A Method for Solving Linear Programming Problems with Unknown Parameters

By

Nobuo SANNOMIYA*, Yoshikazu NISHIKAWA*, and Kyuill LEE*

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Abstract

A method is proposed for solving a problem of linear programming with unknown constraints. The form of the unknown constraints needs to be identified by a proper choice of the observation data. The present method is based upon a bicriterion formulation to the joint identification and optimization problem. A parametric approach is used to obtain an efficient solution to the bicriterion problem. Further, a decomposition into subproblems easily solvable is introduced. The interaction between subproblems is coordinated by an adjustment of a scalar parameter varying over the unit interval.

1. Introduction

This paper presents an algorithm for solving a minimization problem with unknown constraints. The objective function and the known constraints are assumed to be linear. Although the unknown constraints may be nonlinear, an "appropriate" linear approximation to them leads to a linear programming problem with unknown parameters. Then it is necessary to find both the parameters to be identified and the solution to the corresponding linear programming problem. The form of the unknown constraints is not available and the convexity of the total constraint sets is not assured. Therefore, the common approximation methods for nonlinear problems, such as inner linearization or outer linearization[1], cannot be applied directly.

There have been few investigations on the interaction between the identification and optimization problems. Haimes *et al.*[2-4] studied the coupling of the two problems and introduced the bicriterion formulation to the combined identification and optimization problem. The joint approach is required for the following cases: [1] when the identification problem does not have a unique solution, and [2] when the opti-

* Department of Electrical Engineering

mization problem has no feasible solution. Several computational approaches[3,4] have been developed mainly to solve the problem of Case [1], *i.e.*, the problem having many equally valid solutions to the identification problem due to the effects of measurement noise, computational inaccuracies, and inexact modeling.

The problem treated here is that of Case[2]. The mathematical model of the unknown constraints is based upon the available observation data, and generally is identified under the nonoptimal condition. Then the identified problem of linear programming does not have a truely optimal solution as a whole. Sometimes it has no feasible solution. Accordingly, a joint treatment of the identification and the optimization is essential, and a suitable choice of the observation data is necessary.

In this paper, a parametric approach is used for the bicriterion formulation of the combined problem. Since the combined problem is a nonconvex programming, it is decomposed into two subproblems easy to solve. One of the subproblems is an unconstrained minimization. The other is a linear programming problem, the non-feasibility of which is checked by an adjustment of a parameter varying over the unit interval. The observation data used for the identification problem is replaced iteratively until the optimal solution to the joint problem can be obtained. Examples of small dimension are presented to illustrate the necessity and the effectiveness of the proposed algorithm.

2. Problem formulation

Consider the following optimization problem:

$$\min_{x} c'x \tag{1}$$

subject to the constraints

$$Ax-b\leq 0 \tag{2}$$

$$f(x)\leq 0 \tag{3}$$

where x is an n-vector, f is an m-vector function, A is an $l\times n$ matrix, and c and b are vectors of appropriate diemnsion. A prime denotes transposition of a vector or a matrix. The problem (1)–(3) is assumed to have a feasible solution.

The constraint (2) is known a priori, while the form of the function $f(x)$ is not known exactly. Therefore, a mathematical model of $f(x)$ should be found prior to solving the problem (1)–(3). We define the following linear model:

$$h(x,\beta)\triangleq B(\beta)x-d(\beta)\leq 0 \tag{4}$$

with a p-vector β of model parameters remaining to be identified. The linear model is introduced because of the easiness of the constrained optimization procedure. With the structure of (4), the identification problem is to determine the parameter β so as to minimize the deviation between the model and the real system responses to a given class of inputs $\hat{x}^j (j=1, 2, \ldots, N;\ N \geq p)$. A noiseless observation of $f(x)$ is assumed in this paper. The identification problem is then presented as

$$\min_{\beta} G(\beta;\hat{x}^j) \tag{5}$$

where

$$\left.\begin{aligned} G(\beta;\hat{x}^j) &\triangleq \sum_{j=1}^{N} \|h(\hat{x}^j,\beta)-f^j\|^2 \\ f^j &\triangleq f(\hat{x}^j) \end{aligned}\right\} \tag{6}$$

The problem formulated above has been solved by first minimizing G and then solving the problem (1), (2) and (4). However, as pointed out in Refs. 2) and 3), system identification interacts strongly with system optimization. The present problem requires a joint approach to the identification and the optimization for the following reason: The structure of $f(x)$ may be nonlinear in x. Due to the inexact modeling, the parameter β depends upon the available observation data $\hat{x}^j$ and f^j, and generally is identified under the nonoptimal condition. Then, the solution generated from such a system model is nonoptimal. Furthermore, the problem (1), (2) and (4) may have no feasible solutions even when the original problem (1)–(3) has them.

