Locking Equations for Microwave Circuits

By

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Abstract

By extending an ordinary admittance expression, the simplified equation for an oscillator has been obtained, which as yet is in the most general form within the first order approximation. Then, this equation has been transformed into the amplitude and the phase equations suitable for analyzing a microwave circuit involving oscillators. It has been found that the amplitude equation can be ignored when the oscillator is adjusted to produce the maximum output power. An example of application has also been given.

Introduction

Recently many studies have been presented relevant to the locking phenomena of solid-state microwave oscillators. Although these oscillators are usually operated in distributed constant circuits, the analyses have been performed exclusively by the use of the so-called Adler equation[1], after the transformation of the associated microwave distributed circuits into the lumped equivalent. Since microwave circuits can not always be simply expressed in the lumped equivalent form, it is desirable to develop equations suitable to microwave circuitry. This will not only simplify the analytical treatment but also give a clear physical picture.

Basic Equations

We will first derive the lumped-constant locking equations by extending an ordinary admittance expression:

$$Ie^{j\omega t} = Y(j\omega)Ve^{j\omega t}\,. \qquad (1)$$

If the voltage V is varied quasi-stationarily as

$$V(t) = V_0 e^{\sigma(t)+j\phi(t)}\,, \qquad (2)$$

the current I also varies with it and the admittance Y should be replaced by

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$$Y = Y\{j\omega + d(\sigma + j\phi)/dt\}, \tag{3}$$

since the instantaneous frequency is altered by the time derivative of the phase $\sigma + j\phi$. Here it should be noted that the frequency is generalized to be complex. We expand the above equation in Taylor's series about the free-running angular frequency ω_0 and retain only the first order term:

$$Y = Y(j\omega_0) + Y'\{j\Delta\omega + d(\sigma + j\phi)/dt\}, \tag{3$'$}$$

where $\Delta\omega = \omega - \omega_0$ and Y' is the derivative of Y. The Y' can be set equal to $\partial Y/\partial(j\omega)\,|_{\omega=\omega_0} \equiv Y_{j\omega}$, since the Cauchy-Riemann condition

$$Y' = \frac{\partial Y}{\partial(\dot{\sigma} + j\dot{\phi})} = \frac{\partial Y}{\partial\dot{\sigma}} = \frac{\partial Y}{\partial(j\dot{\phi})} = \frac{\partial Y}{\partial(j\omega)}$$

holds for the regular function of frequency Y. The dots over σ and ϕ denote the time derivatives. The free-running oscillation condition will be defined in Appendix I. Since the time derivative of Eq. (2) is

$$dV/dt = Vd(\sigma + j\phi)/dt, \tag{4}$$

we can write Eq. (1) as

$$I(t) = \{Y(j\omega_0) + Y_{j\omega}j\Delta\omega + Y_{j\omega}\,d/dt + Y_V\Delta|V|\}V(t), \tag{5}$$

where the term associated with $Y_V (\equiv \partial Y/\partial|V|\,|_{V=V_0})$ has been introduced in order to represent the first order nonlinearity of admittance*. The value $\Delta|V| = |V| - |V_0|$ is the increment of voltage amplitude from the free-running state. Within the first order approximation, Eq. (5) is of the most general form describing the behavior of the perturbation of an oscillator.

We assume in Fig. 1 a van der Pol type oscillator** as

$$Y = -G(|V|) + j\omega C + 1/j\omega L, \tag{6}$$

so that we have

$$Y_{j\omega} = 2C = 2Y_0 Q_{ex}/\omega_0, \quad Y_V = -G'(|V_0|) > 0, \tag{7}$$

where $\omega_0 = 1/\sqrt{LC}$, Y_0 is the characteristic admittance of the transmission line, and $Q_{ex} = \omega_0 C/Y_0$ is the external Q of the oscillator. Inserting Eqs. (4), (7) and (A2) into Eq. (5) and separating it into real and imaginary parts, we find

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* The time derivative of σ is not a function of the amplitude of voltage, as seen from Eq. (4); $\dot{\sigma} = (1/V)dV/dt - j\dot{\phi}$. Therefore the admittance of Eq (3) is linear.

