Random Response Analysis of a Long-span Suspension Bridge Tower and Pier with Consideration of Nonlinear Foundation

By

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(Received June 29, 1974)

Synopsis

The nonlinear foundation effect upon the response of the tower and pier system of a long-span suspension bridge is investigated by the random vibration theory. As the equivalent linearlization yields the nonproportional damping matrix, the exact response is evaluated both through the frequency domain analysis and the complex mode analysis. Also discussed is the approximate response analysis using classical normal modes for the practical design procedure.

1. Introduction

The random response analysis of the towar and pier system of a long span suspension bridge is dealt with in this paper. This system, comprising the very flexible towar shaft and the rigid body pier, shows such a complicated vibrational behavior that the classical normal modes do not exist due to the drastically divergent damping distribution along the system. That is, the damping effect at the tower shaft ascribes mainly to the structural one of small magnitude, while that of the pier coming from the interaction between this and soil might be great upon consideration of the latter's geometric and hysteretic energy dissipation. This makes the so-called non-proportional damping matrix for the system.

To solve the above model, two different methods of analysis are considered; namely, the frequency domain analysis and the complex modal analysis. The primary interest is placed on the evaluation of the nonlinear soil effect upon the system response and on the proper assignment of the clasical modal damping factor for a practical design procedure. Throughout this invstigation, the stationary white noise process is used as an excitation. This would be acceptable to detect the

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general dynamic trend of the system.

2. Governing Equation

A dynamic modelling is made for the present structure, such that the tower shaft is replaced by the eight-lumped mass system, and the pier is a rigid body free to rotate and translate around the rotational center, R. The illustration is given in Fig. 1. The corresponding governing equation is then expressed in a matrix-vector form as, when the system is subjected to the base acceleration of $\ddot{z}_0(t)$,

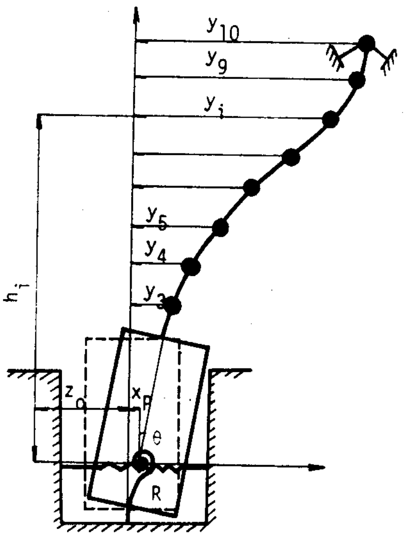


Fig. 1 System Considered.

$$[m]\{\ddot{y}\}+[c]\{\dot{y}\}+[k]\{y\} = \{F\}\ddot{z}_0(t) \qquad (2.1)$$

where $\{y\}$ is the response vector, representing the pier's rotation and translation together with the tower's displacement and $\{F\}^T = -\{0\ 1 \cdots 1\}[m]$; and $[m]$, $[c]$ and $[k]$ denote, respectively, mass, damping and stiffness matrices. For the elements of $[m]$ and $[k]$, the authors' previous papers[1),2)] can be referred to, and for those of $[c]$ the following consideration is made.

So far, the so-called proportional damping matrix has frequently been assumed for response analysis to take advantage of the classical normal mode method. This would be acceptable for the relatively uniform structure. However, for the present system, the interaction between soil and pier being an important aspect, imposing the damping effect independently upon the tower and the pier seems to be preferable. This results in a system of the so-called non-proportional damping matrix.

Expressing the vector $\{y\}$ in terms of pier's displacements $\left\{\begin{matrix}\theta \\ x_p\end{matrix}\right\}$ and tower's deflection $\{x\}$, one can get

$$\{y\} = [[\{h\} \mid \{1\}] \mid [I]]\left\{\begin{matrix}\left\{\begin{matrix}\theta \\ x_p\end{matrix}\right\} \\ \hline \{x\}\end{matrix}\right\} \tag{2.2}$$

where $\{h\}$ is the vector of the mass height from the rotation center R, $\{1\}$ is the vector of 1 and $[I]$ is the unit matrix. The use of Eq. (2.2) for the undamped system, with the following partioning of the mass and the stifiness matrices

$$[m] = \left[\begin{array}{c|c}[m]_P & [0] \\ \hline [0] & [m]_T\end{array}\right] \qquad [k] = \left[\begin{array}{c|c}[k]_P & [k]_{PT} \\ \hline [k]_{TP} & [k]_T\end{array}\right]$$

yields the governing equation of

$$\left[\begin{array}{c|c}[m]_P & [0] \\ \hline [[m]_T\{h\} \mid [m]_T\{1\}] & [m]_T\end{array}\right]\left\{\begin{matrix}\left\{\begin{matrix}\ddot{\theta} \\ \ddot{x}_p\end{matrix}\right\} \\ \hline \{\ddot{x}\}\end{matrix}\right\} + \left[\begin{array}{c|c}[k]_P + [k]_{PT}[\{h\} \mid \{1\}] & [k]_{PT} \\ \hline [k]_{TP} + [k]_T[\{h\} \mid \{1\}] & [k]_T\end{array}\right]\left\{\begin{matrix}\left\{\begin{matrix}\theta \\ x_p\end{matrix}\right\} \\ \hline \{x\}\end{matrix}\right\}$$

$$= -\left[\begin{array}{c|c}[m]_P & [0] \\ \hline [0] & [m]_T\end{array}\right]\left\{\begin{matrix}\left\{\begin{matrix}0 \\ 1\end{matrix}\right\} \\ \hline \{1\}\end{matrix}\right\}\ddot{z}_0 \tag{2.3}$$

From this, the quation of motion for the pier part is found to be

$$[m]_P\left\{\begin{matrix}\ddot{\theta} \\ \ddot{x}_p\end{matrix}\right\} + ([k]_P + [k]_{PT}[\{h\} \mid \{1\}])\left\{\begin{matrix}\theta \\ x_P\end{matrix}\right\} = [k]_{PT}\{x\}$$
$$-[m]_P\left\{\begin{matrix}0 \\ 1\end{matrix}\right\}\ddot{z}_0 \tag{2.4.a}$$

and that for the tower part, upon consideration of $[k]_{TP} + [k]_T[\{h\} \mid \{1\}] = [0]$, is to be

$$[m]_T\{\ddot{x}\} + [k]_T\{x\} = -[m]_T(\{h\}\ddot{\theta} + \{1\}\ddot{x}_p + \{1\}\ddot{z}_0) \tag{2.4.b}$$

Now introduce the damping effect for the pier part of

$$c_\theta = 2\beta_\theta\sqrt{K_\theta I_p} \qquad \text{for rotation}$$
$$c_h = 2\beta_h\sqrt{K_h m_p} \qquad \text{for translation}$$

where β_θ and β_h are the damping factors for rotation and translation, respectively, and I_p is the inertia of moment of the pier around axis R, and m_p is the mass of the pier. The associated motion is then governed by

$$[m]_P\begin{Bmatrix}\ddot{\theta}\\ \ddot{x}_p\end{Bmatrix}+\begin{bmatrix}c_\theta & 0\\ 0 & c_h\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ \dot{x}_p\end{Bmatrix}+([k]_P+[k]_{PT}[\{h\}|\{1\}])\begin{Bmatrix}\theta\\ x_p\end{Bmatrix} = [k]_{PT}\{x\} -[m]_P\begin{Bmatrix}0\\ 1\end{Bmatrix}\ddot{z}_0 \quad (2.5.a)$$

The tower part, on the other hand, is assumed to have the classical normal modes. This leads to the following equation in normal mode co-ordinates $\{q\}$, when Eq. (2.4.b) is transformed by the modal matrix $[V]$ for the tower part only.

$$[I]\{\ddot{q}\}+[\diagdown 2\beta_i\omega_i \diagdown]\{\dot{q}\}+[\diagdown \omega_i^2 \diagdown]\{q\} = -[V]^T[m]_T(\{h\}\ddot{\theta}+\{1\}\ddot{x}_P+\{1\}\ddot{z}_0) \quad (2.5.b)$$

where $[V]^T[m]_T[V] = [I]$, $[V]^T[k]_T[V] = [\diagdown \omega_i^2 \diagdown]$

and $[V]^T$ is the transposition of $[V]$. In Eq. (2.5.b), ω_i and β_i denote, respectively, the natural frequency and damping factor for the tower's i-th mode. The entire system equation thus obtained becomes the form of Eq. (2.1) in the physical co-ordinates with the damping matrix of

$$[c] = \begin{bmatrix}\begin{bmatrix}c_\theta & 0\\ 0 & c_h\end{bmatrix} & [0]\\ [0] & ([V]^T)^{-1}[\diagdown 2\beta_i\omega_i \diagdown][V]^{-1}\end{bmatrix} \quad (2.6)$$

This kind of damping matrix has been used for the entire system analysis by the authors[3]. Recently, it is also proposed by Pajuhesh and Hadjian[4] as the composite damping matrix for the analysis of soil-structure interaction. Eq. (2.6) includes the damping matrix proposed in Ref. (5) as a special case, where the pier's motion is restricted only to rocking about the R axis and the tower's damping is such that $\beta_i=\beta_T$ (const.).

