Perturbation of Sounds in Liquid Helium II by Relative Motion between Super and Normal Fluids

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Abstract

Liquid helium II is represented by Landau's two-fluid model and the sounds are defined as the singular surfaces in it. The propagation velocities of the first, second and fourth sounds are perturbed by the relative motions between the super and normal fluids. The perturbed velocities are calculated whithin the first order approximation of the relative velocity.

1. Introduction

The phenomenon of the *superfluidity* of *liquid helium II* at a temperature below 2.19 K was first analyzed by Landau[1] in 1941; and he introduced the *two-fluid model* for it, where helium II is regarded as a mixture of two different liquids, that is, a super fluid and a normal fluid. By means of this model, Landau showed theoretically the existence of the *first* and *second sounds* in liquid helium II, where two fluids were assumed to have no relative motion.

In sufficiently narrow capillaries, the super fluid may flow through, but the normal fluid must be stationary. Then there is a peculiar sound called the *fourth sound*, which was studied by Atkins.[2]

For references on the subject, refer to Landau and Lifshitz,[3] and Khalatnikov.[4),5]

In general, we have theoretically three kinds of waves, that is, the *singular surface*, the *characteristic*, and the *harmonic oscillation*, which give the same propagation speeds in many cases. The method of the singular surface has been applied, in the past several decades, to the sound propagation in the large variety of continua. For general reference on the subjects, refer to Truesdell and Toupin,[6] and Truesdell and Noll.[7]

The author[8] applied the theory of singular surface to the two-fluid model of liquid helium II, and he investigated theoretically the sounds in it.

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In this paper, the analysis adopted in the former article[8] is applied to the sound propagation in liquid helium II with a small relative motion between the super fluid and the normal fluid. The perturbed propagation velocities of the first, second and fourth sounds are calculated.

2. Easic Equations

The two-fluid model has two densities ρ_s and ρ_n, and two velocities $\boldsymbol{v}_s$ and $\boldsymbol{v}_n$ at every point. Also, there can exist two simultaneous but independent motions, where suffixes s and n denote, respectively, the super and the normal parts of the fluid. We assume that the magnitude of the relative velocity

$$\boldsymbol{w} \equiv \boldsymbol{v}_s - \boldsymbol{v}_n \tag{1}$$

is not so large that the theory is physically significant.

There are four kinds of field equation: the equation of continuity, the equation of conservation of entropy, the equation of potential flow for the super fluid, and the equation of conservation of momentum. They are expressed as

$$\frac{\partial \rho}{\partial t} + \rho_s \nabla \cdot \boldsymbol{v}_s + \rho_n \nabla \cdot \boldsymbol{v}_n + \boldsymbol{w} \cdot \nabla \rho_s = 0\,, \tag{2}$$

$$s\frac{\partial \rho}{\partial t} + \rho \frac{\partial s}{\partial t} + \rho s \nabla \cdot \boldsymbol{v}_n = 0\,, \tag{3}$$

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s + \nabla \mu = 0\,, \tag{4}$$

$$\rho_s \frac{\partial \boldsymbol{v}_s}{\partial t} + \rho_n \frac{\partial \boldsymbol{v}_n}{\partial t} + \nabla p + \boldsymbol{w}\left(\frac{\partial \rho_s}{\partial t} + \rho_s \nabla \cdot \boldsymbol{v}_s\right) = 0 \tag{5}$$

in a coordinate system which moves with a constant velocity being equal to that of the normal fluid at a point and a time concerned, where $\rho \equiv \rho_s + \rho_n$: the density, s: the entropy density, μ: the chemical potential, p: the pressume, and p, s, μ, and ρ_s are assumed to be functions of the independent state variable ρ, the temperature T, and $\boldsymbol{w}$. From the familiar relations of the thermodynamics and the property of isotropic function we have

$$dp = u^2 d\rho + \rho n^2 \alpha dT + \frac{1}{2} a dw^2\,, \tag{6}$$

$$ds = -\frac{u^2 \alpha}{\rho} d\rho + \frac{c_V}{T} dT + \frac{1}{2} b dw^2\,, \tag{7}$$

