amplitude r_{10} and frequency ω_1 are obtained for n_2/n_1 greater than the value of c. Stable solutions with r_{20} and ω_2 are also obtained for n_2/n_1 less than d. For n_2/n_1 lying

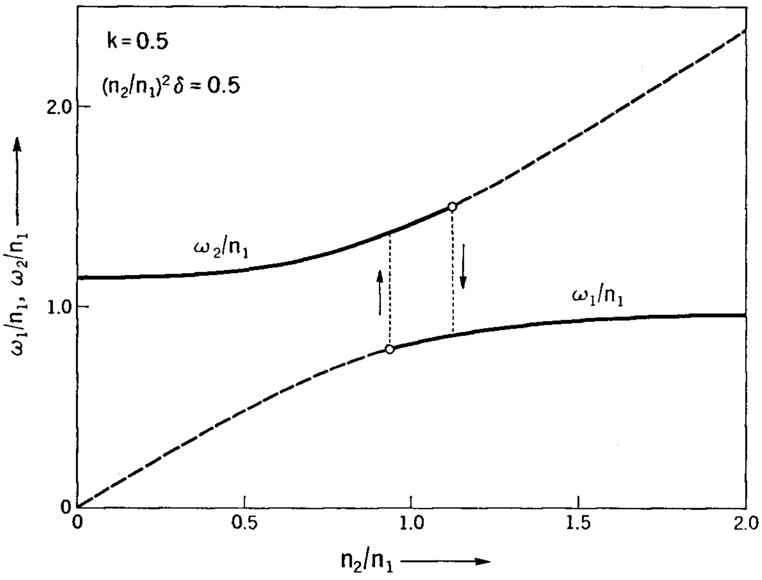

Fig. 2. Frequency characteristic of the self-excited oscillation.

[2] The coupling k and the parameters

$$\frac{n_2}{n_1} = \sqrt{\frac{L_1C_1}{L_2C_2}}, \quad \left(\frac{n_2}{n_1}\right)^2\delta = \frac{L_1}{L_2}\frac{C_1R_2}{MS_1-C_1R_1}$$

are varied independently by changing the values of N, C_2, and R_2, respectively.

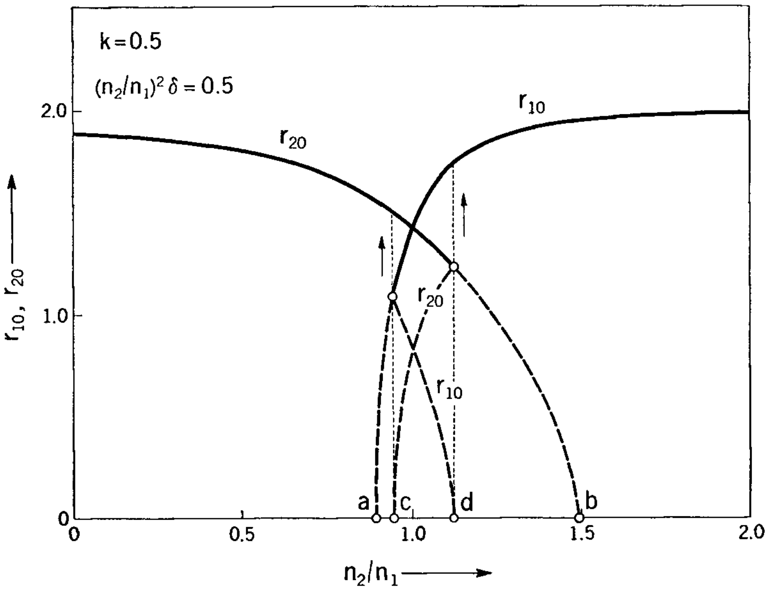

Fig. 3. Amplitude characteristic of the self-excited oscillation.