3. Exact Response Analysis

The response of the system considered can either be obtained by solving Eq. (2.5.a) and (2.5.b), or by Eq. (2.1). The former approach, which is first considered, is to use the frequency domain analysis[6] and the latter, which is taken secondly, is to lead the complex eigenvalue problem.[7]

Taking the Fourier transform of Eq. (2.5.b), one can get

$$\{Q\} = [\diagdown H_{T_i}(\omega)\diagdown][V]^T[m]_T\{(\{h\}\Theta+\{1\}X_P)\omega^2+\ddot{Z}_0\} \quad (3.1)$$

where $\{Q\}\subset\{q\}$ $\Theta\subset\theta$ $X_p\subset x_p$ $\ddot{Z}_0\subset\ddot{z}_0$

and the notation $\subset$ means the Fourier transform, and $H_{T_i}(\omega)$ is the frequency response function for the tower modes as defined by

$$H_{T_i}(\omega) = \frac{1}{\omega_i^2 - \omega^2 + i2\beta_i\omega_i\omega} \qquad i\text{: imaginary unit} \tag{3.2}$$

Substituting Eq. (3.1) into the Fourier transform of Eq. (2.5.a), and solving for $\begin{Bmatrix} \Theta \\ X_p \end{Bmatrix}$, one can get

$$\begin{Bmatrix} \Theta \\ X_P \end{Bmatrix} = \begin{Bmatrix} H_\theta(\omega) \\ H_{x_p}(\omega) \end{Bmatrix} \ddot{Z}_0 \tag{3.3}$$

where $H_\theta(\omega)$ and $H_{x_p}(\omega)$ are, respectively, the frequency response functions for rotation and translation of the pier; and they are calculated as follows

$$\begin{Bmatrix} H_\theta(\omega) \\ H_{x_p}(\omega) \end{Bmatrix} = [\omega^2(-[m]_P + [k]_{PT}[V][\diagdown H_{T_i}(\omega)\diagdown][V]^T[m]_T[[h]|\{1\}]) + i\omega\begin{bmatrix} c_\theta & 0 \\ 0 & c_h \end{bmatrix} + \\ + [k]_P + [k]_{PT}[\{h\}|\{1\}]]^{-1} \cdot ([k]_{PT}[V][\diagdown H_{Ti}(\omega)\diagdown][V]^T[m]_T\{1\} - [m]_P\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}) \tag{3.4}$$

In doing operation in Eq. (3.4), one may reasonably save the computing time by trunacting the number of modes, instead of taking all the modes. In this case

$$[V][\diagdown H_{T_i}(\omega)\diagdown][V]^T = [\sum_{l=1}^{M} H_{T_l}(\omega) V_{il} V_{jl}] \tag{3.5}$$

where M is the number of modes used ($M<N$: N is the number of degrees of freedom).

The Fourier transform of tower modes' response is found as, when Eq. (3.3) is substituted into Eq. (3.1), i.e.

$$\{Q\} = \{H_{q_i}(\omega)\} \ddot{Z}_0 \tag{3.6}$$

where

$$\{H_{q_i}(\omega)\} = [\diagdown H_{T_i}(\omega)\diagdown][V]^T[m]_T([\{h\}|\{1\}]\begin{Bmatrix} H_\theta(\omega) \\ H_{x_P}(\omega) \end{Bmatrix}\omega^2 + \{1\}) \tag{3.7}$$

When the excitation is given by a random process, the primary interest is to obtain the response covariance matrix, i.e.

$$E[\{y\}\{y\}^T] = E\left[\begin{Bmatrix} \theta \\ x_p \\ \{y\} \end{Bmatrix} \{\theta x_p \{y\}^T\}\right] = [T]E\left[\begin{Bmatrix} \theta \\ x_p \\ \{q\} \end{Bmatrix} \{\theta x_p \{q\}^T\}\right][T]^T$$

where $E[\cdot]$ denotes the expectation of random quantity and

$$[T] = \left[\begin{array}{c|c} [I] & [0] \\ \hline [\{h\}|\{1\}] & [V] \end{array}\right]$$

For the stationary random process, the covariance matrix is related to the power spectral density matrix through the Wiener-Kintchine formula, such that

$$E[\{y\}\{y\}^T] = \int_{-\infty}^{\infty} [S_{yy}]\, d\omega \tag{3.8}$$

The power spectral density matrix $[S_{yy}]$ is given from Eqs. (3.3) and (3.6) as

$$[S_{yy}] = [T]\left[\begin{array}{c|c} [\begin{Bmatrix} H_\theta \\ H_{x_p} \end{Bmatrix}\{H_\theta^* H_{x_p}^*\}] & \begin{Bmatrix} H_\theta \\ H_{x_p} \end{Bmatrix}\{H_q^*\}^T] \\ \hline [\{H_q\}\{H_\theta^* H_{x_p}^*\}] & [\{H_q\}\{H_q^*\}^T] \end{array}\right][T]^T S_{\ddot{z}_0}(\omega) \tag{3.9}$$

Where $S_{\ddot{z}_0}(\omega)$ is the power spectral density of the input acceleration $\ddot{z}_0(t)$, and the asterisc in the above means the complex conjugate. In particular, the variances of the pier's rotation and translation are from Eq. (3.3)

$$\begin{aligned} \sigma_\theta^2 &= E[\theta^2] = \int_{-\infty}^{\infty} |H_\theta(\omega)|^2 S_{\ddot{z}_0}(\omega)\, d\omega \\ \sigma_{x_p}^2 &= E[x_p^2] = \int_{-\infty}^{\infty} |H_{x_p}(\omega)|^2 S_{\ddot{z}_0}(\omega)\, d\omega \end{aligned} \tag{3.10}$$

The alternative method for the response covariance matrix is to solve Eq. (2.1) by converting it into the first order differential equation as suggested by Foss.[7] Introducing the state vector $\{u\}^T = \{\{\dot{y}\}^T | \{y\}^T\}$ into Eq. (2.1), one can get the governing equation of

$$[A]\{\dot{u}\} + [B]\{u\} = \{P\} \tag{3.11}$$

where

$$[A] = \left[\begin{array}{c|c} [0] & [m] \\ \hline [m] & [c] \end{array}\right] \quad [B] = \left[\begin{array}{c|c} -[m] & [0] \\ \hline [0] & [k] \end{array}\right] \quad [P] = \left[\begin{array}{c} \{0\} \\ \hline \{F\}\ddot{z}_0 \end{array}\right]$$

or pre-multiplying Eq. (3.11) by $[A]^{-1}$,

$$\{\dot{u}\} + [D]\{u\} = [A]^{-1}\{P\} \tag{3.12}$$

where

$$[D] = \left[\begin{array}{c|c} [m]^{-1}[c] & [m]^{-1}[k] \\ \hline -[I] & [0] \end{array}\right] \tag{3.12}$$