** The van der Pol equation is referred to as the nonlinear differential equation of the second order having a damping term proportional to $|V|^2$. Consequently, the admittance of Eq. (6) will lead to a more generalized van der Pol equation.

$$2C\frac{d\sigma}{dt} = Re\left(\frac{I}{V}\right) + G(|V|)\,, \tag{8a}$$

$$2C\frac{d\phi}{dt} = Im\left(\frac{I}{V}\right) - 2C\varDelta\omega\,, \tag{8b}$$

where $G(|V_0|)+G_V'\varDelta|V|$ has been replaced by $G(|V|)$. These are the simplified equations determining the amplitude and phase of an oscillator. If we set $I=|I|e^{j\theta}$, the phase equation (8b) is reduced to

$$\frac{d\phi}{dt} = -\frac{\omega_0}{2Q_{ex}}\left|\frac{I}{Y_0 V}\right| \sin(\phi-\theta) - \omega + \omega_0\,, \tag{9}$$

which is the Adler equation[1] expressed in our symbols.

If we consider the voltage dependence also for the susceptance terms in Eq. (6), we may obtain more general equations.

Microwave Locking Equations

We define

$$a(t) = |a(t)|e^{j\alpha(t)}\,,\quad b(t) = |b(t)|e^{j\beta(t)} \tag{10}$$

to be the emergent wave from the oscillator and the incident wave into it, respectively. We have then the relations

$$V = (b+a)/\sqrt{Y_0}\,,\quad I = (b-a)\sqrt{Y_0}\,. \tag{11}$$

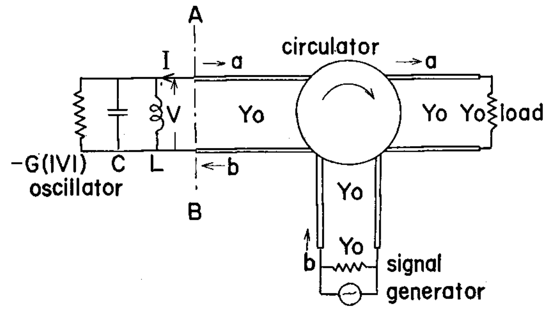

Fig. 1. Locking oscillator model in microwave circuitry.

Substituting them into Eq. (5) yields

$$\begin{aligned}&\{Y_0+Y(j\omega_0)+Y_{j\omega}j\varDelta\omega+Y_{j\omega}\,d/dt+Y_V\varDelta|V|\}a(t)\\ &=\{Y_0-Y(j\omega_0)-Y_{j\omega}j\varDelta\omega-Y_{j\omega}\,d/dt-Y_V\varDelta|V|\}b(t)\,.\end{aligned} \tag{12}$$

Since the increment of the voltage $\varDelta|V|$ due to the small injection signal b is approximated to, as shown in Appendix III,

$$\Delta|V| \approx \{\Delta|a| + |b|\cos(\beta-\alpha)\}/\sqrt{Y_0}\,. \tag{13}$$

Eq. (12) becomes, ignoring the second higher order terms,

$$[Y_V\{\Delta|a| + |b|\cos(\alpha-\beta)\}/\sqrt{Y_0} + Y_{j\omega} j\Delta\omega + Y_{j\omega}\, d/dt]a = 2Y_0 b\,,$$

where the free-running condition (A1) has been used in order to simplify the equation. Inserting Eq. (7) into the above equation and splitting it into real and imaginary parts, we obtain

$$\frac{1}{|a_0|}\cdot\frac{d|a|}{dt} = \frac{\omega_0}{Q_{ex}}\left\{1+\frac{G'(|V_0|)}{2Y_0\sqrt{Y_0}}|a_0|\right\}Re\left(\frac{b}{a}\right)+\frac{\omega_0}{Q_{ex}}\cdot\frac{G'(|V|)}{2Y_0\sqrt{Y_0}}(|a|-|a_0|)\,, \tag{14}$$

$$\frac{d\alpha}{dt} = \frac{\omega_0}{Q_{ex}}\cdot Im\left(\frac{b}{a}\right)-\omega+\omega_0\,. \tag{15}$$

