A Study on the Bandwidth Reduction of the Stiffness Matrix of a Framed Structure

By

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Summary

How to give the numerical number for joints of a framed structure in order to reduce the bandwidth of its stiffness matrix, has been studied and a number of methods have been proposed already. Though the minimum bandwidth is wholly governed by the connectivity relationship between joints, only a few studies have taken that fact into the consideration.

The connectivity relationship of a framed structure can be sufficiently shown by the use of a linear graph, which is composed of nodes and lines. Its characteristics can be understood by introducing the concept of graph theory.

In this paper, the authors propose a graphical method to obtain the minimum bandwidth of tree structures. This is a kind of statically determinate system including no closed path, by the application of some definitions in graph theory. By use of the this sequential file method, all the nodal sequences are efficiently filed along the longest nodal sequence among the graph. The number of rows of filed nodal sequences present the bandwidth of the original structure.

1. Introduction

One of the problems in the field of the structural analysis by the matrix methods is how to solve simultaneous equations efficiently in the range of the ability of an electronic computer. For example, if we analyze a spacial structure precisely, over hundreds or thousands of nodes must be taken into consideration. By the multiplication of the degree of freedom, the order of the equations become over thousands, necessarily. However, all the coefficients of the equations can't be stored in the main memory storage of the computer at one time.

Fortunately, we know that the larger the order of the equations, the greater the decrease of the ratio of the nonzero elements in the matrix. Also, they gather in a narrow band along the main diagonal line of the matrix, if the numbering of nodes are appropriately done. Thus, the bandwidth of the matrix depends on the number-

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ing of nodes; and we have to solve the problem how to give the optimum numerical number, if we intend to use a computer.

The bandwidth under optimum state is governed by the maximum value of the difference of the number of two nodes, which are directly connected by a member. Therefore, we need not pay attention to other factors, except the connectivity relationship of the framed structures. This leads us to the conclusion that we need not treat the actual structure, but treat a linear graph, which consists of nodes and branches showing the actual connectivity relationship between members.

In past studies in this field, the problem was not regarded as a kind of graph problem. Accordingly, the actual connectivity relationship was ignored or almost ignored. In the past, the optimization of the numerical number of nodes was done by using a successive rearrangement of rows and columns of the stiffness matrix[2]. Or, it was obtained by dividing a system into an appropriate number of subgroups of joints located at the same distance from a starting joint group for numbering[3]. The actual connectivity relationship is not at all considered for the former; and for the latter the concept of distance in graph theory is introduced, though the word "distance" is not found in the paper. However, it can be said that the latter is insufficient for a true optimization of numbering, because we can easily find some examples which can't be optimally numbered by the direct use of the method. Thus, in addition to the concept "distance", some other factors which influence on the bandwidth must be found for a better optimization.

In this paper, the authors introduce the graph theory for the optimum joint numbering and search for some concepts, which are defined in the theory and influence on the bandwidth reduction.

Following this, by the application of the concepts, they propose a new method to obtain the minimum bandwidth, and also to give the optimum joint numbering of a framed system.

The method is a graphical one and is easily applied to some kinds of structures (i.e. tree structures). By use of the method, called "Sequential File Method", all of the nodal sequences are efficiently filed along the longest sequence in order to minimize the number of rows of filed sequences.

In the last section of this paper, some examples of the application of the method are presented; and the authors also try to give some considerations of the method of bandwidth reduction for general framed systems in accordance with the concepts of the graph theory.

2. Framed Structure and Its Linear Graph[1]

A linear graph, G, is a configuration which is drawn only by nodes and lines.

Joints and members of a framed system are presented by nodes and lines in a graph, respectively.

In the graph theory, a group of divided subgraphs is also treated as a graph, but in structural analysis they are treated as independent structures. Thus, a framed structure is drawn by a connected graph.

As the problem in consideration is about the bandwidth of stiffness matrix, the existence of three members (indicated by *a*, *b* and *c* branches in Fig. 1) is ignored for the problem, because the influence of the members is restricted only to A, B and C joints, respectively. The other joints have no influence on the removal of the members. It concludes that the members which connect structural joints and datum node (i.e. the ground) are allowed to be removed for the problem of brandwidth. Therefore, the actual system in Fig. 1–a is transformed into a graph shown in Fig. 1-c.

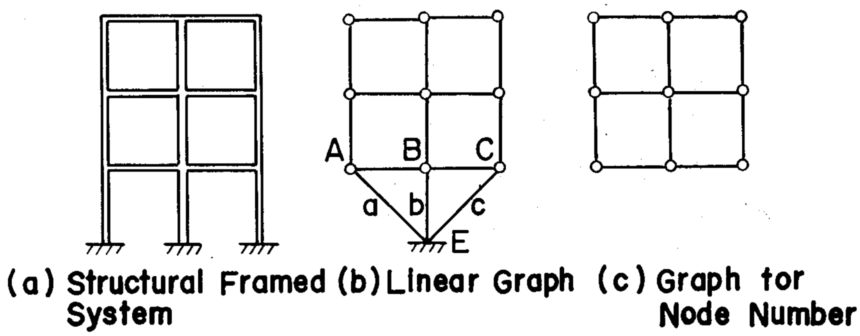


Fig. 1. A Framed System and its Linear Graph.

A graph, $G(n,m)$, consists of n nodes and m lines. If an original frame has m_d members which are connected to the datum node, we have the following relation between the original system and its graph;

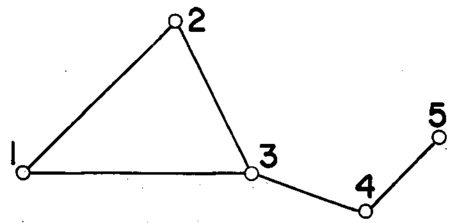
$$\text{Original System } (n,\ m+m_d) \rightarrow G(n,\ m)$$

Following this, the authors explain some concepts used in the graph theory which are necessary for the understanding of this paper.

* Distance (denoted by "d")

The distance between two nodes, A and B, is defined as the number of lines which are located on the path connecting them. Thus, the distance between two nodes, 3 and 5, in Fig. 2 is obtained as

$$d(3,5) = 2$$



Maximum Distance from Each Node in Above Graph

Initial Node	Max. Distance
1	3
2	3
3	2
4	2
5	3

Diameter (d_0) = 3
Radius (r_0) = 2
Center of Graph : Node 3 and 4
Maximum Degree = 3 (at Node 3)

Fig. 2. Diameter, Radius of Graph and Degree of Node.