The orthogonal co-ordinates' transformation of

$$\{u\} = [\Phi]\{r\} \tag{3.13}$$

leads Eq. (3.11) into

$$\{\dot{r}\} + [\diagdown \lambda_i \diagdown]\{r\} = [\diagdown 1/\gamma_i \diagdown][\Phi]^T\{P\} = [\tilde{\Phi}]^T\{P\} \tag{3.14}$$

where λ_i represents the eigenvalues obtained from $|[D] - \lambda[I]| = 0$, which are in general a complex form of

$$\lambda_i = \mu_i + i\nu_i \ (\mu_i > 0) \tag{3.15}$$

and $[\Phi]$ is the associated complex modal matrix which satisfies

$$[\Phi]^T[A][\Phi] = [\diagdown r_i \diagdown], \quad [\Phi]^T[B][\Phi] = [\diagdown \lambda_i r_i \diagdown]$$

The response covariance matrix in this case takes the form of

$$E[\{u\}\{u\}^T] = \left[\begin{array}{c|c} E[\{\dot{y}\}\{\dot{y}\}^T] & E[\{\dot{y}\}\{y\}^T] \\ \hline (E[\{\dot{y}\}\{y\}^T])^T & E[\{y\}\{y\}^T] \end{array}\right] \tag{3.16}$$

This is evaluated from Eqs. (3.13) and (3.14) through the spectral approach in Appendix *A*.

Since for the present system, the rotational spring has great influence upon the entire system behavior[2], only the nonlinearity of the surrounding soil effect is considered here. Furthermore, the simplification of the restoring moment -rotational angle is made such as the bilinear hysteretic type of yield level θ_Y and of the slope ratio α, as shown in Fig. 2. This would be substatiated from Appendix *B* in one sense. The difficulty about analysing this nonlinear system is overcome herein by the equivalent linearization technique. The method taken is the one developed by one of the authors and et al.[8] which, for prediction of any response variance, is superior to the commonly used Caughey's method.[9] According to this, the equivalent rotational spring for the nonlinearity response state σ_θ/θ_Y is found by

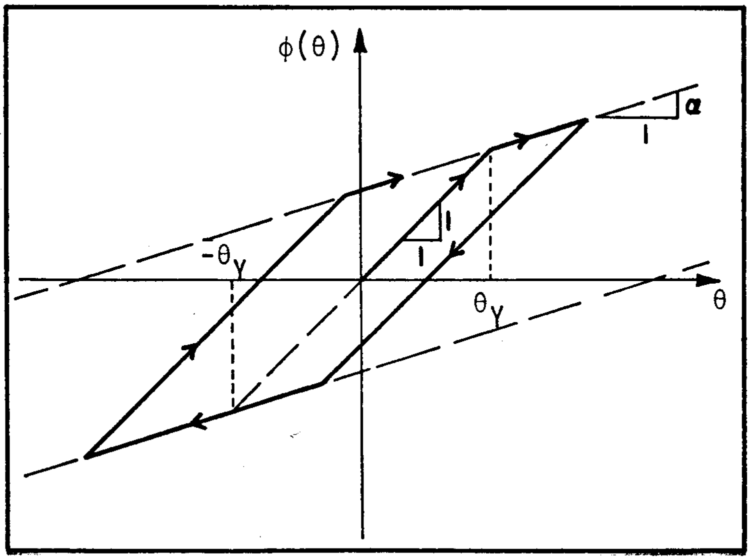


Fig. 2, Foundation Characteristic

$$K_{r,eq} = I_p\omega_{r,eq}^2 = I_p\omega_\theta^2\left\{\alpha+(1-\alpha)f_n\left(\frac{\sigma_\theta}{\theta_Y}\right)\right\} \tag{3.17}$$

where $\omega_\theta = \sqrt{K_r/I_p}$

$$f_n\left(\frac{\sigma_\theta}{\theta_Y}\right) = \left(\frac{\theta_Y}{\sigma_\theta}\right)^2\int_0^\infty \frac{(\theta/\theta_Y)}{(\sigma_\theta/\theta_Y)^2}\cdot\exp\left[-\left(\frac{\theta/\theta_Y}{\sqrt{2}\,\sigma_\theta/\theta_Y}\right)^2\right]\psi\left(\frac{\phi}{\theta_Y}\right)d\left(\frac{\theta}{\theta_Y}\right)$$

$$\psi\left(\frac{\theta}{\theta_Y}\right) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\theta}{\theta_Y}\right)^2 & \text{for } \dfrac{\theta}{\theta_Y}\leq 1 \\ 1+\dfrac{1}{\pi}\left\{\dfrac{3\theta}{2\theta_Y}\left(\dfrac{\theta}{\theta_Y}-\dfrac{4}{3}\right)\cos^{-1}\left(1-2\dfrac{\theta_Y}{\theta}\right)-\left(3\dfrac{\theta}{\theta_Y}-2\right)\left(\dfrac{\theta}{\theta_Y}-1\right)^{1/2}\right\} & \text{for } \dfrac{\theta}{\theta_Y}\geq 1 \end{cases}$$

The equivalent linear damping factor, on the other hand, is found by

$$\beta_{r,eq} = \left(\frac{\omega_\theta}{\omega_{r,eq}}\right)^3\left\{\beta_\theta\left(\frac{\omega_{r,eq}}{\omega_\theta}\right)^2+\sqrt{\frac{\alpha}{\pi}}\,(1-\alpha)\left(\frac{\theta_Y}{\sqrt{2}\,\sigma_\theta}\right) erfc\left(\frac{\theta_Y}{\sqrt{2}\,\sigma_\theta}\right)\right\} \tag{3.18}$$

where erfc ( ) denotes the complementary error function of

$$erfc\ (v) = 1-\frac{2}{\sqrt{\pi}}\int_0^v \exp\ (-\eta^2)\, d\eta$$

The numerical computation is made for the typical cases[2] where the undamped system modes are close to each other (at A ergion in Fig. 3), and the undamped modes are well separted (at B region in the same figure). The damping factor for the tower shaft is assumed as 2% of the critical value for each tower mode, while that of the pier is assumed, based on the solution of machine-foundation on the half space[10], as 5% of the critical value for rotation and as 20% for translation. Fig. 4 gives the equivalent linear rotational spring costant $K_{r,eq}$ and the equivalent linear damping factor $\beta_{r,eq}$ versus the nonlinear response level for $\alpha=0.1$ and 0.5. Note that the equivalent damping factor increases to the amount usually encountered at field tests[11]. Since the response of the bilinear hysteretic system shows the na-arrow band process for $\alpha \lll 1$, the above linearization might well explain the field test results. Fig. 5 shows the rms responses for the white noise excitation input, i.e.

$$E[\ddot{z}_0(t)\,\ddot{z}_0(t+\tau)] = 2\pi S_0\delta(\tau) \quad \delta(\cdot)\text{: Dirac's delta function} \tag{3.19}$$

versus the nonlinear response level. Although the actual earthquake motion has very different frequency contents than the white noise[12], this might be useful for its advantage in simplifying the subsequent analysis and yielding the general

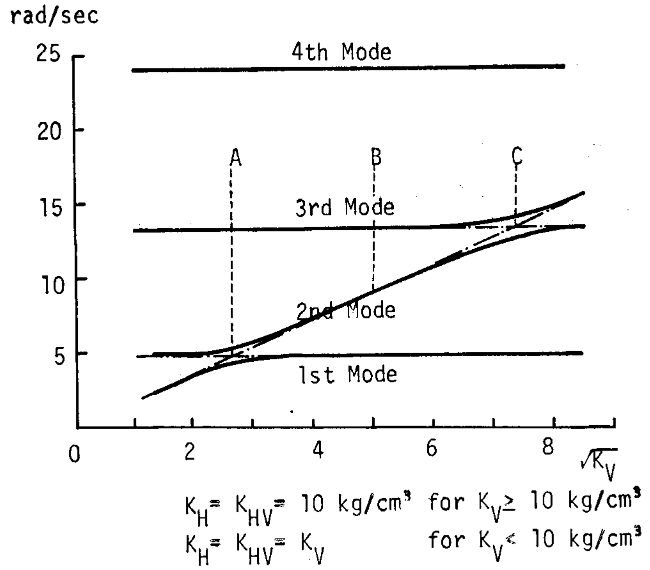
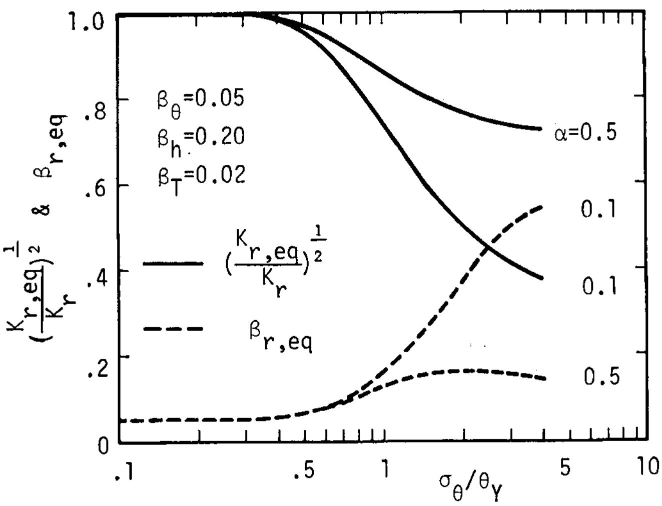