The joint problem is given by

$$\min_{x,\beta} \begin{bmatrix} c'x \\ G(\beta;\hat{x}^j) \end{bmatrix} \quad \text{subject to (2) and (4)} \tag{7}$$

The problem (7) is a bicriterion minimization problem and the solution is defined as

Definition 1 (x^*, β^*) *is an optimal solution to the problem* (7) *if*

[i] $(x^*, \beta^*) \in E$, *where* E *is the efficient set of* (7).

[ii] $G(\beta^*;\hat{x}^j)=\min_{\beta\in E} G(\beta;\hat{x}^j)$, *with the input data satisfying, for a small* ε (>0),

$$\|\hat{x}^j - x^*\| < \varepsilon, \qquad j=1, 2, \cdots, N \tag{8}$$

As to the concept of the efficient set, refer to Appendix. It is noted that minimizing G with respect to $\beta \in E$ implies solving the identification problem under the additional constraint assuring the feasibility of the problem (1), (2) and (4).

Y. Y. Haimes *et al.*[2–4] have treated problems similar to (7). However, in their case, the identification problem is based upon the observation data given a priori, and then the condition (8) is not included in their definition of the solution. The purpose of this paper is to obtain an optimal solution satisfying all the conditions in Definition 1. The algorithm proposed includes the process of changing the observation points used for the parameter identification.

3. Method of approach

Several approaches[2–6] have been proposed for an efficient solution to a vector minimization problem. Of these, a parametric approach[3,5,6] is to construct a scalar minimization problem with parameters, and then to solve it as the parameters vary over an interval. The parametric solution is proved to satisfy a necessary condition for efficiency of the vector minimization. Under convexity assumptions, the equivalence is shown between a vector minimization problem and the corresponding scalar minimization problem. The results of the previous works are summarized in Appendix.

The present problem (7) is replaced by the following scalar minimization problem:

$$\min_{x,\beta}\ [\theta G(\beta;\hat{x}^j)+(1-\theta)c'x] \qquad \text{subject to (2) and (4)} \tag{9}$$

where θ is a parameter such that $0<\theta<1$. Note that (9) is not a convex programming problem in general because of the nonconvexity of $h(x,\beta)$. The Kuhn-Tucker conditions[7] are necessary for the solution to (9).

The joint problem (9) requires the constrained minimization with respect to the $(n+p)$-vector $(x',\beta')'$. Then it is larger and more complex than either the identification or the optimization problem taken separately. Therefore, from a computational viewpoint, a multilevel technique is used to solve the problem (9).

The decomposition of (9) is achieved by introducing the new vector α into the terms where x and β are coupled in (9). Then, define the Lagrangian function with an additional constraint $\alpha=\beta$:

$$\begin{aligned} L(x,\beta,\alpha,\eta,\lambda,\xi,\theta) \triangleq\ & \theta G(\beta)+(1-\theta)c'x+\eta'(Ax-b) \\ & +\lambda'[B(\alpha)x-d(\alpha)]+\xi'(\alpha-\beta) \end{aligned} \tag{10}$$

where η, λ and ξ are, respectively, vectors of Lagrange multipliers for appending the corresponding system constraints. For the time being, the observation points $\hat{x}^j$ are assumed to be fixed, and $\hat{x}^j$ in G is omitted.