These are the simplified equations in terms of traveling waves. When the oscillator is adjusted to generate the maximum power, the first term on the right-hand side of Eq. (14) vanishes, as verified in Appendix II. Under this condition, the amplitude $|a|$ of the output wave becomes independent of the incident wave b. Such a situation never occurs in the lumped circuit equation Eq. (8a), because the voltage continues to increase as the incident power increases. (See Appendix IV.) Consequently, for a microwave oscillator, as long as the maximum output power condition is satisfied, it is sufficient for us to consider only the phase equation (15), which is also written as

$$\begin{aligned}\frac{d\alpha}{dt} &= \frac{\omega_0}{j2Q_{ex}}\left(\frac{b}{a}-\frac{b^*}{a^*}\right)-\omega+\omega_0 \\ &= -\frac{\omega_0}{Q_{ex}}\left|\frac{b}{a}\right|\sin(\alpha-\beta)-\omega+\omega_0\,.\end{aligned} \tag{16}$$

This equation bears close resemblance in form to the Adler equation (9), but the meaning is different.

Application

As an example, if we use the above-derived equation, the relation between the pulling figure and the external Q of an oscillator will easily be obtained. In Fig. 2, the pulling figure is commonly defined to be the total frequency shift of the oscillator as a 1.5–VSWR load is moved through more than a half wavelength. Expressing by Γ the reflection coefficient of the load at a proper

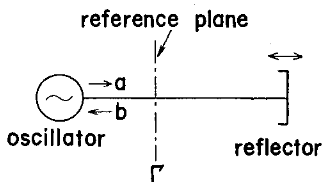

Fig. 2. Relation between the pulling figure and the external Q.

reference plane, we get

$$b = \Gamma a = |\Gamma| e^{j\psi} a .$$

We assume that the oscillator is operated in the maximum output power state, and insert the above into Eq. (16), obtaining

$$\frac{d\alpha}{dt} = \frac{\omega_0}{Q_{ex}} |\Gamma| \sin \psi - \omega + \omega_0 .$$

When the phase ψ is varied quasi-statically ($d\alpha/dt \approx 0$) with $|\Gamma|$ held constant, the maximum angular frequency shift is

$$\omega_{max} - \omega_{min} = 2\omega_0 |\Gamma| / Q_{ex} . \qquad (17)$$

The pulling figure, therefore, becomes $f_{max} - f_{min} = 0.4 f_0 / Q_{ex}$, since $|\Gamma| = 0.2$ for a 1.5–VSWR load.

Conclusion

In the beginning, the simplified equations which are general in the first order approximation have been developed, and the relationship of the so-called Adler equation to them has been clarified. By the aid of these, the locking equations appropriate to microwave circuitry have been derived. When the oscillator circuit is roughly adjusted to produce the highest power, we can disregard the amplitude equation and yet can obtain a good result by using only the phase equation. The utilization of this equation will make the analysis involving microwave oscillators more simple and comprehensive. Also it will make it possible to analyze the circuit which can not easily be expressed in a lumped equivalent form.[2)]

Acknowledgment

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References

1) R. Adler; Proc. IRE, **34**, 351 (1946).
2) M. Nakajima; Proc. IEEE, **61**, 242 (1973).
3) J. Ikenoue and M Nakajima; Jour. IECE (Japan) **54**, 693 (1971).

Appendix

I. Free-Running Oscillation Condition

When the oscillator is coupled to a load Y_0 with no signal injected, the following free-running equation holds:

$$Y+Y_0=0\,. \tag{A1}$$

Substituting Eq. (6) into Eq. (A1), we get

$$-G(|V_0|)+Y_0=0\,,\quad j\omega_0C+1/j\omega_0L=0\,, \tag{A2}$$

from which the free-running amplitude $|V_0|$ and frequency $\omega_0/2\pi$ are determined.

II. Maximum Output Power Condition

In Fig. 1 the free-running, steady-state power dissipated at the load $P_{out}=|a_0|^2=Y_0|V_0|^2$ is maximized when

$$G'(|V_0|)=-2Y_0/|V_0|=-2Y_0\sqrt{Y_0}/|a_0|\,, \tag{A3}$$

which is easily found by the differentiation of P_{out} with respect to Y_0 and by use of Eq. (A2). The variation of the load admittance Y_0 can be realized equivalently by adjusting the coupling of the oscillator to the transmission line.