Fig. 3. Natural Frequencies for Linear System



Fug. 4. Equivalent Foundation Characteristic

response feature. It is known that the bilinear hysteretic system has a least reponse at a certain nonlinearity state[8]. The same thing is also observed for the system of linear modes being close to each other, and that about half of the linear response is attained in the region of $\sigma_\theta/\theta_Y=1.5\sim3$. On the other hand, for the system of linear modes being well separted, the nonlinear response differs by

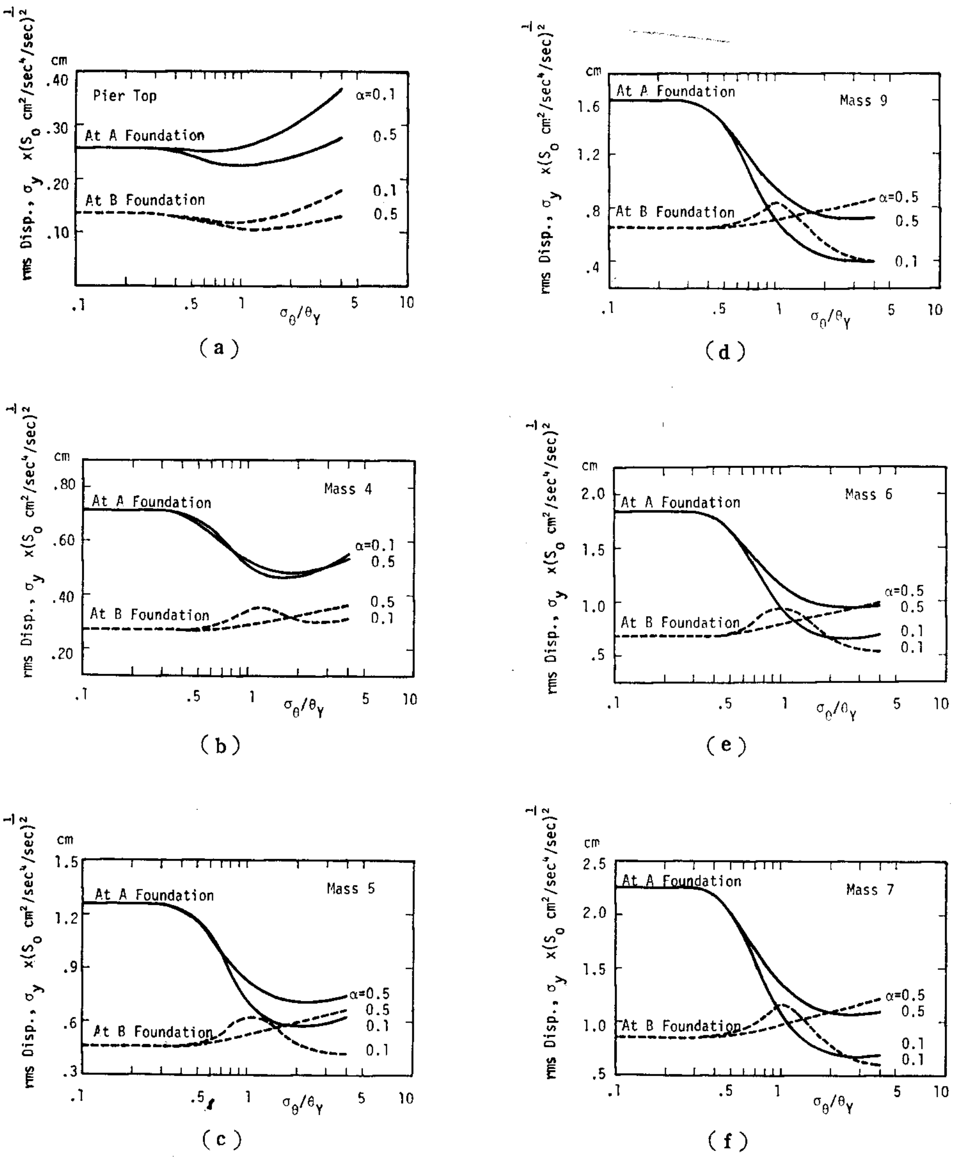


Fig. 5. Variation of rms Displacement with Foundation Nonlinearity

the value of α. In contrast to the former case, the drastic reduction of nonlinear response does not occur in this case.

4. Approximate Response Analysis[13)]

Although the previous two methods of analysis give the exact response of the system, they consume relatively much computing time. Furthermore, from the practical design procedure, the use of the classical normal modes with good accuracy for the entrie system is greatly desired as an approximate analysis. In what follows, such will be the interest of this investigation.

Introducing the undamped normal modes in Eq. (2.1) and expressing them by the associated co-ordinates with use of $\{y\}=[\Psi]\{q\}$, one can get

$$[I]\{\ddot{q}\}+[\tilde{c}]\{\dot{q}\}+[\diagdown\tilde{\omega}_i^2\diagdown]\{q\} = [\Psi]^T\{F\}\ddot{z}_0(t) \tag{4.1}$$

where $[\Psi]^T[m][\Psi] = [I]$, $[\Psi]^T[c][\Psi] = [\tilde{c}]$, $[\Psi]^T[k][\Psi] = [\diagdown\tilde{\omega}_i^2\diagdown]$

and $\tilde{\omega}_i$ are the undamped natural frequencies of the entire system. The damping matrix $[\tilde{c}]$ is not diagonal in general as is true in the case of the proportional damping system. By subtstuting $\{q\}=\{Q\}e^{-\lambda t}$ into homgeneous equation of Eq. (4.1), this results in the frequency equation of,

$$\Delta = \begin{vmatrix} \Delta_1 & -\tilde{c}_{11}\lambda & & -\tilde{c}_{1N}\lambda \\ -\tilde{c}_{21}\lambda & \Delta_2 & \cdots & \vdots \\ -\tilde{c}_{N1}\lambda & \cdots & \cdots & \Delta_N \end{vmatrix} = 0 \tag{4.2}$$

where $\Delta_i = \lambda^2-\tilde{c}_{ii}\lambda+\tilde{\omega}_i^2$ and

for general systems, the off-diagonal elements are small. The expansion of Eq. (4.2) is then

$$\Delta = \prod_{i=1}^{N}\Delta_i+0\ (\tilde{c}_{ij}\lambda) \tag{4.3}$$

where the second term means the effect of the off-diagonal elements which appear at least by the second power. If all the diagonal elements were neglected in Eq. (4.2), the solution would be for a small $\tilde{c}_{ii}$

$$\lambda_i = \tilde{\beta}_i\tilde{\omega}_i+i\tilde{\omega}_{id} \tag{4.4}$$

where $\tilde{\beta}_i = \tilde{c}_{ii}/2\omega_i$, $\tilde{\omega}_{id} = \sqrt{1-\tilde{\beta}_i^2}\,\tilde{\omega}_i$

On the other hand, the exact solution of Eq. (4.2) is given by Eq. (3.15). From the comaprision between this and Eq. (4.4), the undamped frequencies are given by $\sqrt{\mu_i^2+\nu_i^2}$ and the damping factors by $\mu/\sqrt{\mu_i^2+\nu_i^2}$ from the complex eigenvalues. Further, generally, $\mu_i \ll \nu_i$ so that these are well approximated by μ_i and μ_i/ν_i, respectively. It must be noted that these undamped frequencies are clearly different from $\tilde{\omega}_i$ in Eq. (4.1), since the damped frequencies from Eq. (3.15) get closer between coupled modes, so that one of these undamped frequencies becomes greater than the corresponding frequencies in Eq. (4.1), as indicated in Table 1.