The multilevel solution procedure is essentially to relax, in the first-level calculation, one or more of the necessary conditions for optimality and then to satisfy this condition at the second level. Then, the system Lagrangian L may be decomposed into the following sublagrangians.

$$\left.\begin{aligned} & L(x,\beta,\alpha,\eta,\lambda,\xi,\theta)=L_1(x,\eta,\lambda;\alpha,\theta)+L_2(\beta;\xi,\theta) \\ \text{where}\quad & \\ & L_1(x,\eta,\lambda;\alpha,\theta)\triangleq(1-\theta)c'x+\eta'(Ax-b)+\lambda'[B(\alpha)x-d(\alpha)]+\xi'\alpha \\ & L_2(\beta;\xi,\theta)\triangleq\theta G(\beta)-\xi'\beta \end{aligned}\right\} \tag{11}$$

At the first level, the parameter θ and the vectors α and ξ are assumed to be known. Then, we obtain the following independent subproblems:

Subproblem 1

$$\max_{\eta,\lambda\geq 0}\ \min_{x} L_1(x, \eta, \lambda;\, \alpha, \theta) = L_1(x^0, \eta^0, \lambda^0;\, \alpha, \theta) \tag{12}$$

or equivalently

$$\left.\begin{array}{l} \min\limits_{x}\,(1-\theta)c'x \\ \qquad\text{subject to}\quad Ax\leq b,\ B(\alpha)x\leq d(\alpha) \end{array}\right\} \tag{13}$$

This is a problem of linear programming. Note that the problem (13) may have no feasible solution unless giving an appropriate value of α. The Kuhn-Tucker stationarity conditions for L_1 are

$$\left.\begin{array}{l} B'\lambda + A'\eta = -(1-\theta)c \\ Ax-b\leq 0,\ \eta'(Ax-b)=0 \\ Bx-d\leq 0,\ \lambda'(Bx-d)=0 \end{array}\right\} \tag{14}$$

It is observed from (14) that $(m+l-n)$ components of the vector $(\eta', \lambda')'$ vanish, and that the remaining n components include the terms in proportion to $(1-\theta)$. The nonzero components of the Lagrange multiplier vector correspond to the binding constraints.

Subproblem 2

$$\min_{\beta} L_2(\beta;\, \xi, \theta) = L_2(\beta^0;\, \xi, \theta) \tag{15}$$

The solution β^0 is obtained by an unconstrained minimization technique. Note that when $\xi\neq 0$, β^0 is not identical with β_s, where β_s is the solution to the identification problem treated separately, namely

$$\min_{\beta} G(\beta) = G(\beta_s) \tag{16}$$

If the function $G(\beta)$ is quadratic in β, that is,

$$G(\beta) = (\beta-\beta_s)'Q(\beta-\beta_s) + G_{\min}, \qquad Q>0 \tag{17}$$

then the solution to (15) is given by

$$\beta^0 = \beta_s + \frac{1}{2\theta}Q^{-1}\xi \tag{18}$$

The task of the second level is to determine the vectors α and ξ so as to satisfy the constraint $\alpha=\beta$, or equivalently so that

$$\max_{\xi}\ \min_{\alpha} L(x^0, \beta^0, \alpha, \eta^0, \lambda^0, \xi, \theta) \tag{19}$$

for some fixed θ. The stationarity conditions lead to

$$\alpha = \beta^0 \tag{20}$$

$$\xi = -\frac{\partial}{\partial a}[B(a)x^0 - d(a)]'_{a=\beta^0}\lambda^0 \tag{21}$$

The parameter θ is determined at the third level as follows. Define the set of the parameter β:

$$\Omega \triangleq \{\beta \mid \text{by which there exists an } x \text{ satisfying both (2) and (4)}\}. \tag{22}$$

If $\beta_s \in \Omega$, a separate treatment, *i.e.,* identification followed by optimization yields the solution in the sense of Definition 1. However, if $\beta_s \notin \Omega$, the joint approach is necessary to obtain a feasible solution. From Theorem A1 in Appendix, there exists a parameter θ with $0<\theta<1$ such that the minimum of (9) is achieved at an efficient point of (7). Accordingly, the task of the third level is to determine such a θ. The θ is found by a one-dimensional search so as to assure the feasibility of Subproblem 1 under the decomposition of (9). For simplicity, consider the case where $G(\beta)$ is given by (17). Substituting (21) into (18) and using the solution λ^0 of (14), the following is obtained

$$\beta^0 = \beta_s + \frac{1-\theta}{\theta}\Delta\beta \tag{23}$$

Then θ $(0<\theta<1)$ is found to satisfy $\beta^0 \in \Omega$, where β^0 is given by (23) and $\beta_s \notin \Omega$.