* Degree (denoted by deg.)
The degree of a node is the number of lines which are connected to the node. For the graph in Fig. 2, the degree of node 3 is equal to 3. As every node has two terminal points, and they are connected to two different nodes, we have following equation for a graph, $G(n,m)$.

$$\sum_{i=1}^{n} (\text{the degree of the } i\text{-th node}) = 2m$$

* Diameter and Radius (denoted by d_0 and r_0, respectively)
For a graph, $G(n,m)$, the longest distance among the shortest paths from the i-th node to all the other nodes is designated by d_i. This operation is repeated for all nodes included in the graph, and we obtain the result $(d_1, d_2, \cdots, d_i, \cdots, d_n)$. Among them, the longest and the shortest values are called its diameter and radius, respectively. In general, the following relation exists between the number of nodes, diameter and radius.

$$n \geqslant 2r_0 \geqslant d_0 \geqslant r_0$$

* Complete Graph (denoted by G_c)

If the distance between every two nodes is equal to 1, the graph is considered to be a complete graph. In other words, every two nodes in the graph are connected by a line, and the diameter and the radius are equal to 1. Between the number of nodes and lines of a complete graph, we have a relation;

$$m = n(n-1)/2$$

A triangle with 3 nodes and 3 lines is a simple complete graph, and it is indicated by $G_c(3,3)$. Among complete graphs, a graph with less than four nodes can be drawn as a plane graph. However, graphs with more than five nodes become non-plane graphs; and it is obvious that we have to fix handles on the plane to draw the graphs without crossing the lines.

* Complement Graph (denoted by $\bar{G}(n, m)$

The complement graph, $\bar{G}$, of a graph, $G(n, m)$, has the same number of nodes in G and has lines which connect every two nodes that are not connected directly in G. $\bar{G}$ is obtained by use of G and its complete graph, G_c.

$$\bar{G}(n, m) = G_c(n, n(n-1)/2) - G(n, m)$$

3. Factors Which Influence on Bandwidth

In this section, we investigate the factors which influence the bandwidth of an arbitrary frame by use of simple graphs.

Consider a graph, $G(n,m)$, which includes n nodes and m lines. After giving an arbitrary node-numbering, we can make a branch node incidence matrix which is denoted by $A(m \times n)$ matrix and gives whether is an arbitrary pair of nodes are directly connected.

Dividing this matrix into n column matrices, it is shown as

$$A = [A_1 A_2 \cdots A_i \cdots A_n] \tag{3–1}$$

Instead of drawing the graph, we can present it by the following equation:

$$K = \begin{pmatrix} A_1^t A_1 & A_1^t A_2 \cdots A_1^t A_i \cdots A_1^t A_n \\ A_2^t A_1 & A_2^t A_2 \cdots A_2^t A_i \cdots A_2^t A_n \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ A_i^t A_1 & A_i^t A_2 \cdots A_i^t A_i \cdots A_i^t A_n \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ A_n^t A_1 & A_n^t A_2 \cdots A_n^t A_i \cdots A_n^t A_n \end{pmatrix} = A^t A \tag{3–2}$$

In this equation, we obtain a plus integer number at the main diagonal elements, and a minus one or zero at off-diagonals. The former indicates the number of lines which are connected to the node, that is, the degree of the node. The latter shows that the two nodes are directly (or not) connected each other, respectively. They are summarized as follows.

$$\begin{cases} A_i^t A_j = 0\,, & \text{if} \quad d(i,\, j) > 1 \\ A_i^t A_j \neq 0\,, & \text{if} \quad d(i,\, j) \leqslant 1 \end{cases}$$

And, we have a relation for every row or column,

$$\sum_{j=1}^{n} A_i^t A_j = \sum_{j=1}^{n} A_i^t A_j = 0 \tag{3–3}$$

Consider a complete graph, G_c. The degree of every node is equal to $(n-1)$. This means that the distance between the arbitrary two nodes is equal to 1, thus the diameter of the graph$=1$.

If we employ the equation (3–2) for this complete graph, the matrix does not include a zero element. Hence, we can know that the half bandwidth (denoted by H.B.W.) of the matrix$=n$, if the graph has n nodes. It leads us to the conclusion that the node-numbering for a complete graph may be arbitrary.

If an arbitrary line is removed from the complete graph, $G(n,\, m)$, the distance between the two nodes from where a line is removed becomes 2, and the others do not change. Thus, $d_0=2$ and $r_0=1$ for the new graph, $G(n,\, m-1)$. If one of two nodes is numbered to 1 and n, and the other is numbered arbitrarily, and the eq. (3–2) is set up for this graph, the $(1,\, n)$ and $(n,\, 1)$ elements of K are equal to zero.

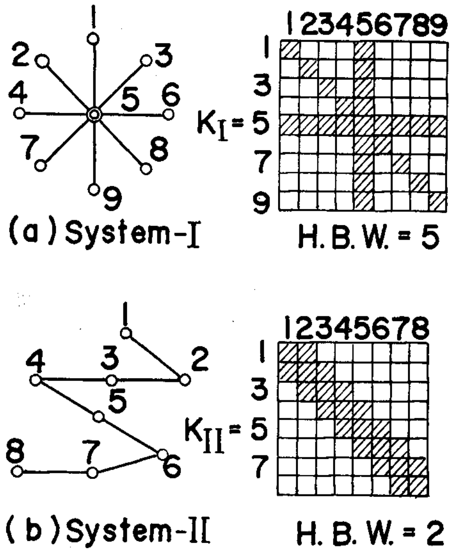


Fig. 3. Typical Two Tree Systems under Optimum Node-Numbering

Thus, the bandwidth decreases by one, compared to the complete graph; and we obtain that H.B.W.$=n-1$. Moreover, we know that the nodes with a comparatively less degree have a higher probability to compose the end points of diameter.

The above considerations for G_c and $G(n, m-1)$ show that the degree and the diameter of a graph may be the important factors when investigating the bandwidth.

Following this, some typical and simple graphs are given to clarify the influence of the factors selected above. Fig. 3-a is an example of a tree with one centre of lines. The characteristic of the graph is that the centre is complete, but the others are with deg.$=1$. Thus, $d_0=2$ and $r_0=1$. The optimum numerical numbering is easily obtained and shown in the same figure. In this case, the number of the centre is decided by the equation as follows.

$$\text{Number of the centre} = [n/2] \quad \text{or} \quad n-[n/2] \tag{3-4}$$

In the equation, [ ] indicates the Gaussian symbol. The number of the other nodes are arbitrary; and K of this graph is also presented as K_{I} in Fig. 3-a.