Table 1. Modal Frequencies and Damping Factors

At A Foundation, $\alpha=0.5$, $\sigma_\theta/\theta_Y=0.3$

Modes	$\tilde{\omega}_i$	$\tilde{\beta}_i$	$\hat{\omega}_i$	$\hat{\beta}_i$	$\check{\beta}_i$	$\sqrt{\mu_i^2+\nu_i^2}$	$\mu_i/\sqrt{\mu_i^2+\nu_i^2}$
1	4.468	0.0370	4.456	0.0383	0.0380	4.476	0.0370
2	5.315	0.0339	5.335	0.0352	0.0356	5.308	0.0339
3	13.38	0.0200	13.37	0.0202	0.0200	13.38	0.0200
4	24.16	0.0204	23.84	0.0284	0.0204	24.16	0.0202

At B Foundation, $\alpha=0.1$, $\sigma_\theta/\theta_Y=1.5$

Modes	$\tilde{\omega}_i$	$\tilde{\beta}_i$	$\hat{\omega}_i$	$\hat{\beta}_i$	$\check{\beta}_i$	$\sqrt{\mu_i^2+\nu_i^2}$	$\mu_i/\sqrt{\mu_i^2+\nu_i^2}$
1	4.441	0.1505	4.386	0.1769	0.1706	4.814	0.2289
2	5.289	0.1333	5.420	0.1610	0.1738	4.887	0.0564
3	13.39	0.0206	13.45	0.0214	0.0206	13.38	0.0207
4	24.17	0.0200	24.20	0.0209	0.0201	24.16	0.0201

Sometimes the damping factors for classical modes are evaluated by[14]

$$h_i = \frac{1}{4\pi}\cdot\frac{\text{Dissipated energy per cycle by the i-th mode}}{\text{Maximum energy stored by the same mode}} \tag{4.5}$$

This is equivalent to finding the damping factors of β_i in Eq. (4.4). A claer demonstration is given in Appendix C.

When the off-diagonal elements in Eq. (4.2) grows as to induce the intermodal coupling motion strongly in the system, this must be taken into account for a response evaluation. Herein, the following two approaches are considered. The first is the application of the linearization technique of nonlinear systems,[15] that is, rewriting Eq. (4.1) in the form of

$$[I]\{\ddot{q}\}+[\diagdown 2\hat{\beta}_i\hat{\omega}_i\diagdown]\{\dot{q}\}+[\diagdown\hat{\omega}^2\diagdown]\{q\}+\{\varepsilon(\{\dot{q}\},\ \{q\})\} = \{\Psi\}^T\{F\}\ddot{z}_0 \tag{4.6}$$

to choose the equivalent damping factors $\hat{\beta}_i$ and equivalent modal frequencies $\hat{\omega}_i$ so as to minimize the error vector $\{\varepsilon\}$ by means of

$$\frac{\partial E[\{\varepsilon\}^T\{\varepsilon\}]}{\partial(2\hat{\beta}_i\hat{\omega}_i)} = 2E[\varepsilon_i\dot{q}_i] = 0,\ \frac{\partial E[\{\varepsilon\}^T\{\varepsilon\}]}{\partial\hat{\omega}_i^2} = 2E[\varepsilon_i q_i] = 0 \tag{4.7}$$

These requirements yield, when the stationarity of response is assumed

$$\hat{\omega}_i^2 = \hat{\omega}_i^2+\frac{\sum_{l=1}^{N}\tilde{c}_{il}E[\dot{q}_l q_i]}{E[q_i^2]},\quad \hat{\beta}_i = \frac{\sum_{l=1}^{N}\tilde{c}_{il}E[\dot{q}_l\dot{q}_i]}{2\hat{\omega}_i E[q_i^2]} \tag{4.8}$$

The second method is to use the dynamic properties of the system considered.

Among the important modes from earthquake-resistant design, only one strong coupling appears between adacent modes[2]. This leads to the following expressions for Eq. (4.1), namely,

for closely correlated modes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_i \\ \ddot{q}_j \end{Bmatrix} + \begin{bmatrix} \tilde{c}_i & \tilde{c}_{ij} \\ \tilde{c}_{ji} & \tilde{c}_{jj} \end{bmatrix} \begin{Bmatrix} \dot{q}_i \\ \dot{q}_j \end{Bmatrix} + \begin{bmatrix} \tilde{\omega}_i^2 & 0 \\ 0 & \tilde{\omega}_j^2 \end{bmatrix} \begin{Bmatrix} q_i \\ q_j \end{Bmatrix} = \begin{Bmatrix} F_i' \\ F_j' \end{Bmatrix} \ddot{z}_0 \quad (4.9.a)$$

where $\{F'\} = \{\Psi\}^T \{F\}$ and

for other modes

$$[I]\{\ddot{q}\} + [\diagdown \tilde{c}_{ll} \diagdown]\{\dot{q}\} + [\diagdown \tilde{\omega}_l^2 \diagdown]\{q\} = \{F'\}\ddot{z}_0 \quad (4.9.b)$$

In solving Eq. (4.9.a),one's assuming the white noise of Eq. (3.19) for $\ddot{z}_0(t)$ would be acceptable because of the filtering nature of the lightly damped system with $\omega_i \approx \omega_j$.

The resultant response variance is

$$E[q_i^2] = \frac{\pi F_i'^2 S_0}{\tilde{\omega}_i^2} \cdot \frac{\tilde{c}_{jj}(\tilde{\omega}_i^2 - \tilde{\omega}_j^2)^2 + (\tilde{c}_{ii} + \tilde{c}_{jj})\left\{\left(\tilde{c}_{jj} - \frac{F_j'}{F_i'}\tilde{c}_{ij}\right)^2 \tilde{\omega}_i^2 + (\tilde{c}_{ii}\tilde{c}_{jj} - \tilde{c}_{ij}\tilde{c}_{ji})\omega_j^2\right\}}{\tilde{c}_{ii}\tilde{c}_{jj}(\tilde{\omega}_i^2 - \tilde{\omega}_j^2)^2 + (\tilde{c}_{ii} + \tilde{c}_{jj})(\tilde{c}_{jj}\tilde{\omega}_j^2 + \tilde{c}_{ii}\tilde{\omega}_j^2)(\tilde{c}_{ii}\tilde{c}_{jj} - \tilde{c}_{ij}\tilde{c}_{ji})} \quad (4.10)$$

Matching this with the response variance of the equivalent mode system of

$$\ddot{q}_i + 2\check{\beta}_i \omega_i \dot{q}_i + \tilde{\omega}_i^2 q_i = F_i' \ddot{z}_0 \quad (4.11)$$

i.e.

$$E[q_i^2] = \frac{\pi F_i'^2 S_0}{2\check{\beta}_i \omega_i^3} \quad (4.12)$$

the equivalent damping factor $\check{\beta}_i$ is found to be

$$\check{\beta}_i = \frac{1}{2\tilde{\omega}_i} \cdot \frac{\tilde{c}_{ii}\tilde{c}_{jj}(\tilde{\omega}_i^2 - \tilde{\omega}_j^2)^2 + (\tilde{c}_{ii} + \tilde{c}_{jj})(\tilde{c}_{jj}\tilde{\omega}_i^2 + \tilde{c}_{ii}\tilde{\omega}_j^2)(\tilde{c}_{ii}\tilde{c}_{jj} - \tilde{c}_{ij}\tilde{c}_{ji})}{\tilde{c}_{jj}(\tilde{\omega}_i^2 - \tilde{\omega}_j^2)^2 + (\tilde{c}_{ii} + \tilde{c}_{jj})\left\{\left(\tilde{c}_{jj} - \frac{F_j'}{F_i'}\tilde{c}_{ij}\right)^2 \omega_i^2 + (\tilde{c}_{ii}\tilde{c}_{jj} - \tilde{c}_{ij}\tilde{c}_{ji})\omega_j^2\right\}} \quad (4.13.a)$$

Similarly, the equivalent damping factor $\check{\beta}_j$ is obtained by interchanging the above suffix i and j in Eq. (4.13.a). The equivalent damping factors from Eq. (4.9.b) are given by

$$\check{\beta}_l = \tilde{c}_{ll}/2\tilde{\omega}_l \quad (4.13.b)$$

Table 1 indicates the comparison of such obtained equivalent modal frequencies and damping factors. Note that the difference between $\hat{\beta}_i$ and $\check{\beta}_i$ is very small.