$$d\mu = \frac{u^2}{\rho} d\rho - (s - u^2 \alpha) dT + \frac{1}{2} c dw^2\,, \tag{8}$$

$$d\rho_s = e d\rho + f dT + \frac{1}{2} g dw^2\,, \tag{9}$$

where we put

$$u\equiv\left(\frac{\partial p}{\partial \rho}\right)^{1/2}_{T,w}, \qquad \alpha\equiv-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p,w}, \qquad c_{\mathrm{V}}\equiv T\left(\frac{\partial s}{\partial T}\right)_{\rho,w}, \tag{10}$$

$$a\equiv2\left(\frac{\partial p}{\partial w^2}\right)_{\rho,T}, \qquad b\equiv2\left(\frac{\partial s}{\partial w^2}\right)_{\rho,T}, \qquad c\equiv2\left(\frac{\partial \mu}{\partial w^2}\right)_{\rho,T}, \tag{11}$$

$$e\equiv\left(\frac{\partial \rho_{\mathrm{s}}}{\partial \rho}\right)_{T,w}, \qquad f\equiv\left(\frac{\partial \rho_{\mathrm{s}}}{\partial T}\right)_{\rho,w}, \qquad g\equiv2\left(\frac{\partial \rho_{\mathrm{s}}}{\partial w^2}\right)_{\rho,T}, \tag{12}$$

α and c_{V} are called, respectively, the thermal expansion coefficient and the specific heat at constant volume.

Now we define a surface as the *sound* in liquid helium II if the following two conditions hold:

(i) *ρ, T, $\boldsymbol{v}_{\mathrm{s}}$ and $\boldsymbol{v}_{\mathrm{n}}$ are continuous everywhere,*

(ii) *the first derivatives of them have jump discontinuities across the surface but are continuous everywhere else.*

The geometrical and kinematical compatibility conditions of the first order of ψ are given by[6]

$$[\nabla\psi]=\boldsymbol{n}\bar{\psi}, \quad \left[\frac{\partial\psi}{\partial t}\right]=-U\bar{\psi}, \quad \bar{\psi}\equiv\boldsymbol{n}\cdot[\nabla\psi], \tag{13}$$

where $\boldsymbol{n}$ is the normal of the sound surface and U is the normal velocity of it.

Applying the compatibility conditions (13) to the field equations (2)-(5) and to the condition of potential flow

$$\nabla\times\boldsymbol{v}_{\mathrm{s}}=0, \tag{14}$$

for the super fluid, and neglecting the second and higher order terms of $\boldsymbol{w}$, we have five independent field equations

$$Q_{KM}(U,\boldsymbol{w})a_M=0 \qquad (K=1,\cdots,5), \tag{15}$$

where

$$Q_{KM}(U,\boldsymbol{w})\equiv$$
$$\left\|\begin{matrix} -U+ew_l & fw_l & \rho & -\rho_{\mathrm{n}} & 0 \\ -U(s-u^2\alpha) & -U\dfrac{\rho c_{\mathrm{V}}}{T} & \rho s & -\rho s-U\rho bw_l & -U\rho bw_t \\ \dfrac{u^2}{\rho} & -(s-u^2\alpha) & -U+w_l & cw_l & cw_t \\ u^2-Uew_l & \rho u^2\alpha-Ufw_l & -U\rho+2\rho_{\mathrm{s}}w_l & U\rho_{\mathrm{n}}+aw_l & aw_t \\ -Uew_t & -Ufw_t & \rho_{\mathrm{s}}w_t & 0 & U\rho_{\mathrm{n}} \end{matrix}\right\|, \tag{16}$$

$$a_K \equiv (\bar{\rho},\ \bar{T},\ \bar{v}_s,\ \bar{w}_l,\ \bar{w}_t)\ . \tag{17}$$

Here, the bar denotes the jump discontinuity of the respective quantity, the subscripts l and t denote, respectively, the normal and the tangential components for the singular surface; and the above quantities are evaluated at $\boldsymbol{w}=\boldsymbol{0}$.

