Multiple-Criteria Optimization for Environment Development System —Constrained Case—

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Abstract

This paper deals with the multiple-criteria optimization problem for environment development systems, where the global amounts of each exogenous factor are limited. The model of an environment development system, which was introduced by R. Kulikowski, is described by a system of nonlinear differential equations which include interconnected n endogenous factors and $m\times n$ exogenous factors. The problem of multiple-criteria optimization for an environment development system is formulated. A main difficulty of multiple-criteria optimization is that it is no longer clear what one means by an optimal solution. A possible remedy for this situation is to introduce an objective function which is expressed as some function of various criterions. Given the specific objective function, we first optimize the system with respect to another criterion which is a linear combination of the given criteria. For the special case when the systems have similar nonlinearities, the solution of the linear combination problem is obtained in an explicit manner, in terms of the weighting factors in the linear combination functional. A search procedure is then used to determine the optimum values of these weighting factors for the specified objective function.

1. Introduction

A simple model of a development system (such as national economy, ecological system etc.) has been proposed by R. Kulikowski[1,2].

The model consists of n endogenous factors (such as industrial production or biological species) with intensities $x_i(t)$, $i=1, \cdots, n$, and $m\times n$ exogenous, controlled factors (such as raw materials and resources, capital, etc.) with intensities $u_{ij}(t)$, $j=1, \cdots, m$. The set of n differential equations which are linear with respect to $(dx_i/dt)/x_i$ and $(du_{ij}/dt)/u_{ij}$ relates the endogenous and exogenous factors. The solution of these equations:

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$$x_i(t)=x_i(0)\exp\left[\int_0^t \gamma_i(\tau)d\tau\right]\prod_{j=1}^{m}\left[\frac{u_{ij}(t-t_j)}{u_{ij}(-t_j)}\right]^{\gamma_{ij}}, \tag{1-1}$$

where the continuous function $\gamma_i(\tau)$ and the positive numbers $x_i(0)$, t_j, γ_{ij}, $u_{ij}(-t_j)$ $i=1, \cdots, n$, $j=1, \cdots, m$, are given, can be called the system development characteristics.

In the previous paper[3], we considered an unconstrained multiple-objective optimization problem for LRD systems, and proposed the computational procedures for calculating a set of so-called Pareto-optimal solutions.

In this paper, we consider the multiple-objective optimization problem, where the global amounts of each exogenous factor are limited, i. e.

$$\sum_{i=1}^{n} u_{ij}(t)\leq U_j(t) \qquad j=1, 2, \cdots, m \tag{1-2}$$

2. Problem Formulation

Consider the following multiple-objective optimization problem. Maximize

$$J(u)=[J_1(u_1), \cdots, J_n(u_n)]^T \tag{2-1}$$

subject to

$$\sum_{i=1}^{n} u_{ij}(t)\leq U_j(t) \qquad j=1, \cdots, m, \tag{2-2}$$

where

$$u_i=(u_{i1}, u_{i2}, \cdots, u_{im}) \tag{2-3}$$

$$u=(u_1, u_2, \cdots, u_n)\geq 0 \tag{2-4}$$

$$J_i(u_i)=\int_0^T x_i(t)dt \tag{2-5}$$

(T is a given optimization horizon.)

$$x_i(t)=A_i\exp\left[\int_0^t \gamma_i(\tau)d\tau\right]\prod_{j=1}^{m}\left[\frac{u_{ij}(t-t_j)}{u_{ij}(-t_j)}\right]^{\gamma_{ij}} \qquad (A_i=x_i(0)) \tag{2-6}$$

Let us define the following function:

$$\phi=\phi(J_1, J_2, \cdots, J_n),$$

where ϕ is of class C^2. We shall refer to ϕ as the objective function.

As mentioned in the previous paper[3], a main difficulty of multiple-objective optimization is that it is no longer clear what one means by an optimal solution. For solving this problem, we first optimize the system with respect to another criterion which is a linear combination of the given criteria. The optimal control for this criterion is obtained in terms of the weighting factors in the linear combination functional. A search procedure is then used to determine the optimal values of these weighting factors for the specified objective function.

In summary, the technique for solving the problem under discussion consists of two steps[4,5].

Step 1

The problem is solved using the linear combination functional

$$\sum_{i=1}^{n} \mu_i J_i$$

as the objective functional. The optimal strategy u^* is determined as a function of the weighting factors $\mu_i's$ or ratios of weighting factors:

$$\mu_2/\mu_1,\ \mu_3/\mu_1,\ \cdots,\ \mu_n/\mu_1$$

Step 2

The optimal values of the ratios of weighting factors which maximize the given ϕ are then determined by using a search technique.