The second example is a series of lines and presented in Fig. 3-b. If the graph contains n nodes, it can be expressed by $G(n, n-1)$ and includes the least number of lines of n-node connected graphs. K_{II} in Fig. 3-b shows K of the graph and it is obvious from this example that the fact, H.B.W.$=2$, indicates the graph being the simplest one with minimum bandwidth. This is because a graph with H.B.W.$=1$ is not a connected graph but a group of nodes without lines. In the above example, the diameter is equal to $(n-1)$ and the graph is the diameter, itself. The degree of both ends is equal to one, and all the other nodes have the degree of 2. These facts appreciate the correctness of the consideration done for the tree graph with one centre.

Summarizing the above considerations, we can conclude that two items,

(i) the diameter of a graph and

(ii) the maximum degree of a node,

may play a most important roll in the research for the minimum bandwidth of graphs.

4. Sequential File Method for Tree Graph

The tree graph is one of the simplest graphs and has no closed paths. In this section, the authors investigate a general method to obtain minimum bandwidth of tree systems by using simple tree graphs at the beginning of the investigation. The method proposed here is a graphical one which removes the complexity of the connectivity relationship of graph.

4.1 In the case of a tree graph with one centre ($d_0=2$ and $r_0=1$)

An example of this tree graph is presented in Fig. 3-a. If it includes n nodes, the maximum degree of the centre is $(n-1)$. In this case, it was shown that the bandwidth will reduce to a minimum value when the numerical number of the centre is ordered to be equal to the middle one within n nodes. Thus, H.B.W. is obtained by following equation.

$$\text{H.B.W.} = m-[m/2]+1\,, \tag{4–1}$$

where m indicates the maximum degree of the graph.

Following this, we consider how to represent H.B.W. by a graphical method. According to the above considerations, the number of nodes included between the first and the centre nodes should be equal to the rest. Using this suggestion, we can give a graphical representation to minimize H.B.W., as shown by the following steps. (The procedures can be easily understood by refering to Fig. 4.)

(1). Selecting two arbitrary nodes except the centre.

(2). These two nodes with the centre compose the diameter and they are arranged on a lateral thick line with length 2 in the figure. This line shows the diameter, d_0.

(3). As three nodes among 7 in the example are already chosen, the other four nodes have to be placed in the new graph. Two of them are placed above the centre and the other two above the node on d_0, which is directly connected to the centre by $d=1$, and placed on the left side of the centre. Thus, they are rearranged in two columns with the same height, i.e. 2. If there are x nodes as the rest, the height of two columns will be $[x/2]$ and $x-[x/2]$.

(4). The four nodes rearranged in the previous step are newly connected to the centre. The new graph seems different from the original one at a glance, but both are topologically the same, and have the same connectivity relationship.

(5). Labeling of number for nodes is performed in accordance with the arrow presented in Fig. 4. That is, it is done from the right side to the left, and also from top to bottom. By this operation, the center is ordered as desired.

In this graphical method, the length and the width of a tree graph are represented in lateral and vertical directions, respectively. For this example, the maximum length and width are governed by the diameter and the degree of the centre, respectively. Furthermore, H.B.W. of the graph is obtained as the maximum width of the rearranged graph.

By use of the method, the maximum difference value of the two node-numbers concerned with H.B.W. is found between two nodes connected by a lateral line at the location with maximum rows. In above example, it is found between two nodes on the diameter.

Furthermore, H.B.W. can be calculated only by counting the maximum width of the graph (i.e. the number of rows) and using the equation;

$$\text{H.B.W.} = (\text{H.B.W. of diameter}) + (\text{number of rows}) \tag{4–2}$$

H.B.W. of any diameter is always equal to 2. Thus,

$$\text{H.B.W.} = 2 + (\text{number of rows}) \tag{4–3}$$

In the example, maximum width$=2$ and H.B.W. of the graph is obtained to be equal to 4. This coincides with the fact.

Fig. 4 explains the graphical representation of a tree graph with one centre and $(2n+1)$ nodes. The degree of the centre is equal to $2n$, and it is obvious that n rows, including the diameter, are necessary to draw a rearranged graph. We notice from this fact that H.B.W. will increase at least by one, when the degree of the center increases by two.

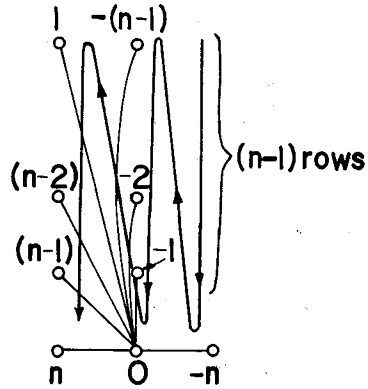


Fig. 4. Arrangement of Nodes around a Node with Multi Degree.

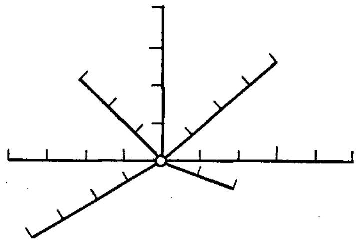
The allowable directions of a connecting line are restricted to be lateral, vertical, and obliquely descending to the right or ascending to the left as Fig. 4. Thus, every two nodes in neighbouring columns can be connected in the range of allowable directions; and two nodes in the same column can be connected without restrictions.

4.2 In the case of a general tree graph with one center

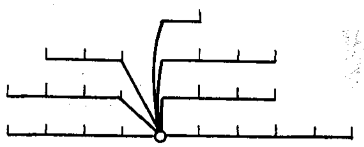
The above example in Section 4–1 is restricted to the case of $d_0=2$ and $r_0=1$. In this section, we treat one whose nodal sequences can have an arbitrary length.

Consider a center having m degree. In order to obtain the diameter, we compare m nodal sequences gathering at the center and select the longest two sequences, which include d_1 and d_2 nodes (except the center), respectively. These two sequences with the center compose the diametre, namely, $d_0=d_1+d_2$.

By the application of the graphical representation to this case, the general tree with one center can be drawn as shown in Fig. 5. An arbitrary nodal sequence can be ploeed in a row along the diameter. Thus, it concludes that H.B.W. of a general tree graph is decided by the number of nodes which locate at $d=1$ from the center, that is by the degree of the centre. Therefore, this case is just the same as the case of a tree with $d_0=2$ and $r_0=1$.



(a) A Tree Graph
(A Node with multi Degree)



(b) Graphical Expression

Fig. 5. Graphical Determination of Half Bandwidth.

4.3 In the case of a tree with two centers

An example of a tree graph with two centers is shown in Fig. 6. As the graph is a tree, there exists only one line between the two centers, and its length is denoted by d_2.

The degrees of the two centers are denoted by m_2 and m_3. The longest sequences among m_2 and m_3 sequences are selected (except d_2), and are denoted by d_1 and d_3, respectively. The nodal sequence, $(d_1+d_2+d_3)$, is selected as a temporary diameter of the graph.