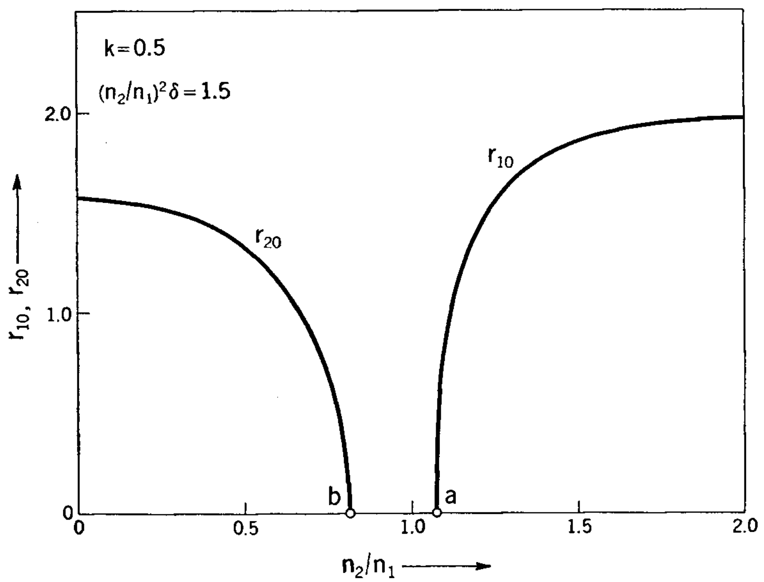

Fig. 4. Amplitude characteristic of the self-excited oscillation—continued.

between c and d, there are almost periodic solutions with two frequency components ω_1 and ω_2. They are, however, unstable. In Fig. 4, two characteristic curves are separated. No oscillation exists for n_2/n_1 between a and b.

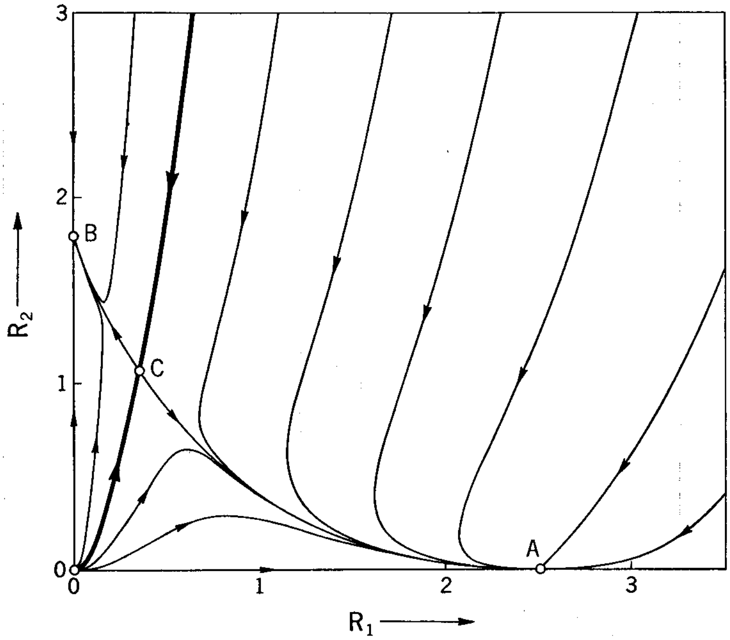

Fig. 5. Phase-plane portrait of Eq. (3.13), the system parameters being $k=0.5$, $(n_2/n_1)^2\delta=0.5$, and $n_2/n_1=1.05$.

As one sees in Fig. 3, two kinds of periodic oscillations are stably sustained for the values of n_2/n_1 between c and d. It depends on the initial condition as regards which kind of oscillations occurs. This situation is made clear from the phase-plane analysis of the equation

$$\frac{dR_2}{dR_1}=-\frac{\omega_2^2k_1}{\omega_1^2k_2}\frac{(\rho_2-2R_1-R_2)R_2}{(\rho_1-R_1-2R_2)R_1} \tag{3.13}$$

which is readily derived from Eqs. (2.19). Figure 5 shows an example of the phase-plane portrait where $n_2/n_1=1.05$ and, as before, $k=0.5$ and $(n_2/n_1)^2\delta=0.5$. The stable oscillations are represented by singular points A and B, whose amplitudes are given by $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$, respectively. Singular point C represents an almost periodic oscillation which has two frequency components ω_1 and ω_2. It is, however, unstable. The origin is also an unstable singularity. A representative point on the integral curves moves in the direction of the arrows with increasing time. The integral curves (heavy line) that approach C divide the R_1R_2 plane into two domains of attraction, in one of which all integral curves tend to the singularity A, and in the other to the singularity B. Hence, once the initial conditions of the system (2.5) are given, the initial values of R_1 and R_2 are calculated, and therefore, the resulting response is found.