Once the equivalent modal frequencies $w_{i,eq}$ and the damping factors $\beta_{i,eq}$ are found, the response analysis is straightforward by the usual classical normal modes method. The spectral analysis relates the response-excitation relation in the frequency domain as[16]

$$[S_{yy}] = [R][S_F][R^*]^T \tag{4.14}$$

where $[R]$ is the receptance matrix of the system, $[R^*]^T$ is its conjugate transpose, and $[S_F]=[\{F\}\{F\}^T]\ S_{\ddot{z}_0}(\omega)$. The existance of the classical modes yields

$$[R] = [\Psi][\diagdown H_i \diagdown][\Psi]^T \tag{4.15}$$

where $[\diagdown H_i \diagdown]$ is the d·agonal matrix of the frequency response function of each mode, i.e.

$$H_i(\omega) = \frac{1}{\omega^2_{i,eq}} \cdot \frac{1}{\left\{1-\left(\frac{\omega}{\omega_{i,eq}}\right)^2\right\}+i2\beta_{i,eq}\left(\frac{\omega}{\omega_{i,eq}}\right)} \tag{4.16}$$

For the white noise excitation of Eq. (3.19), through the Wiener-Khintchine relationship for Eq. (4.14), one can get the response covariance matrix as

$$E[\{y\}\{y\}^T] = [\Psi]([\Psi]^T[\{F\}\{F\}^T][\Psi]\circledast[J])[\Psi]^T \tag{4.17}$$

where the symbol $\circledast$ means the multiplication of elementwise only, and

$$J_{ij} = \frac{4\pi S_0(\beta_{i,eq}\omega_{i,eq}+\beta_{j,eq}\omega_{j,eq})}{(\omega^2_{i,eq}-\omega^2_{j,eq})^2+4(\beta_{i,eq}\omega_{i,eq}+\beta_{j,eq}\omega_{j,eq})(\beta_{i,eq}\omega_{j,eq}+\beta_{j,eq}\omega_{i,eq})\omega_{i,eq}\omega_{j,eq}} \tag{4.18}$$

For computation of $E[\{\dot{y}\}\{y\}^T]$ and $E[\{\dot{y}\}\{\dot{y}\}^T]$, one must replace Eq. (4.18), respectively, by

$$\dot{J}_{ij} = \frac{2\pi S_0(\omega^2_{i,eq}-\omega^2_{j,eq})}{(\omega^2_{i,eq}-\omega^2_{j,eq})^2+4(\beta_{i,eq}\omega_{i,eq}+\beta_{j,eq}\omega_{j,eq})(\beta_{i,eq}\omega_{j,eq}+\beta_{j,eq}\omega_{i,eq})\omega_{i,eq}\omega_{j,eq}} \tag{4.19}$$

$$\ddot{J}_{ij} = \frac{4\pi S_0(\beta_{i,eq}\omega_{j,eq}+\beta_{j,eq}\omega_{i,eq})\omega_{i,eq}\omega_{j,eq}}{(\omega^2_{i,eq}-\omega^2_{j,eq})^2+4(\beta_{i,eq}\omega_{i,eq}+\beta_{j,eq}\omega_{j,eq})(\beta_{i,eq}\omega_{j,eq}+\beta_{j,eq}\omega_{i,eq})\omega_{i,eq}\omega_{j,eq}} \tag{4.20}$$

It must be noted here that the phase of each mode should be discarded in Eq. (4.17), since the equivalent modal frequencies and damping factors are obtained

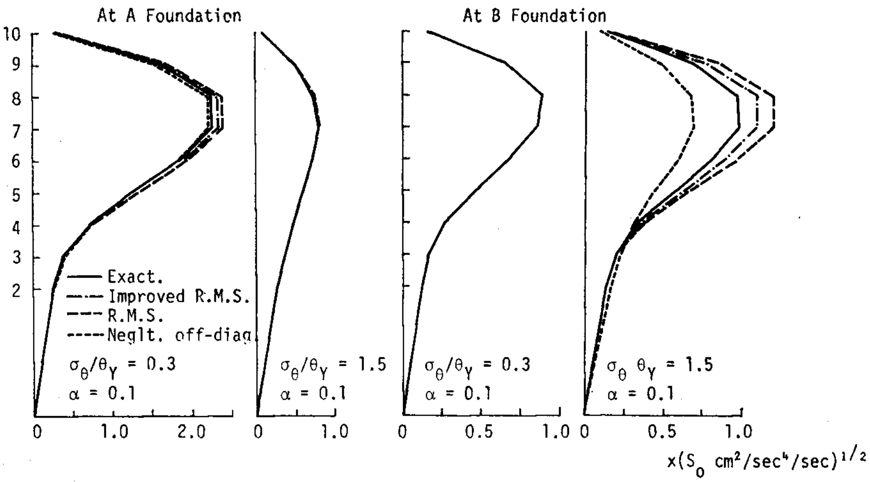


Fig. 6. Exact and Approximate Responses

by presuming the complete modal independence. This is tantamount to the concept of the root-mean-square value of each modal response (*R.M.S.* response).

Fig. 6 shows the comparison of the exact r.m.s. displacement response and the approximate responses using the modal frequencies of $\tilde{\omega}_i$ and the damping factors $\tilde{\beta}_i$ and $\check{\beta}_i$. Note here that when the modes are well separted and/or the damping effect $\beta_{r,eq}$ is small, all the approximate responses are very close to the exact one, while when $\beta_{r,eq}$ is great and the modes are in proximaity, only the R.M.S. response superposition using $\tilde{\omega}_i$ and $\check{\beta}_i$ gives the acceptable approximation. It is then concluded that for the present system the R.M.S. response using $\tilde{\omega}_i$ and $\check{\beta}_i$ yields the best approximation for the whole situation.

5. Conclusions

Through the investigation reported herein, the following conclusions may be drawn:

1. The proper determination of the damping matrix for the entire system is made by taking account of the properties of each composing part.
2. The linearization of the nonlinear foundation increases the damping effect and decreases the stiffness of the foundation.
3. The combined damping matrix makes the so-called nonproportional form which violates the classical normal mode analysis. The resulting effect of the intermodal coupling is clearly investigated.
4. The exact response analysis shows that the least nonlinear response of about half

the linear response can exist for the system whose linear modes are in proximity in the region of $\sigma_\theta/\theta_Y=1.5\sim3$; while the nonlinear response does not diverge so much from the linear response for the system whose linear modes are well separated.

5. The approximate response analysis is proposed for the practical design purpose, taking the R.M.S. response superposition of each mode. This works well and gives the safe-side response for the system having strongly coupled modes.