In Fig. 6, 1 and 4 present the end nodes of the diameter, and 2 and 3 are the centres. d_2^i $(i=1,2,\cdots,m_2-2)$ and d_3^j $(j=1,2,\cdots,m_3-2)$ present the length of nodal sequences from 2 and 3 nodes, respectively. d_1, d_2 and d_3 are not included among them.

The word "temporary diameter" is used, because there is a posibility that the nodal sequence, $(d_1+d_2+d_3)$, will not compose the diameter, when $d_2^1>d_2+d_3$ or $d_3^1>d_1+d_2$. The authors give consideration to these cases at the end of this section. At this stage, the sequence, $(d_1+d_2+d_3)$, is supposed to be the longest one, and it is treated as a true diameter.

$$d_0 = d_1+d_2+d_3 \tag{4-4}$$

Also, we have the following relations between nodal sequences.

$$\begin{aligned} &d_1 \geqslant d_2^i \geqslant d_2^{i+1} \qquad (i = 1, 2, \cdots, m_2-2) \\ &d_3 \geqslant d_3^j \geqslant d_3^{j+1} \qquad (j = 1, 2, \cdots, m_3-2) \end{aligned} \tag{4-5}$$

In the case of a tree graph with one center, H.B.W. is decided only by the maximum degree. In the case with two centers, the problem is much more complicated and we have to introduce another concept which is called the nodal capacity.

When all of the nodes included in the tree graph in Fig. 6 are optimally numbered, all of the sequences are filed within h rows (except the diameter) by use of the sequential file method. The number, h, has to satisfy the following relations;

$$\begin{aligned} &h \geqslant m_2-[m_2/2]-1 \\ &h \geqslant m_3-[m_3/2]-1 \end{aligned} \tag{4-6}$$

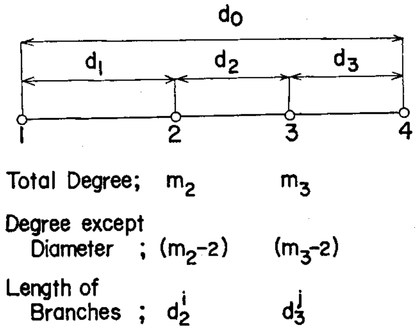


Fig. 6. An Example of Tree Graph with Two Centres.

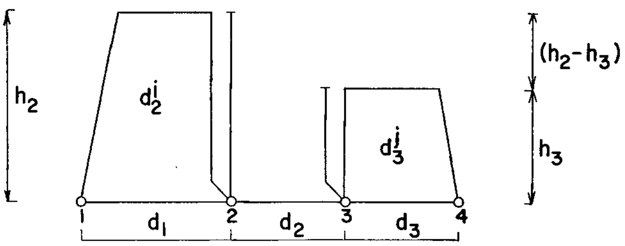


Fig. 7. Initial State of Sequential File Method for Tree Graph with Two Centres.

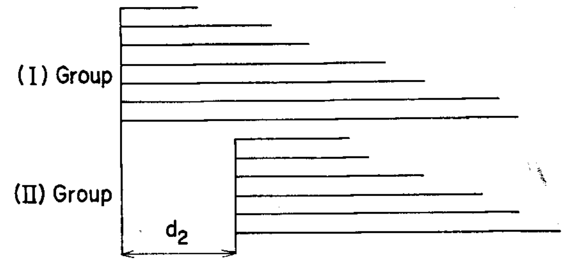


Fig. 8. Comparison Table for Residual Nodal Sequences.

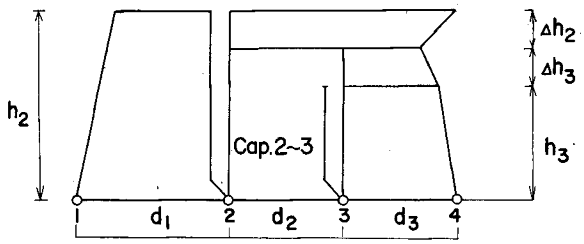


Fig. 9. Last State of Sequential File Method for Tree Graph with Two Centres.

At this stage, we call the number of nodes, which can be stored between nodes 1 and 2, 2 and 3, and nodes 3 and 4, the nodal capacity. They are described as Cap. 1~2, Cap. 2~3 and Cap. 3~4, respectively.

$$\begin{aligned} \text{Cap. } 1\sim2 &= h\cdot d_1 \\ \text{Cap. } 2\sim3 &= h\cdot d_2 \\ \text{Cap. } 3\sim4 &= h\cdot d_3 \end{aligned} \tag{4–7}$$

Any nodal sequence, which is placed on the left side of 2–node and also on the right side of 3–node, can be filed in a row. However, the sequences which should be filed between the two centers may occupy more than two rows, if their lengths are longer than d_2. Thus, it is obvious that the region between the two centers requres enough rows in order to place the nodes which are connected to the centre by $d=1$. Furthermore, the area must be larger than the total number of nodes which belong to the sequences placed in the central region. This can be presented by following equation.

$$\text{Cap. } 2\sim3 \geqslant \sum d_2^i + \sum d_3^j \tag{4–8}$$

This equation gives the suggestion that the sequence to be placed in the central region should be selected successively from the shortest one among the sequences.

The above considerations show that the minimum bandwidth of tree graphs with two centers can be obtained not only to consider the maximum degree of a center, but also to pay attention to the length of sequences and the nodal capacity between the two centers.

Following this, the authors propose some graphical steps to obtain the minimum width of a tree graph with two centres. The steps are for the general tree graph as shown in Fig. 6.

[Step–1]. Selection of a temporary diameter of the graph.

The temporary diameter is selected, as shown at the beginning of this section. This diameter is drawn by a straight line in Fig. 7.

[Step–2]. Calculation of the initial width of the tree graph.

After Step–1, we consider only the nodes which are connected to the centers by $d=1$, and take enough width to place them in the new graph. Thus, the method proposed for the case of one center is directly applied for Step–2. Then, we calculate the initial width (i.e. the number of rows) above the centres, 2 and 3 nodes, and they are described by h_2 and h_5, respectively. They are calculated as follows:

$$\begin{aligned} h_2 &= m_2-[m_2/2]-1 \\ h_3 &= m_3-[m_3/2]-1 \end{aligned} \tag{4–9}$$

These values are drawn by vertical lines above the centers and above the nodes locating on the left side of them by $d=1$. (Fig. 7)

[Step–3]. Selection of nodal sequences to be located on the left side and the right side of the centres, 2 and 3 nodes, respectively.