After the convergence of the multilevel procedure mentioned above, the observation points used for the parameter identification should be changed so as to satisfy the condition (8). Therefore,

$$\hat{x}^j + \gamma(x^0 - \hat{x}^j) \rightarrow \hat{x}^j, \qquad j=1, 2, \cdots, N \tag{24}$$

where $0<\gamma\leq 1$. In order to obtain a globally optimal solution, the points $\hat{x}^j$ should be changed by (24) with a small γ and with the initial points scattered uniformly in the constraint sets given by (2).

4. Computational scheme

The computational procedure of the proposed algorithm is summarized in Fig. 1. The procedure consists of two parts. Part I calculates the solution obtained by a separate treatment. That is, first, an unconstrained minimization technique is used to obtain the solution β_s to the problem (5). Secondly, the simplex method is applicable to the problem of linear programming, (1), (2) and (4) with $\beta=\beta_s$. Phase 1 of the simplex method[7] finds a feasible solution or gives the information that none exists. If a solution does exist, Phase 2 uses this solution as a starting point and obtains the optimal solution x^0. If no feasible solution exists, the constraints (2) and (4) with $\beta=\beta_s$ are inconsistent. In that case, we require the joint treatment and proceed to Part II.

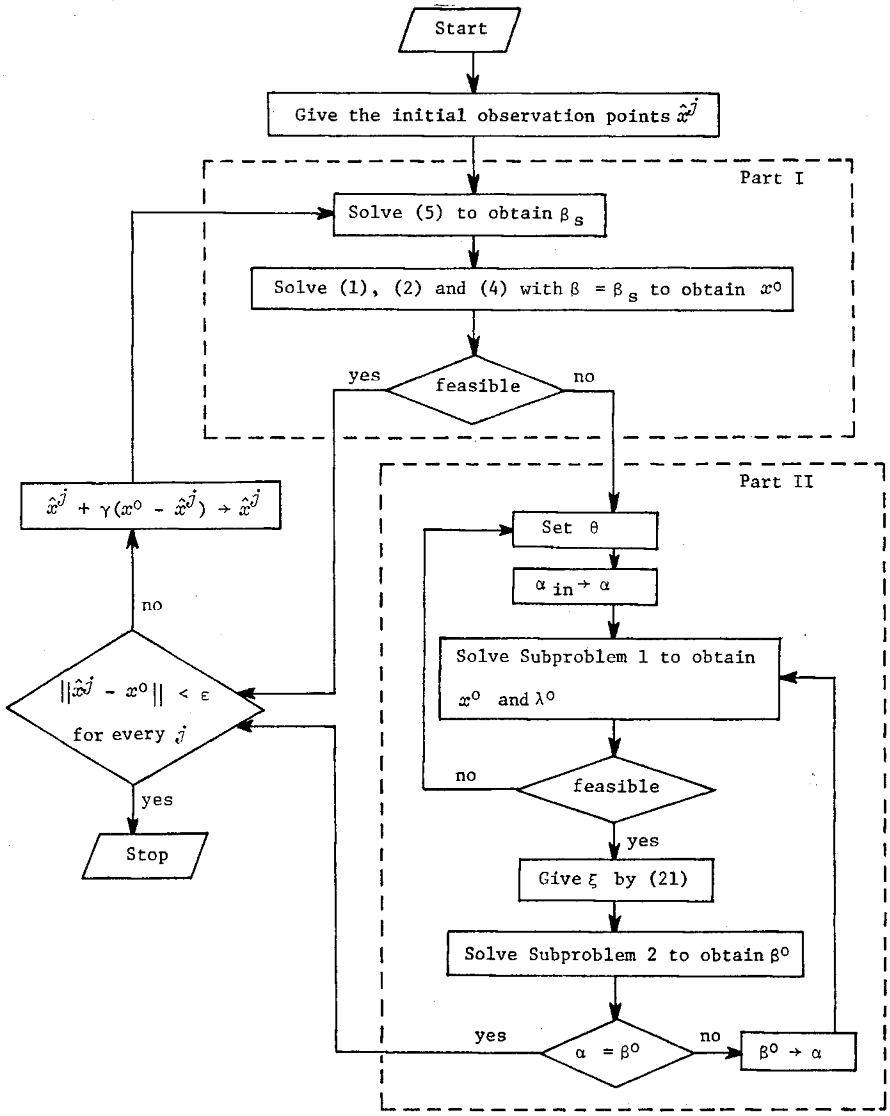

Fig. 1. The proposed algorithm.