4. Internal resonance

4.1 Internal resonance which occurs when $3\omega_1 \simeq \omega_2$

As mentioned in Section 2, the entrainment of frequency occurs when $3\omega_1$ is in the neighborhood of ω_2. To investigate this phenomenon, we make use of the expansion

$$\begin{aligned} \omega_1 &= \omega_{10} + \mu\omega_{11} + \cdots \\ \omega_2 &= \omega_{20} + \mu\omega_{21} + \cdots \end{aligned} \tag{4.1}$$

The entrained oscillation consists of two components of frequencies ω_{10} and ω_{20}, which are in the neighborhood of ω_1 and ω_2, respectively, and are related by $3\omega_{10} = \omega_{20}$.

Substituting Eqs. (4.1) into Eqs. (2.11) gives

$$\begin{aligned} \ddot{x} + \omega_{10}^2 x &= \mu[f(x, y, \dot{x}, \dot{y}) - 2\omega_{10}\omega_{11}x] + 0(\mu^2) \\ \ddot{y} + \omega_{20}^2 y &= \mu[g(x, y, \dot{x}, \dot{y}) - 2\omega_{20}\omega_{21}y] + 0(\mu^2) \end{aligned} \tag{4.2}$$

We assume that the solution of Eqs. (4.2) takes the form

$$\begin{aligned} x(t) &= r_1(t) \cos [\omega_{10}t + \theta_1(t)] \\ y(t) &= r_2(t) \cos [\omega_{20}t + \theta_2(t)] \\ \dot{x}(t) &= -\omega_{10}r_1(t) \sin [\omega_{10}t + \theta_1(t)] \\ \dot{y}(t) &= -\omega_{20}r_2(t) \sin [\omega_{20}t + \theta_2(t)] \end{aligned} \tag{4.3}$$

Hence the solution of Eqs. (2.5) is written as

$$\begin{aligned} u(t) &= r_1(t) \cos [\omega_{10}t + \theta_1(t)] + r_2(t) \cos [\omega_{20}t + \theta_2(t)] \\ v(t) &= k_1 r_1(t) \cos [\omega_{10}t + \theta_1(t)] + k_2 r_2(t) \cos [\omega_{20}t + \theta_2(t)] \end{aligned} \tag{4.4}$$

Substituting Eqs. (4.3) into Eqs. (4.2) and using the averaging method[3] as we have done in Section 2, we obtain the autonomous system

$$\begin{aligned} \dot{r}_1 &= \frac{\mu\omega_1^2}{8n_1} \frac{k_2}{k_2 - k_1} [(\rho_1 - r_1^2 - 2r_2^2)r_1 - r_1^2 r_2 \cos (3\theta_1 - \theta_2)] \\ \dot{r}_2 &= \frac{\mu\omega_2^2}{8n_1} \frac{k_1}{k_1 - k_2} [(\rho_2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3} r_1^3 \cos (3\theta_1 - \theta_2)] \\ r_1\dot{\theta}_1 &= \mu[\omega_{11}r_1 + \frac{\omega_1^2}{8n_1} \frac{k_2}{k_2 - k_1} r_1^2 r_2 \sin (3\theta_1 - \theta_2)] \\ r_2\dot{\theta}_2 &= \mu[\omega_{21}r_2 - \frac{\omega_2^2}{24n_1} \frac{k_1}{k_1 - k_2} r_1^3 \sin (3\theta_1 - \theta_2)] \end{aligned} \tag{4.5}$$

[3] We made use of the relationship $3\omega_{10} = \omega_{20}$ in the derivation of Eqs. (4.5).