According to the consideration which is done in this section, they should be selected successively from the longest nodal sequence among (m_2-2) and (m_3-2), respectively. After the selection of h_2 and h_3 sequences, they are filed in rows along the diameter. Their initial nodes (i.e. the nodes with $d=1$ from the centres) are placed in columns which locate above the node neighbouring to the centre (i.e. 2–node), and above the centre (i.e. the 3–node), respectively.

The state after this step is presented in Fig. 7.

At this step, nodal sequences, $(d_2^1, d_2^2, \ldots, d_2^{h_3})$ and $(d_3^1, d_3^2, \ldots, d_3^{h_3})$, are filed and the remaining sequences are left for the following steps.

[Step–4]. Filing operation of $|h_2-h_3|$ nodal sequences.

At this step–4, we select $|h_3-h_3|$ sequences to be placed between (h_3+1) and h_2 rows, if $h_2 \geqslant h_3$. By a proper selection of $|h_2-h_3|$ nodal sequences, we can leave the least number of nodes belonging to the sequences which are not filed at this step; but will be filed in Cap. 2∼3 in the following step.

They are selected successively from the longest to the $|h_2-h_3|$-th sequence by use of the comparison table of residual nodal sequences, which is newly introduced and is shown in Fig. 8. As h_2 is larger than, or equal to h_3 in this example, the selected sequences are filed as shown in Fig. 9. That is, the initial nodes which are distant from the centers by $d=1$ are ordered in two columns. The distance between the two columns coincides with the distance of the two centers of the original tree graph, namely d_2 for this example.

After step–4, there are enough unoccupied points left in the two columns above 2–node and next to 3–node for the initial nodes of unfiled nodal sequences being left for step–5.

[Step–5]. Checking whether the remaining nodal sequences can be filed in Cap. 2∼3.

In order to file the remaining sequences of step–4 in the area between the two centers, it is necessary that the nodal capacity of the area is at least equal to, or larger than the summation of the nodes belonging to the sequences. If the summation of nodes is denoted by S, it can be expressed by the following relation.

$$\begin{aligned} &\text{Cap. } 2\sim 3 \geqslant S = \left(\sum_{i=\alpha}^{m_3-2} d_2^i + \sum_{j=\beta}^{m_2-2} d_3^j\right) \\ &\alpha = m_2 - h_2 - \varDelta h_2 - 1 \\ &\beta = m_3 - h_3 - \varDelta h_3 - 1 \end{aligned} \tag{4–10}$$

If Cap. $2\sim 3 < S$, it is obvious that the nodal capacity between $2\sim 3$ is insufficient to file the sequences. In this case, the operation for sequencial filing has to be returned to the previous step–4. Then, by use of Fig. 8, an even number of nodal sequences are newly selected and filed above the h_2-th row, until the relation (4–10) is satisfied. If $2\varDelta h$ sequences are newly selected and filed additionally, the total rows become $(h_2+\varDelta h)$, and this state is shown in Fig. 9.

If the assumption that the nodal sequences, d_2^i and d_3^i, can be filed in Cap. 2∼3 without leaving an unoccupied area is correet, the minimum width of a filed graph can be decided by comparing S with Cap. 2∼3, and by operating the additional filing until the establishment of the relation (4–10). Thus, the half bandwidth is obtained by following equation:

$$\text{H.B.W.} = 2+h \tag{4–11}$$

Following this the authors show the correctness of the above assumption. The area of Cap. 2∼3 is shown in Fig. 14. The critical case of the assumption is that the summation of nodes which belong to the remaining sequences is equal to the nodal capacity. It is expressed by the equation:

$$\text{Cap. } 2\sim 3 = n\cdot d_2 = \bar{S}_2 + \bar{S}_3 = \sum_i d_2^i + \sum_j d_3^j\,, \tag{4–12}$$

where $\bar{S}_2$ and $\bar{S}_3$ are total nodes belonging to d_2^i and d_3^i, respectively.

The area of Cap. 2∼3 is divided into two sub-areas, $\bar{S}_2$ and $\bar{S}_3$, as shown in Fig. 14, in accordance with following operation. At first, the area is divided into two sub-areas at the $[\bar{S}_2/n]$-th column from the left center. By the addition of $(\bar{S}_2\text{-}[\bar{S}_2/n] \times n)$ nodes in the $([\bar{S}_2/n]+1)$-th column to the sub-area on the left side, we can divide the $\bar{S}_2$-area among Cap. 2∼3. Thus, the remaining area is obviously equal to $\bar{S}_3$. In order to file d_2^i in $\bar{S}_2$, we begin to file from the shortest one; and sufficient rows are used to file it in order to avoid the reflection at the border line between $\bar{S}_2$ and $\bar{S}_3$, as shown in Fig. 10. This operation is repeated for all d_2^i and $\bar{S}_2$ is covered by d_2^i, using the allowable directions of connecting lines. This procedure is repeated to file d_3^i in the area of $\bar{S}_3$. Thus, the area of Cap. 2∼3 is just covered by the nodal sequences, if Cap. 2∼3$=\bar{S}_2+\bar{S}_3$.

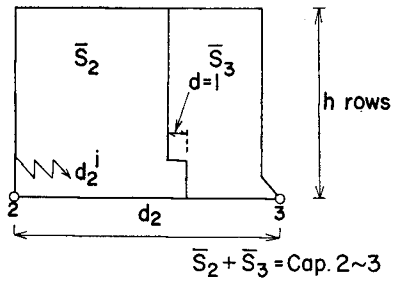


Fig. 10. How to File Cap. 2∼3.

By use of the correctness of the above assumption, the minimum H.B.W. of a tree with two centers is graphically obtained in accordance with the five steps described in this section; and the H.B.W. is given by eq. (4–11). The number, h, is the width of the tree graph.

In the beginning of this section, the authors mentioned the exceptional case where the $(d_1+d_2+d_3)$ sequence does not compose the true diameter. Here is given further consideration about it. The exceptional case happens when two nodal

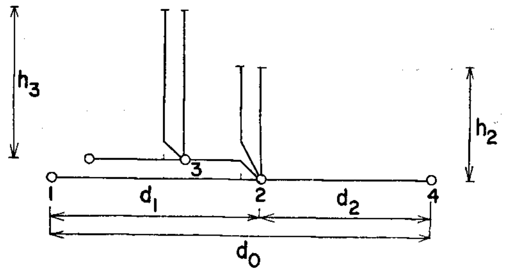


Fig. 11. An Example of Sequential File Method with True Diameter. $(h_3>h_2)$

sequences from one center compose the diameter. If the degree of the center with two nodal sequences, which compose the diameter, is less than that of another centre, it is obvious from Fig. 11 that the direct application of the sequencial file method to the case has the tendency not to induce an accurate result for H.B.W. Rather, it gives a result largeı than the minimum H.B.W. by one.