Part II seeks the parametric solution to the joint problem developed in the previous section. Given θ and α, Subproblem 1 is first solved. Then Subproblem 2 is solved by use of (21). This first-level calculation is repeated until the stationarity condition (20) holds at the second level. If Subproblem 1 is nonfeasible, the parameter θ ($0<\theta<1$) should be changed to make the problem (13) feasible. The initial guess α_{in} of α must be chosen so that Subproblem 1 with $\alpha=\alpha_{in}$ has a feasible solution $(x^{0\prime}, \lambda^{0\prime})'$, where $\lambda^0 \neq 0$. Otherwise no repeated first-level calculation could be attained as a feasible solution. Such an α_{In} is obtained by modifying the result of the feasibility test in Part I.

If the calculation of Part I or Part II is terminated, change the observation points $\hat{x}^j$ by (24) and return to Part I. The process of replacing the observation data is continued until the condition (8) holds.

5. Illustrative examples

Two examples are presented to illustrate the application of the present algorithm.

Example 1

Consider a one-dimensional problem:

$$\min_x 2x \quad \text{subject to } 1 \leq x \leq 2 \text{ and } f(x) \leq 0 \tag{25}$$

We simply assume for $f(x)$:

$$h(x, \beta) = \beta - x \tag{26}$$

with $N=2$, *i.e.*, two observation points $\hat{x}^1$ and $\hat{x}^2$ being available.

The joint problem (9) for this example is convex and the efficient solution is obtained

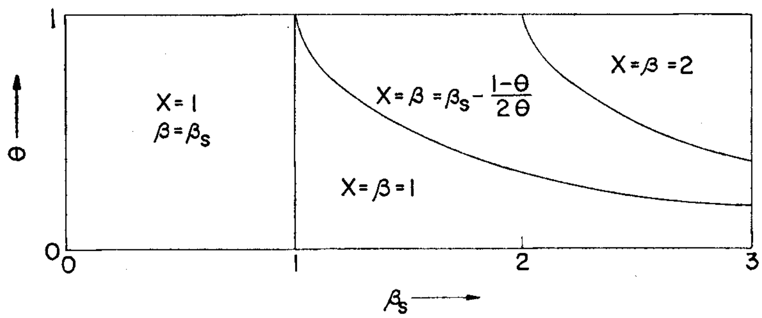

Fig. 2. Efficient solutions for various values of θ and β_s.

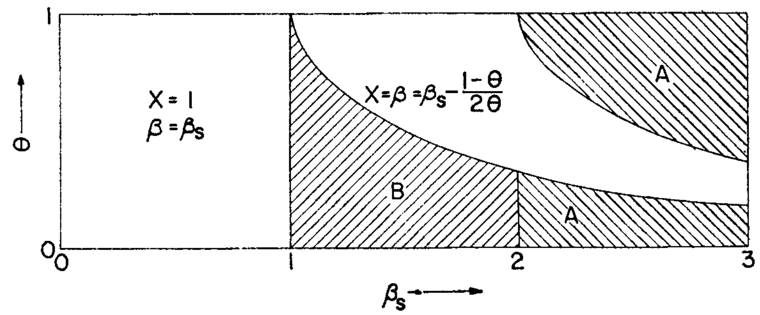

Fig. 3. The solution obtained by the present algorithm. The algorithm has no solution in A and does not converge in B.

exactly. Figure 2 shows the efficient solution to (9) for various values of θ and β_s, where β_s, the solution to (5), is given by

$$\beta_s = \frac{1}{2}(\hat{x}^1 + \hat{x}^2 + f^1 + f^2) \tag{27}$$

Figure 3 illustrates the result obtained by the present algorithm. The solution in the sense of Definition 1 lies on the line segment expressed by $\theta=1$ and $\beta_s \leq 2$. In the hatched region, the present algorithm has no solution (in A) or does not converge (in B). This occurs due to solving the problem (9) by a decomposition technique. However, from Fig. 3, there exists a parameter θ for which the present algorithm has a solution. The solution is identical with that shown in Fig. 2. Therefore, starting at any point on the line $\theta=0$, the algorithm can reach the solution in the sense of Definition 1 by adjusting θ. The line $\theta=1$ corresponds to a separate treatment, *i.e.,* the procedure in Part I. Note that the separate treatment is valid only for $\beta_s \leq 2$.