3. Perturbation of First and Second Sounds

The definition of the singular surface depicted in the above section demands that the surface may exist if the *propagation conditions*:

$$\det Q_{KM}(U,\ \boldsymbol{w}) = 0 \tag{18}$$

holds. The solution U of (18) is called the *propagation velocity*, and the non-vanishing a_K satisfying (15) under the condition (18) is called the *amplitude*.

From (16) we can easily show that

$$\det Q_{KM}(0,\ \boldsymbol{w}) = 0 \tag{19}$$

holds within the second order of velocity. Referring to the matrix (16), the propagation condition (18) and the relation (19), and neglecting the terms of $O(w^2)$, we can easily show that the field equations (15) are reduced to

$$R'_{\Gamma\Delta}(U',\ \boldsymbol{w}')a'_{\Delta} = 0 \qquad (\Gamma = 1, \cdots, 4) \tag{20}$$

for non-vanishing propagation velocity, where

$$R'_{\Gamma\Delta}(U',\ \boldsymbol{w}') \equiv \left\| \begin{matrix} -U'+e'w'_l & \alpha Tf'w'_l & 1 & -\rho'_n \\ U'\left(1-\dfrac{u^2\alpha}{s}\right) & -U'\dfrac{c_V}{s} & 1 & -1-U'b'w'_l \\ 1 & -\left(\dfrac{sT}{u^2}-\alpha T\right) & -U'+w'_l & c'w'_l \\ 1-U'e'w'_l & \alpha T(1-U'f'w'_l) & -U'+2\rho'_s w'_l & U'\rho'_n+a'w'_l \end{matrix} \right\|, \tag{21}$$

$$a'_{\Gamma} \equiv (\bar{\rho}',\ \bar{T}',\ \bar{v}'_s,\ \bar{w}'_l)\ , \tag{22}$$

$$U' \equiv \frac{U}{u}\ , \quad \rho'_s \equiv \frac{\rho_s}{\rho}\ , \quad \rho'_n \equiv \frac{\rho_n}{\rho}\ , \quad w'_l \equiv \frac{w_l}{u}\ , \tag{23}$$

$$\bar{\rho}' \equiv \frac{l}{\rho}\bar{\rho}\ , \quad \bar{T}' \equiv \frac{l}{T}\bar{T}\ , \quad \bar{v}'_s \equiv \frac{l}{u}\bar{v}_s\ , \quad \bar{w}'_l \equiv \frac{l}{u}\bar{w}_l\ , \tag{24}$$

$$a' \equiv \frac{a}{\rho}\ , \quad b' \equiv \frac{u^2}{s}b\ , \quad c' \equiv c\ , \quad e' \equiv e\ , \quad f' \equiv \frac{f}{\alpha\rho}\ , \tag{25}$$

l is a reference length, and the prime indicates a dimensionless quanity.

After a brief manipulation of det $R'_{\Gamma\Delta}(U', \boldsymbol{w}')=0$, we have the propagation condition:

$$U'^4+U'^3Aw'_l-U'^2(1+\beta^2+\tau)+U'Bw'_l+\beta^2=0\,, \tag{26}$$

where

$$\beta\equiv\left(\frac{\rho_s s^2 T}{\rho_n u^2 c_V}\right)^{1/2}, \quad \tau\equiv\frac{u^2\alpha^2 T}{c_V}, \tag{27}$$

$$A\equiv-1+\frac{1}{\rho'_n}\left(a'-\frac{\rho_n}{\rho_s}\beta^2 b'-c'+\sqrt{\frac{\rho_n\tau}{\rho_s}}\beta f'\right), \tag{28}$$

$$B\equiv-A-2\rho'_s+2\rho'_s\left(\beta+\sqrt{\frac{\rho_n\tau}{\rho_s}}\right)\left(\beta-\sqrt{\frac{\rho_s\tau}{\rho_n}}\right)$$
$$+\frac{\tau}{\rho'_n}(\rho'_n-a'+c')+\frac{1}{\rho'_n}\sqrt{\frac{\rho_n\tau}{\rho_s}}\beta(a'+e'-\rho'_s)\,. \tag{29}$$