Thus, it concludes that the application of the temporary diameter instead of the true one gives an accurate result for the sequentical file method in the case of the two centers.

4.4 In the case of a tree graph with three centers.

In this section, the authors explain the sequential file method for a tree graph with three centers; and it is quite similar to the one described in the former section.

An example of a general tree graph is shown in Fig. 12. The centers are described by 2, 3 and 4–node, and their degrees are (m_2+2), (m_3+2) and (m_4+2), respectively.

Between every two centers, there is only one nodal sequence. The longest sequences among (m_2+1) and (m_4+1) are selected. These four sequences compose the temporary diameter of the graph, as shown in Fig. 12.

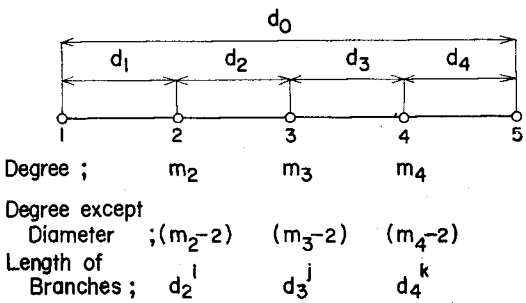


Fig. 12. An Example of Tree System with Trhee Centres.

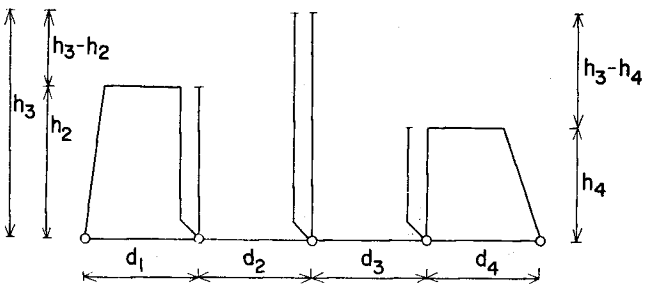


Fig. 13. Initial State of Sequential File Method for Tree System with Three Centres

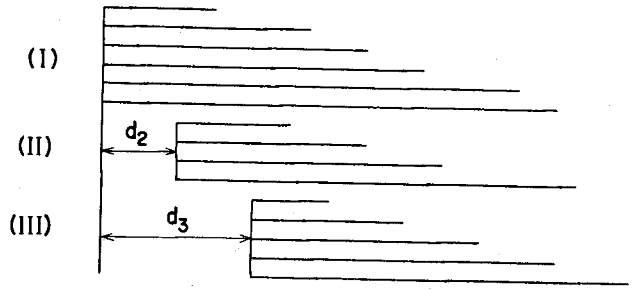


Fig. 14. Comparison of Residual Nodal Sequences.

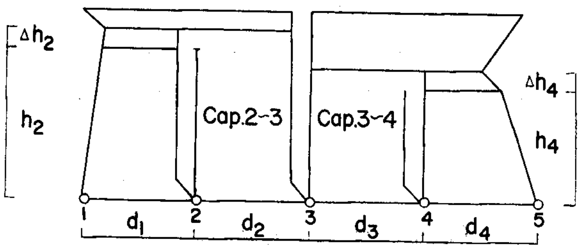


Fig. 15. Sequential File Method for Tree System with Three Centres at Final State.

$$d_0 = d_1 + d_2 + d_3 + d_4 . \qquad (4\text{–}13)$$

The example is the one with $m_3 \geqslant m_2$ and $m_3 \geqslant m_4$. Following that, the procedures for the sequencial file method are described briefly.

[Step–1]. Selection of a temporary diameter of the graph.

d_0 is shown by a straight line in Fig. 13.

[Step–2]. Calculation of the initial width of the graph.

For the reservation of enough rows, necessary for the degree of the centers, following calculations are done.

$$\begin{aligned} h_0 &= m_2 - [m_2/2] \\ h_3 &= m_3 - [m_3/2] \\ h_4 &= m_4 - [m_4/2] \end{aligned} \qquad (4\text{–}14)$$

h_2, h_3 and h_4 rows are reserved above the three centres, respectively. They are shown by the vertical lines in Fig. 13.

[Step–3]. Selection of nodal sequences to be located on the left and the right side of 2– and 4–node, respectively.

Among m_2 sequences, the longest h_2 sequences are selected and filed on the left side of 2–node. The same procedure is done for the area on the right side of 4–node. The state after this step is shown in Fig. 13.

[Step-4]. Selection and filing procedure of (h_3-h_2) and (h_3-h_4) sequences.

At the state after this step, we have to leave a minimum number of nodes in order to file the central area between 2 and 4 nodes. That is, we select the longest sequences at this step. For this purpose, we use the comparison table of the rest of nodal sequences, as shown in Fig. 14. Then, we select (h_3-h_2) and (h_3-h_4) sequences among (I) and (II), and (I) and (III) group, respectively. The sequences selected by the above procedure are filed over the graph as shown in Fig. 13; and the state after this step is presented in Fig. 15. The method to make the comparison table is given in the previous section. Δh_2 among (h_3-h_2) are selected from (II) group, and Δh_4 among (h_3-h_4) are from (III) group.

[Step-5]. Checking whether the remawing sequences can be filed in Cap. 2~4.

The nodal capacities between every two centers are fixed by the previous step; and they are given by the following equations.

$$\begin{aligned} \text{Cap. } 2\sim 3 &= (h_2+\Delta h_2)\, d_2 \\ \text{Cap. } 3\sim 4 &= (h_4+\Delta h_4)\, d_3 \end{aligned} \tag{4–15}$$

and

$$\text{Cap.} 2\sim 4 = \text{Cap. } 2\sim 3 + \text{Cap. } 3\sim 4 \tag{4–16}$$

The total number of nodes belonging to unfiled nodal sequences are counted for every group (I), (II) and (III), and they are described by $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$, respectively.

$$\begin{aligned} \bar{S}_2 &= \sum_{i=h+\Delta h_2+1}^{m_2} d_2^i \\ \bar{S}_3 &= \sum_{j=\gamma}^{m_3} d_3^j \\ \bar{S}_4 &= \sum_{k=h_4+\Delta_4+1}^{m_4} d_4^k, \end{aligned} \tag{4–17}$$

where $\gamma=2h_3-h_2-h_4-\Delta h_2-\Delta h_4+1$.

Thus, $S=\bar{S}_2+\bar{S}_3+\bar{S}_4$ gives the total nodes of unfiled sequences.