The steady-state solutions of Eqs. (4.5) are obtained by equating $\dot{r}_1=\dot{r}_2=\dot{\theta}_1=\dot{\theta}_2=0$. Denoting the steady-state values of these variables by r_{10}, r_{20}, θ_{10}, and θ_{20}, respectively, we obtain

$$
\begin{aligned}
&(\rho_1-r_{10}^2-2r_{20}^2)r_{10}-r_{10}^2r_{20}\cos(3\theta_{10}-\theta_{20})=0\\
&(\rho_2-2r_{10}^2-r_{20}^2)r_{20}-\frac{1}{3}r_{10}^3\cos(3\theta_{10}-\theta_{20})=0\\
&\omega_{11}r_{10}+\frac{\omega_1^2}{8n_1}\frac{k_2}{k_2-k_1}r_{10}^2r_{20}\sin(3\theta_{10}-\theta_{20})=0\\
&\omega_{21}r_{20}-\frac{\omega_2^2}{24n_1}\frac{k_1}{k_1-k_2}r_{10}^3\sin(3\theta_{10}-\theta_{20})=0
\end{aligned}
\tag{4.6}
$$

We see, from Eqs. (4.6), that there are three different states of equilibrium, i.e.,

$$
\begin{aligned}
&(a)\quad r_{10}=0, \qquad r_{20}=0\\
&(b)\quad r_{10}=0, \qquad r_{20}=\sqrt{\rho_2}, \qquad \omega_{21}=0\\
&(c)\quad r_{10}\neq 0, \qquad r_{20}\neq 0
\end{aligned}
\tag{4.7}
$$

The first and second cases are identical with those already discussed in (a) and (c) of Eqs. (3.2), respectively. We are particularly interested in the third case (c) where two frequency components are not zero and are entrained mutually by the relation $3\omega_{10}=\omega_{20}$. As will be mentioned later, this state of equilibrium is stable. Thus, the internal resonance occurs. We see from Eqs. (4.1) that

$$
\begin{aligned}
3\omega_1-\omega_2&=3\omega_{10}+3\mu\omega_{11}-\omega_{20}-\mu\omega_{21}+0(\mu^2)\\
&=\mu(3\omega_{11}-\omega_{21})+0(\mu^2)
\end{aligned}
\tag{4.8}
$$

Solving Eqs. (4.6) and Eq. (4.8) simultaneously gives the amplitudes r_{10}, r_{20}, the phase difference $3\theta_{10}-\theta_{20}$, and the frequencies ω_{11}, ω_{21}. Hence, from Eqs. (4.1), the entrained frequencies ω_{10} and ω_{20} are determined.

The stability of the equilibrium states as given by Eqs. (4.7) is studied by solving the variational equations derived from Eqs. (4.5). Since the first and the second cases of Eqs. (4.7) are identical with (a) and (c) of Eqs. (3.2), the conditions for stability are also given by (3.8) and (3.10), respectively. The stability of the entrained oscillation in which both r_{10} and r_{20} are not zero is tested by making use of the Routh-Hurwitz criterion, and this type of oscillation is found to be stable.

Numerical example

Let us consider a case in which

$$\mu=0.1, \qquad k=0.5, \qquad (n_2/n_1)^2\delta=0.5$$

Substituting $k=\sqrt{\chi_1\chi_2}=0.5$ into Eq. (2.8), we find that $3\omega_1=\omega_2$ provided

$$n_2/n_1 \simeq 0.403 \qquad \text{and} \qquad n_2/n_1 \simeq 2.48$$

The amplitudes r_{10}, r_{20} and the entrained frequency ω_{10} are calculated by using Eqs. (4.6) and Eq. (4.8). Real values of the amplitudes are obtained only for $n_2/n_1 \simeq 2.48$. Figure 6 shows the entrained frequency ω_{10}/n_1 as n_2/n_1 varies. For comparison's sake, ω_1/n_1 curve calculated from Eq. (2.8) is also shown in the figure. Figures 7 and 8 show the amplitude characteristics of the entrained oscillation.[4] We see that, when the entrainment occurs, the amplitude $k_2 r_{20}$ of the third harmonic component of the solution $v(t)$ increases predominantly. The phase characteristic of the entrained oscillation is shown in Fig. 9.