In the case $w'_l=0$, (26) reduces to

$$U_0'^4-U_0'^2(1+\beta^2+\tau)+\beta^2=0\,, \tag{30}$$

which has the solutions:

$$U'_{0(\alpha)}=\sqrt{\frac{1}{2}\{1+\beta^2+\tau\pm\sqrt{(\beta^2-1)^2+2(\beta^2+1)\tau+\tau^2}\}} \quad (\alpha=1,\,2), \tag{31}$$

where the plus and minus signs in (31) refer, respectively, to the first $(\alpha=1)$ and second $(\alpha=2)$ sounds. In the case $w'_l\neq 0$ we can easily obtain the solutions of (26) by the form

$$U'_{(\alpha)}=U'_{0(\alpha)}+U'_{1(\alpha)}w'_l\,, \qquad (\alpha=1,\,2)\,, \tag{32}$$

where

$$U'_{1(1)}\equiv-\frac{U'^2_{0(1)}A+B}{2U'_{0(1)}\sqrt{(\beta^2-1)^2+2(\beta^2+1)\tau+\tau^2}}\,, \tag{33}$$

$$U'_{1(2)}\equiv\frac{U'^2_{0(2)}A+B}{2U'_{0(2)}\sqrt{(\beta^2-1)^2+2(\beta^2+1)\tau+\tau^2}} \tag{34}$$

The ratio of the amplitudes a'_Γ of the perturbed sounds can be calculated easily and given by the ratio of the co-factors of the elements of a row of the matrix (21). Here, for simplicity, the explicit depiction of their expressions is omitted.

4. Perturbation of Fourth Sound

In narrow capillaries, the normal fluid must be stationary. In this case the first order approximation of the field equations are given by[8]

$$R'_{km}(U', \boldsymbol{w}')a'_m = 0 \qquad (k=1, 2, 3)\ , \tag{35}$$

where

$$R'_{km}(U', \boldsymbol{w}') \equiv \left\| \begin{array}{ccc} -U'+e'w'_l & \alpha Tf'w'_l & \rho'_s \\ -U'\left(1-\dfrac{u^2\alpha}{s}\right) & -U'\dfrac{c_V}{s} & -U'b'w'_l \\ 1 & -\left(\dfrac{sT}{u^2}-\alpha T\right) & -U'+(1+c')w'_l \end{array} \right\|, \tag{36}$$

$$a'_k \equiv (\bar{\rho}', \bar{T}', \bar{v}'_s)\ . \tag{37}$$

In the case of non-vanishing velocity the propagation condition $\det R'_{km}(U', \boldsymbol{w}')=0$ reduces to

$$U'^2+U'Cw'_l-\rho'_s-\rho'_n\left(\beta-\sqrt{\frac{\rho_s\gamma}{\rho_n}}\right)^2=0\ , \tag{38}$$

where

$$C\equiv -1-\frac{\rho_n}{\rho_s}\beta\left(\beta-\sqrt{\frac{\rho_s\gamma}{\rho_n}}\right)b'-c'-e'+\sqrt{\frac{\rho_n\gamma}{\rho_s}}\left(\beta-\sqrt{\frac{\rho_s\gamma}{\rho_n}}\right)f'. \tag{39}$$

For the zeroth order approximation $w'_l=0$ we have the propagation velocity:

$$U'_0=\sqrt{\rho'_s+\rho'_n\left(\beta-\sqrt{\frac{\rho_s\gamma}{\rho_n}}\right)^2} \tag{40}$$

and for the first order approximation we have the perturbed propagation velocity:

$$U' = U'_0+U'_1w'_l\ , \tag{41}$$

where

$$U'_1=-\frac{C}{2}\ . \tag{42}$$

The perturbation of the ratio of the amplitude is also easily obtained by the same method mentioned in the last part of the above section. Its explicit depiction is omitted.

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