If $$S > \text{Cap. } 2\sim 4\,, \tag{4–18}$$

the nodal capacity is obviously insufficient for the filing of the residual sequences. Then, step-4 is applied again, till the equation

$$S \leqslant \text{Cap. } 2\sim 4 \tag{4–19}$$

is established by selecting the additional sequences and by filing them over h_3 rows. This procedure is explained in the previous section, and is done by use of the comparison table of sequences. The state where eq. (4–19) is established is presented in Fig. 15.

At this stage, Cap. 2～4 has a sufficient nodal capacity for the remaining sequences. However, the relations between $\bar{S}_2$ and Cap. 2～3, and $\bar{S}_4$ and Cap. 3～4 are not related. Thus, we have following three cases;

Case 1, $\bar{S}_2 \leqslant$ Cap. 2～3 and $\bar{S}_4 \leqslant$ Cap. 3～4

Case 2, $\bar{S}_2 >$ Cap. 2～3 and $\bar{S}_3 + \bar{S}_4 <$ Cap. 2～4$-\bar{S}_2$

Case 3, $\bar{S}_4 >$ Cap. 3～4 and $\bar{S}_2 + \bar{S}_3 <$ Cap. 2～4$-\bar{S}_4$

They are shown in Fig. 16. These three cases are treated by following procedure. The treatment for case 1:

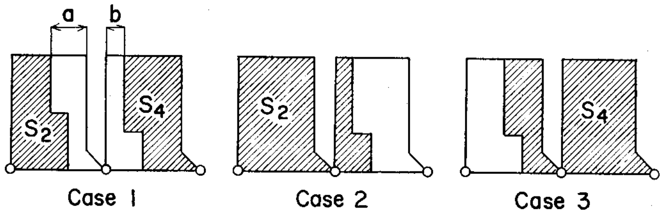


Fig. 16. Possible Cases at Step-5 of Sequential File Method.

If the length, b, is longer than a, the area (Cap. 3～4$-\bar{S}_4$) can store as many sequences among $\bar{S}_2$ as the number of rows of the area. Thus, we select them from the longest one successively and file them in the area. At this state, a number of unoccupied points are left in the area, Cap. 3～4. Then, the last operation for filing Cap. 2～3 is similar to the procedure for the case of two centers. The area, Cap. 2～3, has an insufficient nodal capacity for the filing of $\bar{S}_2$ and the remaining sequences of $\bar{S}_3$. Then, even number of additional sequences is selected among them by use of the comparison table. These sequences are filed additionaly over the rows already filed in the previous steps, until the rest can be stored in the area.
The treatment for case 2:

At first, $\bar{S}_4$ is filed in Cap. 3～4 and we select among $\bar{S}_3$ as many long sequences as the number of rows of Cap. 3～4 and file them in the area. The procedure after this state is just the same as that of case. 1.
Treatment for case 3:

In this case, the area of Cap. 2～3 is filed at first and the treatment after this is just the same as the previous cases.

By use of these five steps in the case of tree graph with three centers, the original graph is modified into a new one with a minimum width without changing the topological property. If it has n rows after the operation, we can conclude that the minimum half bandwidth of the tree graph can be reduced to $(2+n)$.

The above methods are done by use of the temporary diameter. If the true diameter does not pass three centers, but passes less than two centers, and the degree of the center which does not compose the diameter is larger than the others, H.B.W. obtained by use of the true diameter does not give the minimum value but increases by one. That is, the state which appeared in the case of two centres can be found in this case, too.

Therefore, it can be said that we should use the temporary diameter for the application of the sequential file method to tree graphs with more than two centres.

5. Examples of Sequential File Method

In this section, the sequential file method is actually applied to tree systems with three centres. The method will be clarified by these examples.

The examples are shown in Fig's 17 and 18. Both of them consist of the figure of given systems, Comparison Table of Residual Nodal Sequences-(a), the last state of filed sequences-(b) and Calculation Table for Step–4 and Step–5.

The given Systems are analyzed in accordance with the procedure in Section 4–4.

At first, the temporary diameters, d_0, are calculated and are shown at the beginnings of Fig's 17 and 18.

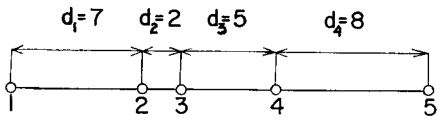
Following this, h_2, h_3 and h_4 are calculated to reserve enough rows for initial nodes of branches from the centers. Among m_2 and m_4, h_2 and h_4 branches are successively selected from the longest nodal sequence and are filed.

Residual sequences are compared with each other in (a) of Fig's 17 and 18.

By use of the comparison table of sequences, we select two sequences at a time and they are filed in a row, till the nodal capacity between 2 and 4 nodes has enough area for the residual sequences. This operation is presented in Calculation Table–(c). In the table, $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$ indicate the number of nodes belonging to the residual sequences from 2, 3 and 4 nodes, respectively. S_{total} means the summation of $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$. Cap. indicates the nodal capacity between 2 and 4 nodes. $\Delta S=(\text{Cap.}-S_{\text{total}})$, and if $\Delta S<0$, the nodal capacity has not enough area for the residual sequences and they can't be filed in the area. Thus, the operation is repeated till ΔS becomes positive or is equal to zero. For the example in Fig. 17, the operation was repeated only twice. It means that four nodal sequences have to be selected, and they are filed over h_2 rows in order to reserve enough nodal capacity between 2 and 4 nodes. For the example in Fig. 18, we have to repeat the operation eight times and select 16 sequences.

The last state of filed nodal sequences is presented in (b) of Fig's 17 and 18.

We can easily examine the possibility of decreasing the number of rows. Taking (b) in Fig. 18 as an example, 15 nodes are occupied on the highest row, but we have only 14 nodes unoccupied under the 14–th row. Thus, we can conclude that it is impossible to further decrease the number of rows. Therefore, the half bandwidth of the two systems are 9 and 17, respectively.