4.2 Internal resonance which occurs when $\omega_1 \simeq \omega_2$

The entrainment of frequency also occurs when $\omega_1 \simeq \omega_2$. As mentioned in Section 2, this type of internal resonance occurs when $n_1 \simeq n_2$ and $N \simeq 0$. Let the entrained frequency be ω_0 which is in the neighborhood of ω_1 and ω_2. Then the solution of Eqs. (2.5) is written as

$$\begin{aligned} u(t) &= r_u(t) \cos [\omega_0 t + \theta_u(t)] \\ v(t) &= r_v(t) \cos [\omega_0 t + \theta_v(t)] \\ \dot{u}(t) &= -\omega_0 r_u(t) \sin [\omega_0 t + \theta_u(t)] \\ \dot{v}(t) &= -\omega_0 r_v(t) \sin [\omega_0 t + \theta_v(t)] \end{aligned} \tag{4.9}$$

Substituting Eqs. (4.8) into Eqs. (2.5) and applying the averaging method as before, we obtain the autonomous system

$$\begin{aligned} \dot{r}_u &= \frac{1}{2\omega_0(1-\chi_1\chi_2)} \Big[\mu\omega_0 n_1\Big(1-\frac{1}{4}r_u^2\Big) r_u + \chi_1 n_2^2 r_v \sin(\theta_u-\theta_v) \\ &\quad -\mu\omega_0\chi_1 \frac{n_2^2}{n_1}\delta r_v \cos(\theta_u-\theta_v)\Big] \\ \dot{r}_v &= \frac{1}{2\omega_0(1-\chi_1\chi_2)} \Big[-\mu\omega_0 \frac{n_2^2}{n_1}\delta r_v - \chi_2 n_1^2 r_u \sin(\theta_u-\theta_v) \\ &\quad +\mu\omega_0\chi_2 n_1\Big(1-\frac{1}{4}r_u^2\Big) r_u \cos(\theta_u-\theta_v)\Big] \\ r_u\dot{\theta}_u &= \frac{1}{2\omega_0(1-\chi_1\chi_2)} \Big\{[n_1^2-(1-\chi_1\chi_2)\omega_0^2] r_u \\ &\quad +\mu\omega_0\chi_1\frac{n_2^2}{n_1}\delta r_v \sin(\theta_u-\theta_v) + \chi_1 n_2^2 r_v \cos(\theta_u-\theta_v)\Big\} \end{aligned} \tag{4.10}$$

[4] One sees from Eqs. (2.6) and Eq. (3.12) that

$$\chi_1 = \sqrt{L_1/L_2}\,(n_1/n_2)^2 k, \qquad \chi_2 = \sqrt{L_2/L_1}\,(n_2/n_1)^2 k$$

To fix the values of χ_1 and χ_2 it is assumed, in this numerical example, that $L_1 = L_2$.

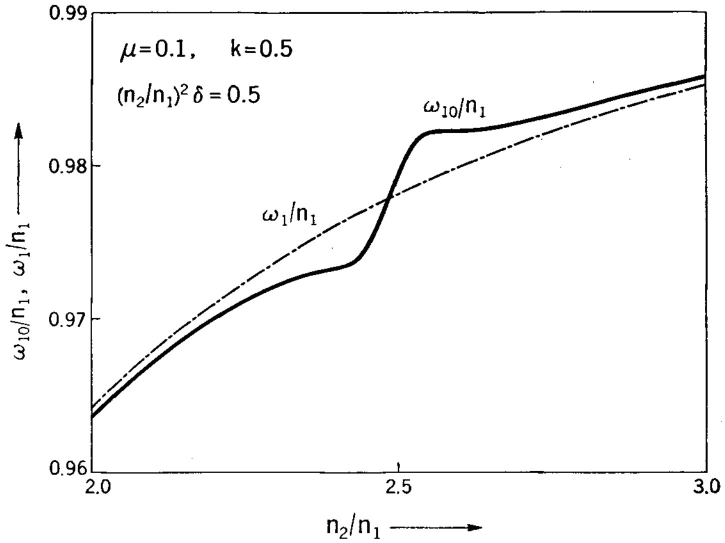

Fig. 6. Frequency characteristic of the entrained oscillation ($3\omega_1 \simeq \omega_2$).