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Appendix A. Response Covariance Matrix

This derivation is given here from the spectral analysis of the system. The corresponding power spectral density matrix for Eq. (3.11), i.e.

$$[S_{uu}] = \left[\frac{[S_{\dot{y}\dot{y}}] \;|[S_{\dot{y}y}]}{[S_{\dot{y}y}]^T|[S_{yy}]}\right] \tag{A.1}$$

is found through the input-output relationship in frequency domain as

$$[S_{uu}] = [\Phi][\diagdown Hr_i \diagdown][\tilde{\Phi}]^T[S_F][\tilde{\Phi}][\diagdown Hr_i^* \diagdown][\Phi]^T \tag{A.2}$$

where $[\diagdown Hr_i \diagdown]$ is the diagonal matrix of the frequency response function of the complex mode system of Eq. (3.14), i.e. $Hr_i(\omega)=\frac{1}{\lambda_i+i\omega}$ and $[\diagdown Hr_i^* \diagdown]$ is its conjugate-matrix and $[S_F]=[\{F\}\{F\}^T]S_{z_0}(\omega)$. Consideration of the nature of the modal matrix $[\Phi]$ which consists of the ones for displacement $[\Phi_d]$ and for velocity $[\Phi_d][\diagdown \lambda_i \diagdown]$, i.e.

$$[\Phi] = \left[\frac{[\Phi_d][\diagdown \lambda_i \diagdown]}{[\Phi_d]}\right] \tag{A.3}$$

leads Eq. (A.2) into

$$\left[\frac{[S_{\dot{y}\dot{y}}] \;|[S_{\dot{y}y}]}{[S_{\dot{y}y}]^T|[S_{yy}]}\right] = \left[\frac{[\Phi_d][\diagdown\lambda\diagdown][\diagdown H_r\diagdown][\tilde{\Phi}_d]^T[S_F][\tilde{\Phi}_d]][\diagdown H_r\diagdown][\diagdown\lambda\diagdown][\Phi_d]^T}{[\Phi_d][\diagdown H_r\diagdown][\tilde{\Phi}_d]^T[S_F][\tilde{\Phi}_d][\diagdown H_r^*\diagdown][\diagdown\lambda\diagdown][\Phi_d]^T}\right| \left.\frac{[\Phi_d][\diagdown\lambda\diagdown][\diagdown H_r\diagdown][\tilde{\Phi}_d]^T[S_F][\tilde{\Phi}_d][\diagdown H_r^*\diagdown][\Phi_d]^T}{[\Phi_d][\diagdown H_r\diagdown][\tilde{\Phi}_d]^T[S_F][\tilde{\Phi}_d][\diagdown H_r^*\diagdown][\Phi_d]^T}\right] \tag{A.2}$$

For the response covariance matrix, one can apply the Wiener-Kintchine formula

$$[R_{uu}] = \int_{-\infty}^{\infty}[S_{uu}]\,d\omega \tag{A.4}$$

As a special case, when the white noise of Eq. (3.19) is assumed for the excitation $\ddot{z}_0(t)$, this integration is easily carried out by the residue theory. The result is

$$E\left[\frac{d^l}{dt^l}\{y\}\cdot\frac{d^m}{dt^m}\{y\}^T\right] = 2\sum_{i=1,3\cdots}^{2N-1}\sum_{j=1}^{2N} R_e\left(I_{ij}\left[\frac{d^l}{dt^l}\{\Phi_d^{(i)}\}\cdot\{\Phi_d^{(i)}\}^T\right][S_F]\left[\{\Phi_d^{(i)}\}\cdot\frac{d^m}{dt^m}\{\Phi_d^{(i)}\}^T\right]\right) \tag{A.5}$$

where the symbol Re denotes taking of the real part of the complex value and l, m mean the integer 0 or 1, $\frac{d}{dt}\{\Phi_d^{(i)}\}=\lambda_i\{\Phi_d^{(i)}\}$, and $I_{ij}=S_0/r_ir_j(\lambda_i+\lambda_j)$.

The above covariance matrix is also obtained by another method elsewhere[13] by the authors.

Appendix B. Modelling of Foundation

Assume the foundation constant under the pier to be represented by a bilinear hysteresis loop. For this, consider the modelling in Fig. B.1.a where each element is continuously distributed. The force-deformation relationship of each element has an initial stiffness of $K^*=k_1^*+k_2^*$, a yield level of $Y=f^*/K^*$ and a slope ratio between the initial stiffness and the stiffness after yeilding of $\alpha=k_2^*/K^*$, so that the hysteresis loop is defined by

$$\left.\begin{array}{lll} \text{for phase I} & f(\eta)=K^*\eta & ;\ \dot{\eta}>0,\ 0\leq\eta\leq f^*/K^* \\ \text{for phase II} & f(\eta)=k_2^*\eta+(1-\alpha)f^* & ;\ \dot{\eta}\geq 0,\ f^*/K^*\leq\eta\leq A \\ \text{for phase III} & f(\eta)=K^*\eta+(1-\alpha)f^*-k_1^*A & ;\ \dot{\eta}\leq 0,\ A-2f^*/K^*\leq\eta\leq A \\ \text{for phase IV} & f(\eta)=k_2^*\eta-(1-\alpha)f^* & ;\ \dot{\eta}<0,\ -A\leq\eta\leq A-2f^*/K^* \end{array}\right\} \quad (B.1)$$

This illustration is given in Fig. B.1.b.

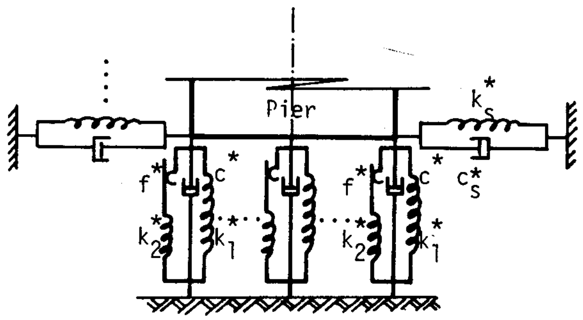

Fig. B. 1. a. Model of Foundation

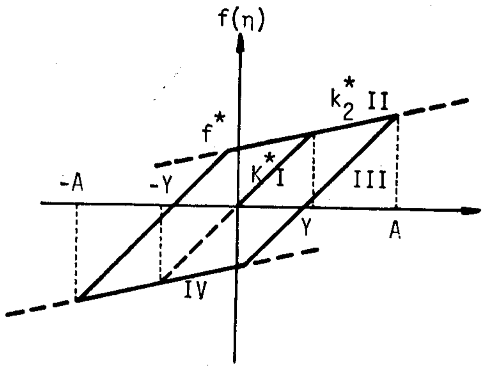

Fig. B. 1. b. Foundation Characteristic

When the pier is rotated around the rotation center R by θ from the state at rest, the total restoring moment is derived from the restoring forces of each element. Referring to the reaction distribution of Fig. B.2.b, one can evaluate this as the sum of the phase I reaction in the region $|\xi| \leq Y/\theta$ and the phase II reaction in $Y/\theta \leq |\xi| \leq a$, i.e.

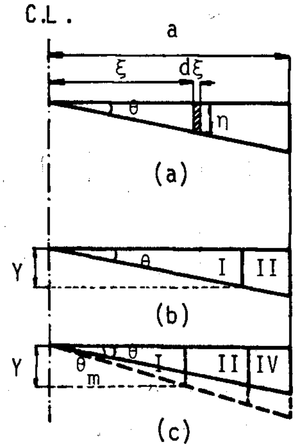

Fig. B.2.b. Distribution of Restoring Forces

$$M(\theta) = 2b\left[\int_0^{min(Y/\theta,a)} \xi \cdot K^*\eta d\xi + \int_{min(Y/\theta,a)}^{a} \xi\{k_2^*\eta+(1-\alpha)f^*\}d\xi\right] \quad](B.2)$$

Where b is the pier width along the rotational axis and the integral bound *min* $(Y/\theta, a)$ means that one should choose the smaller one in the braket. From Eq. (B.2), if $Y/\theta \geq$ a or $\theta \leq Y/a$, all the elements remain in lienar so that

$$M_L = M(\theta) = \frac{2}{3}K^*a^3b\theta = K_R\theta \qquad (B.3)$$

where K_R represents the linear rotational spring constant. On the other hand, if $Y/\theta < a$ or $\theta > Y/a$, referring to Fig. B.2.b, one can get

$$M(\theta) = K_R\phi(\theta)$$

where

$$\phi(\theta) = \alpha\theta + \frac{3(1-\alpha)}{2}\left(\frac{Y}{a}\right) - \frac{(1-\alpha)}{2}\left(\frac{Y}{a}\right)^3 \cdot \frac{1}{\theta^2} \qquad (B.4)$$