Under this condition, the first term on the right-hand side of Eq. (14) vanishes, indicating that the amplitude of the output wave $|a|$ becomes independent of the incident wave b. Noting that $|a|-|a_0|\equiv\Delta|a|$, we can write the solution of Eq. (14) as

$$\Delta|a(t)|=\Delta|a|_0e^{-(\omega_0/Q_{ex})t}\,.$$

In other words, the increment (or decrement) of the amplitude from the steady-state value always decreases with time. Under the maximum power condition, therefore, we can conclude that the amplitude of the output wave $|a|$, after a certain period of time, is equal to the constant value $|a_0|$.

III. Voltage Increment due to Injection Small Signal b

Assuming that $|b|$ and $|\Delta a|$ are small compared with $|a_0|$, we have

$$\begin{aligned}
&2|a_0|\cdot\Delta|a+b|\\
&\quad\approx\{|a_0+\Delta a+b|+|a_0|\}\cdot\{|a_0+\Delta a+b|-|a_0|\}\\
&\quad=|a_0+\Delta a+b|^2-|a_0|^2\\
&\quad=|a_0+\Delta a|^2+2|a_0+\Delta a|\cdot|b|\cos(\alpha-\beta)+|b|^2-|a_0|^2\\
&\quad\approx\{|a_0+\Delta a_0|+|a_0|\}\{|a_0+\Delta a|-|a_0|\}+2|a_0|\cdot|b|\cos(\alpha-\beta)\\
&\quad\approx 2|a_0|\cdot\Delta|a|+2|a_0|\cdot|b|\cos(\alpha-\beta)\,.
\end{aligned}$$

Since $\Delta|V|=\Delta|a+b|/\sqrt{Y_0}$ by Eq. (11), we obtain Eq. (13), dividing the above equation by $2|a_0|\sqrt{Y_0}$.

IV. On the Lumped-Constant Circuit Case

We will show that, in the lumped-constant circuit case, the output power is not saturated with increasing input power. Consider the circuit of Fig. A1, in which a negative conductance oscillator $(-G, C, L)$ is driven by a signal generator (G_g, I_g)

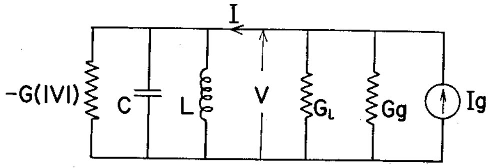

Fig. A1. Output power in the lumped circuit case.

and the output power is dissipated at a load G_l. If we denote the voltage across the oscillator by V, the output power is written,

$$P_{\text{out}} = |V|^2 G_l . \tag{A4}$$

Now, when the frequency of the generator is equal to the free-running frequency of the oscillator, we have the circuit equation,

$$I_g = \{-G(|V|)+G_l+G_g\}V . \tag{A5}$$

We can here assume the voltage V to be positive real without loss of generality. Noting that I_g becomes necessarily real and that $-G(|V|)$ is an increasing function of $|V|$, i.e. $-G'(|V|)>0$, we can express the relation (A5) graphically as in Fig. A2. The free-running condition has been defined as

$$-G(|V_0|)+G_l+G_g = 0$$

with $I_g = 0$. It should be noted that the total conductance $\{-G(|V|)+G_l+G_g\}$ is negative in the range $0<V<|V_0|$, so that the generator can not work as such, since $-I_g/V>0$. It can be shown from the stability criterion[3] that the state corresponding to $0<V<|V_0|$ is unstable in the circuit of Fig. A1. In other words, the range $|V|<|V_0|$ corresponds to an ordinary operation of an oscillator with only a passive load conductance $G_l+(-I_g/V)$. It is therefore concluded that, when the current of signal I_g is increased from zero, the voltage V and hence the output power of Eq. (A4) increases from the free-running state without limit.

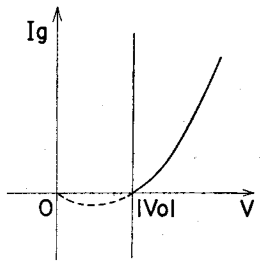

Fig. A2. Relation of the oscillstor voltage amplitude V to the current of the signal generator I_g.