As an example, let $f(x)$ be given by

$$\begin{aligned} f(x) &= 1.9x^2 - 7.4x + 7.1 \\ &= 1.9(x-1.713)(x-2.182) \end{aligned} \tag{28}$$

although the form of (28) is assumed unknown and is not available. In this case, the set of observation points which requires the joint approach is outside the circle in Fig. 4. Starting at a point in the region $\beta_s > 2$, say, $\hat{x}^1=1$ and $\hat{x}^2=2$, the convergence of the observation points is shown in Fig. 4. Figure 5 shows the stepwise paths toward the solution $x^* = \beta^* = 1.713$ on the $\beta_s\theta$ plane.

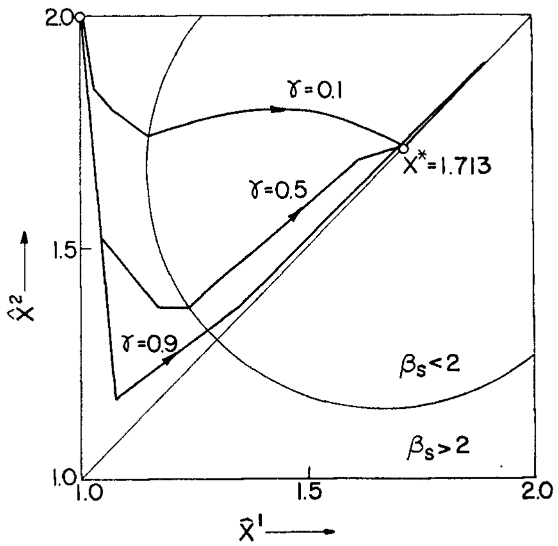

Fig. 4. Convergence of the observation points $\hat{x}^1$ and $\hat{x}^2$.

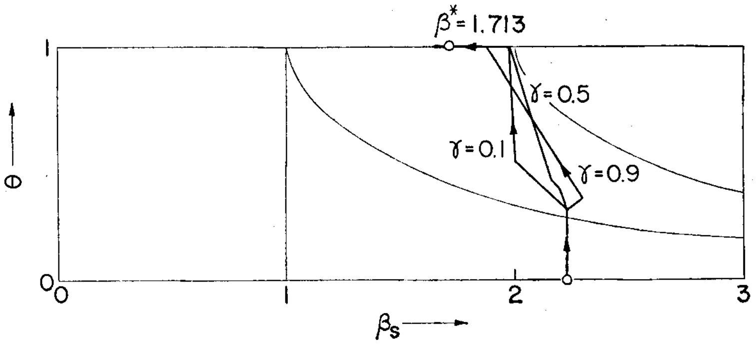

Fig. 5. Convergence of β.

Example 2

Consider a two-dimensional problem:

$$\left.\begin{array}{l} \min\limits_{x_1,x_2} (2x_1+x_2) \\ \text{subject to} \\ 1\leq x_1\leq 3,\ 0\leq x_2\leq 2 \text{ and } f(x_1, x_2)\leq 0 \end{array}\right\} \tag{29}$$

The form of $f(x_1, x_2)$ is assumed to be

$$h(x_1, x_2, \beta_1, \beta_2)=\beta_1 x_1-x_2+\beta_2 \tag{30}$$

The solution obtained by the separate treatment, *i.e.*, the solution corresponding to $\theta=1$ is shown on the $\beta_{1s}\beta_{2s}$ plane in Fig. 6, where (β_{1s}, β_{2s}) is the solution to (5). Equation (18) reduces to

$$\left.\begin{array}{l} \beta_i{}^0=\beta_{is}-\dfrac{\lambda}{2\theta\sigma}\Delta\beta_i \qquad (i=1, 2) \\ \text{where} \\ \Delta\beta_1\triangleq Nx_1{}^0-\sum\limits_{j=1}^{N}\hat{x}_1{}^j \\ \Delta\beta_2\triangleq -x_1{}^0\sum\limits_{j=1}^{N}\hat{x}_1{}^j+\sum\limits_{j=1}^{N}(\hat{x}_1{}^j)^2 \\ \sigma\triangleq N\sum\limits_{j=1}^{N}(\hat{x}_1{}^j)^2-(\sum\limits_{j=1}^{N}\hat{x}_1{}^j)^2>0 \end{array}\right\} \tag{31}$$