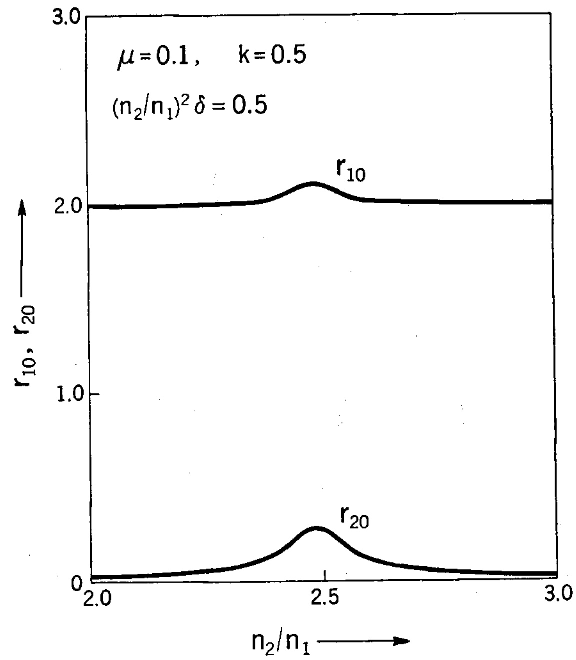

Fig. 7. Amplitude characteristic of the entrained oscillation ($3\omega_1 \simeq \omega_2$).

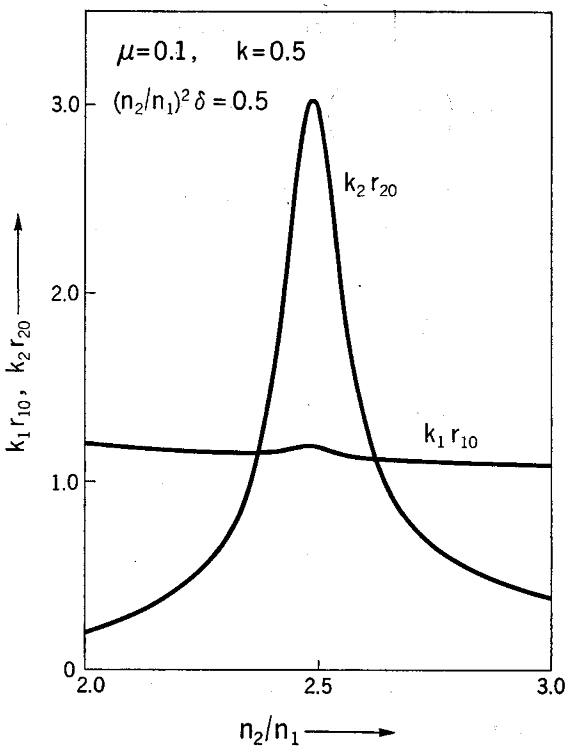

Fig. 8. Amplitude characteristic of the entrained oscillation ($3\omega_1 \simeq \omega_2$) — continued.

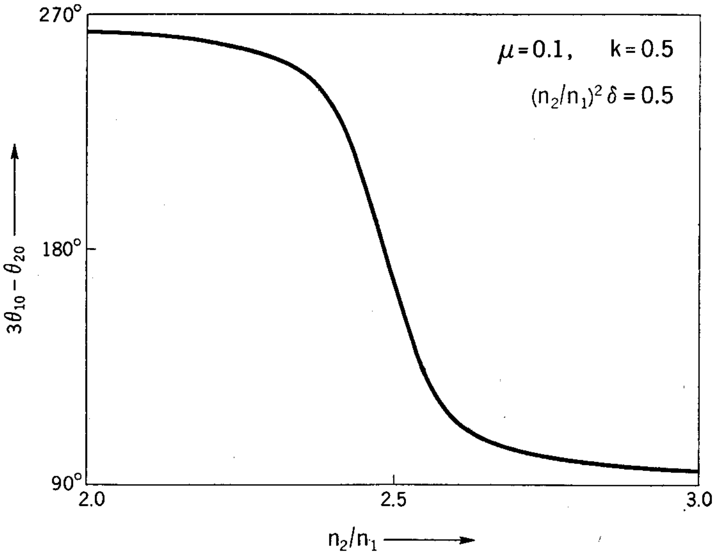

Fig. 9. Phase characteristic of the entrained oscillation ($3\omega_1 \simeq \omega_2$).