$\bar{m}_i = m_i - 2$

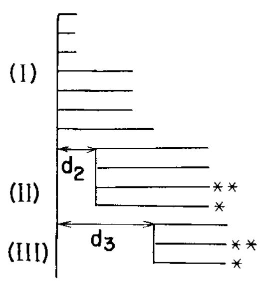
$\bar{m}_2 = 9$; $d_2^i = (7, 7, 7, 7, 6, 6, 6, 6, 6)$

$\bar{m}_3 = 7$; $d_3^j = (5, 4, 4, 4, 1, 1, 1)$

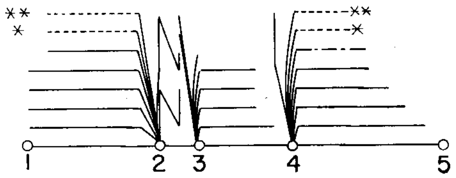
$\bar{m}_4 = 8$; $d_4^k = (8, 7, 6, 5, 5, 4, 4, 4)$

$h_2 = 5$, $h_3 = 4$, $h_4 = 4$

$d_0 = 22$



(a) Comparison of Sequences

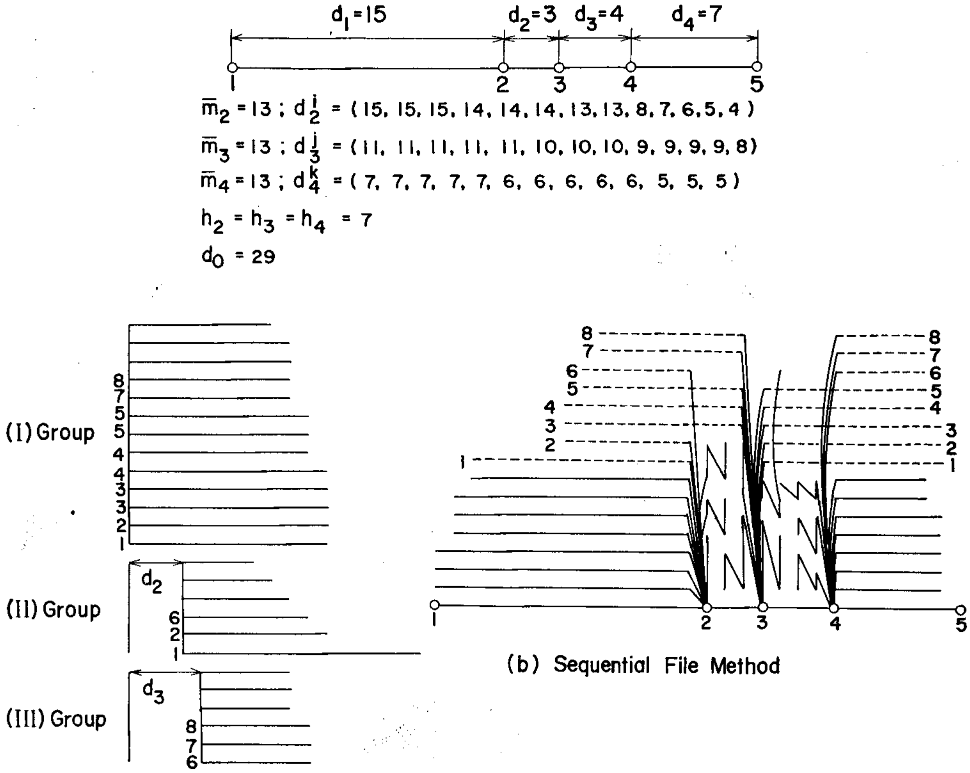


(b) Sequential File Method

		$\bar{S}_2$	$\bar{S}_3$	$\bar{S}_3$	S total	C ap.	ΔS
Step-4		24	20	12	56	35	−21
Step-5	*	−6	20	−4	−10	+7	−4
		18	20	8	46	42	
	**	−6	0	−4	−10	+7	+13
		12	20	4	36	49	

(c) Calculation Table

Fig. 17. An Example of Sequential File Method (1).



(a) Comparison Table of Residual Nodal Sequences

		$\bar{S}_2$	$\bar{S}_3$	$\bar{S}_4$	S total	C ap.	ΔS
Step-4		43	129	33	205	49	−156
Step-5	1	30	118	33	181	52	−129
	2	22	107	33	162	55	−107
	3	22	85	33	140	55	−85
	4	22	64	33	119	55	−64
	5	22	44	33	99	55	−44
	6	15	44	27	86	62	−24
	7	15	35	21	71	66	−5
	8	15	26	15	56	70	+14

(c) Calculation Table

Fig. 18. An Example of Sequential File Method (2).

6. Some Considerations about the Application of Sequential File Method to Framed Structures

The sequential file method is proposed to give numerical numbers for joints of a tree structure but not for a framed system. Thus, the method can't be applied to the latter, directly, because its characteristic is quite different from the former. The biggest difference is that the latter contains closed paths, but not the former.

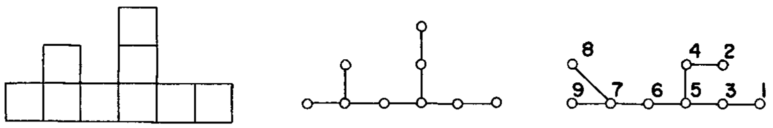
By the existence of the closed paths, a node in a framed system has more than two constraints from the diameter. However, a node in a tree has only one constraint for the reason that the node is connected to the diameter by one path. Thus, in order to define the position of a node in framed system by the graphical method, all of the constraints have to be taken into consideration; and it is very difficult to satisfy the conditions for all nodes in the system by the direct application of the previous method for the tree systems.

Yet, it is possible to apply the method to a kind of framed structure by modifying them.

After that the authors present an example. Fig. 19–a is an example of a framed structure. If we present a mesh by a node, the system can be reduced to a tree system in Fig. 19–b. We can apply the method for the tree system, and the result is shown in Fig. 19–c. Even if we give a numerical number for the nodes in the tree, the number is actually attached to the meshes. According to the order of meshes indicated in Fig. 19–c, we should give a proper numerical number for joints in every mesh, successively.

In this case, we can't obtain the exact numerical number of joints by the application of the method. But, as its tendency is obtained by this application, it is expected that the difference between the result obtained by the above procedure and the actual minimum bandwidth is restricted within a slight error.

For the finite element method, the procedure may be applied just the same as for framed systems. In the case, one finite element with some boundaries and nodes is newly presented by one node.



(a) An Example of Frames (b) Obtained Tree System (c) Sequencial File Method

Fig. 19. Application of Sequential File Method to Framed Strutcure.

7. Conclusion

By the introduction of the graph theory to the investigation of the minimum bandwidth of stiffness matrix, the authors proposed the sequential file method.

The method is a graphical one, and without changing the topological properties of the original systems, the geometrical profile is modified and rearranged so as to have the maximum lateral width and the minimum height along new coordinate systems. The vertical axis of the coordinate system coincides with the bandwidth; and the minimum height of the modified system indicates the minimum bandwidth of original structure.

Comparing the tree system with a framed structure, every node in the former has less constraints than the latter. Therefore, the method proposed in this paper can be easily applied to tree systems.

According to such considerations, we can conclude that the method has to be modified in order to be applied to framed systems, because most of the actual structures contain closed paths in them.

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