$\lambda(\geq 0)$ is the Lagrange multiplier for the constraint $h(x_1, x_2, \alpha_1, \alpha_2)\leq 0$. Note that $\Delta\beta_1<0$ and $\Delta\beta_2<0$ never holds. It then follows that $\beta_1{}^0<\beta_{1s}$ and/or $\beta_2{}^0<\beta_{2s}$. Also note that

$$\left.\begin{array}{ll} |\Delta\beta_2/\Delta\beta_1|>1 & \text{for } \Delta\beta_1<0 \text{ and } \Delta\beta_2>0 \\ |\Delta\beta_2/\Delta\beta_1|<3 & \text{for } \Delta\beta_1>0 \text{ and } \Delta\beta_2<0 \end{array}\right\} \tag{32}$$

Accordingly, the solution to Subproblem 2 can be transferred from the nonfeasible region

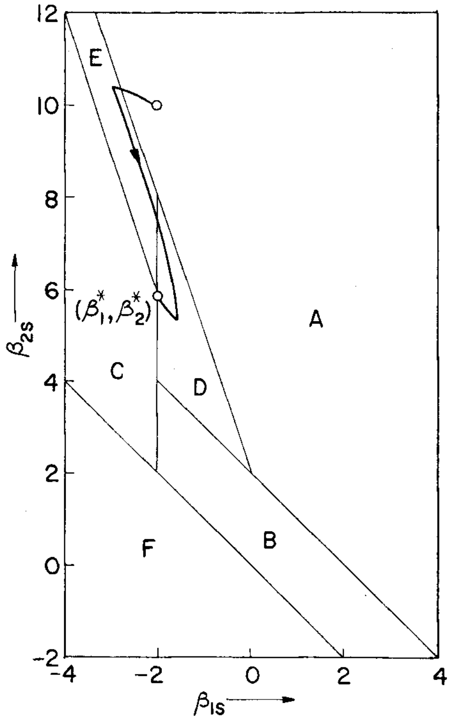

Fig. 6. The solution obtained by the separate treatment: A; nonfeasible, B; $x_1=1$, $x_2=\beta_1+\beta_2$, C; $x_1=-\beta_2/\beta_1$, $x_2=0$, D; $x_1=(2-\beta_2)/\beta_1$, $x_2=2$, E; $x_1=3$, $x_2=3\beta_1+\beta_2$, F; $x_1=1$, $x_2=0$. The thickline shows the stepwise path toward the solution $\beta_1^*=-2$ and $\beta_2^*=5.88$, where $\gamma=0.1$.

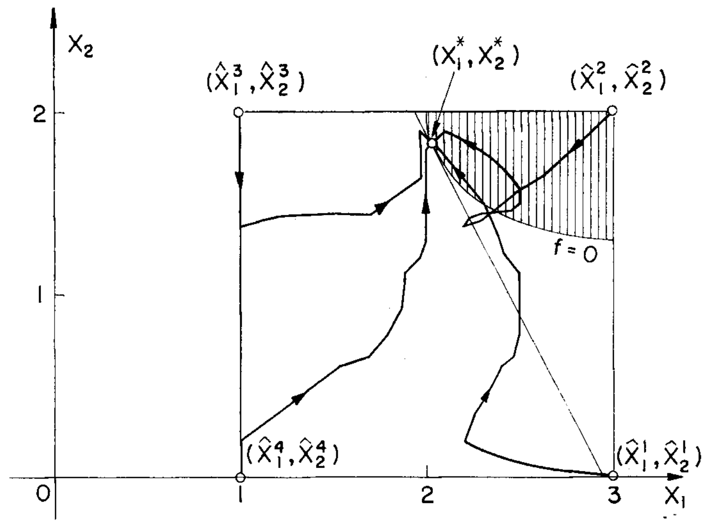

Fig. 7. Convergence of the observation points $(\hat{x}_1^j, \hat{x}_2^j)$. The solution is $x_1^*=2.04$ and $x_2^*=1.84$. The hatched region shows the true constraint set.

to the feasible region in Fig. 6, by an appropriate adjustment of θ.