$$r_v\dot{\theta}_v = \frac{1}{2\omega_0(1-\chi_1\chi_2)}\left\{[n_2^2-(1-\chi_1\chi_2)\omega_0^2]r_v + \mu\omega_0\chi_2 n_1\left(1-\frac{1}{4}r_u^2\right)r_u \sin(\theta_u-\theta_v)+\chi_2 n_1^2 r_u \cos(\theta_u-\theta_v)\right\}$$

The steady-state solutions of Eqs. (4.10) are obtained by equating $\dot{r}_u=\dot{r}_v=\dot{\theta}_u=\dot{\theta}_v=0$. After some algebraic manipulation, we obtain

$$\begin{aligned} r_{u0}^2 &= 4\left(1-\frac{\omega_0^2-n_1^2}{\omega_0^2-n_2^2}\frac{n_2^2}{n_1^2}\delta\right) \\ r_{v0}^2 &= \frac{\chi_2}{\chi_1}\frac{\omega_0^2-n_1^2}{\omega_0^2-n_2^2}r_{u0}^2 \\ \sin(\theta_{u0}-\theta_{v0}) &= -\frac{\mu}{\chi_1\omega_0}\frac{\omega_0^2-n_1^2}{\omega_0^2-n_2^2}\frac{n_2^2}{n_1}\delta\frac{r_{u0}}{r_{v0}} \\ \cos(\theta_{u0}-\theta_{v0}) &= \frac{\omega_0^2-n_1^2}{\chi_1\omega_0^2}\frac{r_{u0}}{r_{v0}} \end{aligned} \qquad (4.11)$$

and

$$[(1-\chi_1\chi_2)\omega_0^4-(n_1^2+n_2^2)\omega_0^2+n_1^2n_2^2](\omega_0^2-n_2^2) + \left(\mu\frac{n_2^2}{n_1}\delta\right)^2(\omega_0^2-n_1^2)\omega_0^2 = 0 \qquad (4.12)$$

Equations (4.11) give the amplitudes r_{u0} and r_{v0} and the phase difference $\theta_{u0}-\theta_{v0}$ of the entrained oscillation. Equation (4.12) determines the entrained frequency ω_0.

The stability of the steady-state solutions is tested as before.

Numerical example

We consider a system in which

$$\mu = 0.1, \qquad k = 0.04, \qquad (n_2/n_1)^2\delta = 0.5$$

Since k is small, the mutual inductance N is also small [see Eq. (3.12)]. Therefore, when $n_1 \simeq n_2$, the difference between ω_1 and ω_2 becomes so small that we may expect the internal resonance. Substituting those values of the parameters into Eq. (4.12) gives the relation between n_2/n_1 and ω_0/n_1. The result is shown in Fig. 10. The entrained frequency ω_0/n_1 varies continuously as indicated in the figure (cf. Fig. 2). The amplitude characteristic is calculated by using Eqs. (4.11) and shown in Fig. 11. To fix the values of χ_1 and χ_2, it is chosen that $L_1=L_2$ (see Footnote 4). The amplitude r_{u0} dips when $n_1 \simeq n_2$, and simultaneously an increase in the amplitude r_{v0} results. The phase characteristic of the entrained oscillation is shown in Fig. 12.