After reaching the maximum angle θ_m, the pier begins to rotate in the opposite direction. In this case, the restorting moment is the sum of the effects of phase I, III and IV reactions as illustrated in Fig. B.2.c. Then,

$$M(\theta) = 2b\Big[\int_0^{min(Y/\theta_m,a)} \xi K^*\eta d\xi + \int_{min(Y/\theta_m,a)}^{min(2Y/(\theta_m-\theta),a)} \xi\{K^*\eta+(1-\alpha)f^*-k_1^*A\}d\xi$$
$$+\int_{min(2Y/\theta_m-\theta),a)}^{a} \xi\{k_2^*\eta-(1-\alpha)f^*\}d\xi\Big] \quad (B.5)$$

If $\theta_m \geq Y/a$ is assumed, this becomes

$$M(\theta) = K_R\phi\theta(\theta)$$

where

$$\phi(\theta) = \begin{cases} \theta-(1-\alpha)\theta_m+\frac{3(1-\alpha)}{2}\left(\frac{Y}{a}\right)-\frac{(1-\alpha)}{2}\left(\frac{Y}{a}\right)^3\frac{1}{\theta_m^2}; \ \theta_m-2\left(\frac{Y}{a}\right)\leq\theta\leq\theta_m \\ \alpha\theta-\frac{3(1-\alpha)}{2}\left(\frac{Y}{a}\right)+(1-\alpha)\left\{\frac{4}{(\theta_m-\theta)^2}-\frac{1}{2\theta_m^2}\right\}\left(\frac{Y}{a}\right)^3; \ -\theta_m\leq\theta\leq\theta_m-2\left(\frac{Y}{a}\right) \end{cases} \quad (B.6)$$

Usually, the value of Y/a is $Y/a \ll 1$, so that one may neglect this higher order term in the above function $\phi(\theta)$. This results in the bilinear hysteretic type of the restoring moment-rotational angle relationship represented by

$$\phi(\theta) = \begin{cases} \theta \quad ;\dot{\theta}>0,\ 0\leq\theta\leq\theta_Y,\ \theta_Y=\frac{3}{2}\left(\frac{Y}{a}\right) \\ \alpha\theta+(1-\alpha)\theta_Y;\ \dot{\theta}\geq0,\ \theta\geq\theta_Y \end{cases} \quad (B.7)$$

in replace of Eq. (B.4), and

$$\phi(\theta) = \begin{cases} \theta-(1-\alpha)\theta_m+(1-\alpha)\theta_Y;\ \dot{\theta}\leq0,\ \theta_m-2\theta_Y<\theta\leq\theta_m \\ \alpha\theta-(1-\alpha)\theta_Y \quad ;\ \dot{\theta}<0,\ -\theta_m\leq\theta\leq\theta_m-2\theta_Y \end{cases} \quad (B.8)$$

in replace of Eq. (B.6). The comparison between the exact and the approximate $M(\theta)$ is shown in Fig. B.3. for some typical cases. Note here that a good approxi-

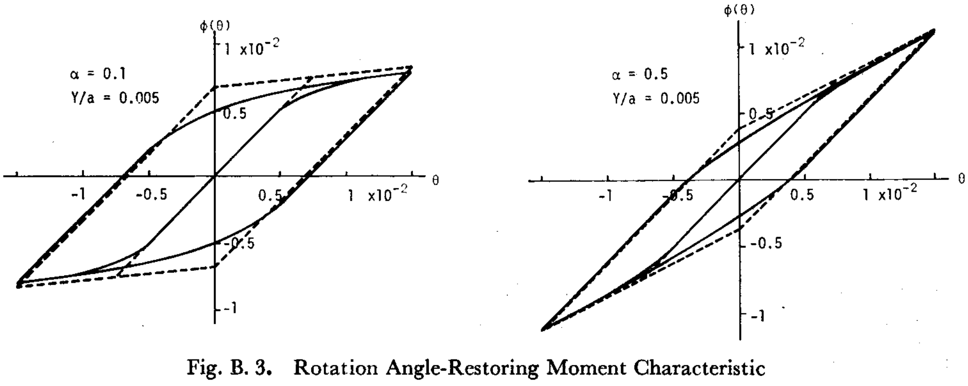

Fig. B. 3. Rotation Angle-Restoring Moment Characteristic mation is attained except $\alpha \ll 1$ cases. For the foundation restoring force-deformation one might assume this condition.

Appendix C. Weighted Average Modal Damping Factors

Presume the system considered is in a resonance state of i-th classical vibration mode $\{\Psi^{(i)}\}$. Then,

$$\ddot{z}_0(t) = \cos \tilde{\omega}_i t,\ \{y\} = \{\Psi^{(i)}\} \cos\left(\tilde{\omega}_i t - \frac{\pi}{2}\right) \tag{C.1}$$

are substituted into Eq. (2.1). Pre-multiplying Eq. (2.1) by $\{\dot{y}\}^T$, and integrating this over one cycle duration $2\pi/\omega_i$, one can get

$$\oint \{\dot{y}\}^T[m]\{\ddot{y}\}\,dt + \oint \{\dot{y}\}^T[c]\{\dot{y}\}\,dt + \oint \{\dot{y}\}^T[k]\{y\}\,dt = \oint \{\dot{y}\}^T\{F\} \cos \tilde{\omega}_i t \tag{C.2}$$

Since the first term in the left hand side can be rewritten as $\oint \{\dot{y}\}^T[m]\cdot d\{\dot{y}\}$, the second term as $\oint d\{y\}^T\cdot[c]\{\dot{y}\}$, and the third term as $\oint d\{y\}^T[k]\{y\}$, they represent the kinetic energy, the dissipative energy, and the strain energy, respectively. As expected, the above integrals of the kinetic and the strain energy vanish.

The weighted average damping factors defined in Eq. (4.5) mean the ratio of the dissipative energy over one cycle duration to the maximum strain energy. The former is evaluated as

$$\begin{aligned} E_d &= \int_0^{2\pi/\tilde{\omega}_i} \{\dot{y}\}[c]\{\dot{y}\}\,dt \\ &= \{\Psi^{(i)}\}^T[c]\{\Psi^{(i)}\}\tilde{\omega}_i^2 \int_0^{2\pi/\tilde{\omega}_i} \cos^2\left(\tilde{\omega}_i t - \frac{\pi}{2}\right)dt \\ &= \tilde{\omega}_i \pi \{\Psi^{(i)}\}^T[c]\{\Psi^{(i)}\} \end{aligned} \tag{C.3}$$

and the latter is evlauated as

$$\begin{aligned} E_s &= \int_0^{\pi/2\tilde{\omega}_i} \{\dot{y}\}^T[k]\{y\}\,dt \\ &= -\{\Psi^{(i)}\}^T[k]\{\Psi^{(i)}\}\tilde{\omega}_i \int_0^{\pi/2\tilde{\omega}_i} \sin\left(\tilde{\omega}_i t - \frac{\pi}{2}\right) \cos\left(\tilde{\omega}_i t - \frac{\pi}{2}\right)dt \\ &= \frac{1}{2}\{\Psi^{(i)}\}^T[k]\{\Psi^{(i)}\} \end{aligned} \tag{C.4}$$

Substituting Eqs. (C.3) and (C.4) into Eq. (4.5) results in

$$h_i = \frac{1}{2}\cdot\frac{\tilde{\omega}_i\{\Psi^{(i)}\}^T[c]\{\Psi^{(i)}\}}{\{\Psi^{(i)}\}^T[k]\{\Psi^{(i)}\}} \tag{C.5}$$

which further leads to the following expression upon consideration of the relationships of $\{\Psi^{(i)}\}^T[c]\{\Psi^{(i)}\}=\tilde{c}_{ii}$ and $\{\Psi^{(i)}\}^T[k]\{\Psi^{(i)}\}=\tilde{\omega}_i^2$,

$$h_i = \tilde{c}_{ii}/2\tilde{\omega}_i \tag{C.6}$$

This is the same as the damping factors to be obtained by neglecting the modal coupling effect in Eq. (4.1.).