By way of an example, take

$$f(x_1, x_2)=(x_1-3)^2+2(x_2-2)^2-1 \tag{33}$$

and choose the following four points:

$$(\hat{x}_1, \hat{x}_2)=(1, 0),\ (1, 2),\ (3, 0) \text{ and } (3, 2) \tag{34}$$

as the initial observation points. Consequently, $\beta_{1s}=-2$ and $\beta_{2s}=10$. The convergence of the solution is illustrated in Figs. 6 and 7. Note that the optimum of the problem (29) and (33) is not at an extreme point of the constraint set. It is observed that the iterative process leads to the point of tangency with the constraint set. At the end of iterations, Subproblem 1 has an infinite number of solutions. In this case the solution is obtained from the corresponding observation points.

6. Conclusion

An algorithm has been developed for solving a problem of linear programming with unknown constraints. A kind of three-level optimization technique is successfully used to obtain an efficient solution to the joint identification and optimization problem. The interaction between subproblems causes the nonfeasibility of the linear programming. The idea of coordination is to hold the feasibility by an adjustment of a scalar parameter varying over the unit interval. The validity of the procedure is shown by taking examples of small dimension. The convergence proof and error estimation of the algorithm are being investigated with application to problems of high dimension.

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Appendix

The present appendix summarizes the theory of the vector minimization problem.[2,5,6] Given a vector-valued criterion function $f(x)=(f_1(x), \cdots, f_r(x))'$ and a subset X of R^n, the vector minimization problem is

$$\min_x f(x) \qquad \text{subject to } x \in X \tag{A1}$$

The solution to the problem posed in (A1) lies in the set of efficient points defined as

Definition A *A point x_e is said to be an efficient point of* (A1) *if $x_e \in X$, and there exists no other feasible point x such that $f(x) \leq f(x_e)$ and $f(x) \neq f(x_e)$.*

The efficient solution is also known as the noninferior solution or Pareto optimal solution in economics and game theory.

The fundamental results characterizing the efficient point are given in the following theorems.

Theorem A1 *If x_e is efficient in* (A1), *then there exists a vector $\lambda=(\lambda_1, \cdots, \lambda_r)'$ with $\lambda_i \geq 0$ and*

$$\sum_{i=1}^{r} \lambda_i = 1$$

such that x_e is optimal in the scalar minimization problem:

$$\min_x \lambda' f(x) \qquad \text{subject to } x \in X \tag{A2}$$

Theorem A2 *Let X be a convex set, and let the $f_i(x)$ be convex on X. Then x_e is efficient in* (A1) *if and only if x_e is optimal in* (A2) *for some λ with strictly positive components.*

As for the proof of the theorems, refer to Ref. 5).

From a computational viewpoint, finding efficient solutions is reduced to a parametric programming problem. In the case of $r=2$, *i.e.*, the bicriterion minimization problem, Haimes *et al.*[2] proposed another method, called the ϵ-constraint approach. The method involves replacing one of the criterion functions by a constraint and constructing an ordinary minimization problem. Their theorem is

Theorem A3 *Let $\varepsilon \geq \min_x f_2(x)$ and let x_e be a solution to*

$$\min_x f_1(x) \qquad \textit{subject to } x \in X \textit{ and } f_2(x) \leq \varepsilon \tag{A3}$$

Further assume that, if x_ε *is not unique, then* x_ε *is an optimal solution to* (A3) *with minimum value of* $f_2(x)$. *Then* x_ε *is efficient in* (A1).

In the problem treated here, $f_1(x)$ is linear and the constraint X is defined by a set of linear inequalities, but $f_2(x)$ is a nonlinear function. Accordingly, applying the ε-constraint approach to the present problem is reduced to a nonlinear programming problem. Then, we have used the parametric approach to solving the problem (7).

Geoffrion[6] has proposed a slightly restricted definition of efficiency, called proper efficiency, in order to eliminate efficient points of a certain anomalous type. As to the problem considered in this paper, there is no significant difference between the two concepts of efficiency. For that reason, the details are omitted here.