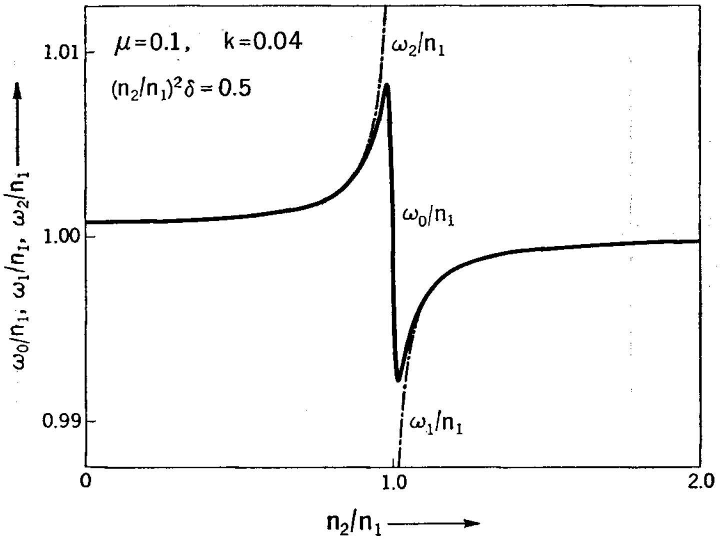

Fig. 10. Frequency characteristic of the entrained oscillation ($\omega_1 \simeq \omega_2$).

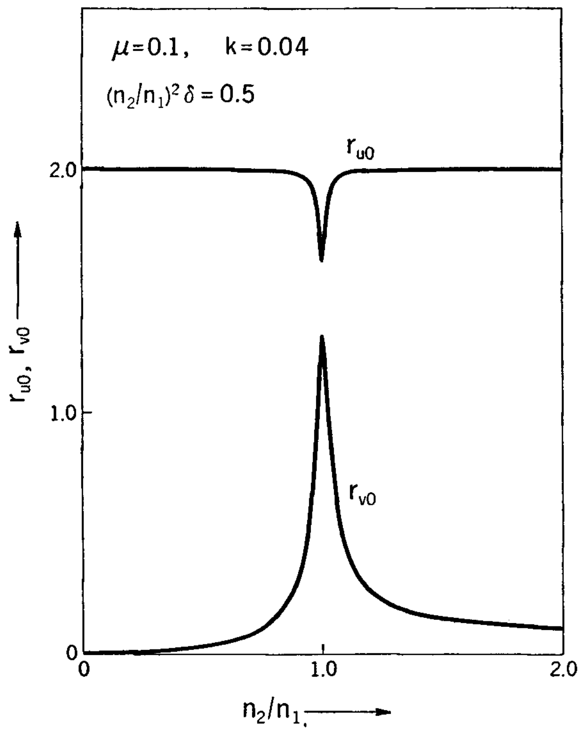

Fig. 11. Amplitude characteristic of the entrained oscillation ($\omega_1 \simeq \omega_2$).

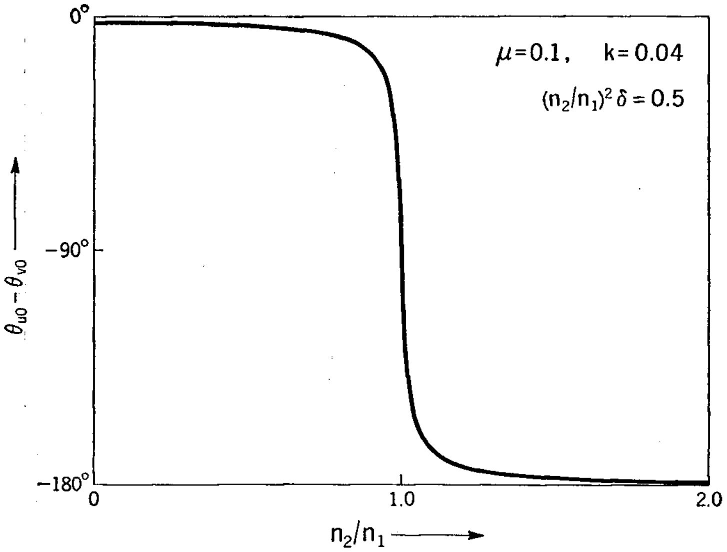

Fig. 12. Phase characteristic of the entrained oscillation ($\omega_1 \simeq \omega_2$).

5. Conclusion

The behavior of a negative-resistance oscillator having two resonant circuits is discussed. It is shown that the self-excited oscillation of the system is periodic. No combination oscillation having two different frequencies is sustained stably.

When a certain relationship exists between the natural frequencies of the two resonant circuits, one may expect the internal resonance between them. Two representative types of internal resonance are discussed. The first case deals with the entrainment of frequency resulting in the build-up of the third harmonic component of the self-excited oscillation. The second case treats the synchronization of the two resonant circuits.

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