3. Method of Solution

In this section, we shall first obtain the solution to the linear combination problem in an explicit manner, and later a search of the optimum values of weighting factors will be explained.

3.1. Solution to the Linear Combination Problem

The linear combination problem can now be concisely stated as follows:

Maximize

$$\sum_{i=1}^{n} \mu_i J_i \tag{3-1}$$

subject to

$$\sum_{i=1}^{n} u_{ij}(t) \leq U_j(t) \qquad j=1, \cdots, m \tag{3-2}$$

$$u \geq 0 \tag{3-3}$$

For the special case where the systems have similar nonlinearities, i. e. when

$$\gamma_{ij}=\delta_j>0, \quad \sum_{j=1}^{m} \delta_j<1 \qquad \text{for} \quad i=1, \cdots, n, \ j=1, \cdots, m$$

the solution of the present linear combination problem can be stated in an explicit manner in the following theorem: (In this theorem, we consider the case where $\mu_i's$ are all positive.)

Theorem

There exists a unique optimum strategy for the LRD systems, described by (3-1), (3-2) and (3-3), with

$$\gamma_{ij}=\delta_j>0,\quad \sum_{j=1}^{m}\delta_j<1 \qquad \text{for} \quad i=1, \cdots, n, \; j=1, \cdots, m.$$

The optimum strategy becomes;

$$u^*_{ij}(t-t_j)=\frac{k_i(t)}{k(t)}U_j(t) \qquad \begin{matrix} i=1, \cdots, n \\ j=1, \cdots, m \end{matrix} \qquad t\in[0, T] \tag{3-4}$$

where

$$k_i(t)=\left\{\mu_i A_i \exp\left[\int_0^t \gamma_i(\tau)d\tau\right]\right\}^{1/q} \qquad i=1, \cdots, n \tag{3-5}$$

$$k(t)=\sum_{i=1}^{n} k_i(t) \tag{3-6}$$

$$q=1-\sum_{j=1}^{m}\delta_j \tag{3-7}$$

and the corresponding maximum value of the linear combination problem is:

$$\sum_{i=1}^{n}\mu_i J_i(u^*)=\int_0^t [k(t)]^q \prod_{j=1}^{m}\left[\frac{U_j(t-t_j)}{U_j(-t_j)}\right]^{\delta_j}dt \tag{3-8}$$

Proof

The proof of this theorem is similar to the proof of the allocation theorem of the appendix, where the instantaneous type of constraints (3-2) is replaced by the integral type of constraints.

Remark

The solution of the above theorem can be viewed as a Pareto-optimal solution for all $\mu_i \geq 0$ $(i=1, \cdots, n)$ and

$$\sum_{i=1}^{m}\mu_i=1.$$

This type of solution has been widely used in the fields of cooperative games and vector-valued criterion problems.

3. 2. A Search of the Optimum Values of Weighting Factors

Having solved the linear combination problem and determined the optimal strategy u^* as a function of weighting factors, we now proceed to search for the optimum values of weighting ratios

$$[(\mu_2/\mu_1)^*, \cdots, (\mu_n/\mu_1)^*], \qquad \mu_1\neq 0,$$

which maximize the given ϕ.

This can be performed using a search technique. In this search, the components of the gradient of ϕ are given by:

$$[\text{grad}\,\phi]_{\mu_2/\mu_1,\cdots,\mu_n/\mu_1} = \begin{pmatrix} (\partial\phi/\partial J_1)\{\partial J_1/\partial(\mu_2/\mu_1)\} + \cdots\cdots + (\partial\phi/\partial J_n)\{\partial J_n/\partial(\mu_2/\mu_1)\} \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ (\partial\phi/\partial J_1)\{\partial J_1/\partial(\mu_n/\mu_1)\} + \cdots\cdots + (\partial\phi/\partial J_n)\{\partial J_n/\partial(\mu_n/\mu_1)\} \end{pmatrix}$$

All the components of the gradient vector are known if μ_i/μ_1, $i=2, \cdots, n$, are specified.

Therefore, the problem is now reduced to the determination of the maximum of a function, assuming that both its value and its gradient at each point of the search region are known. In this regard, many techniques grouped under the broad category of steepest ascent methods are available[6,7].

Remark

The basis for choosing an objective function is very difficult to mention, but when the form of the objective function is changed, there is no need to formulate a new problem, only some algebraic computations are required.

4. Conclusion

The multiple-objective optimization problem for environment development systems, where the global amount of each exogenous factor is limited, has been investigated in this paper. The model consists of n endogenous factors and $m\times n$ exogenous factors with intensities related by a system of nonlinear differential equations. It can be solved effectively for the growth intensities of the endogenous processes and as a result which are completely specified by the exogenous factors. The form of this function can be viewed as a generalization of the Cobb-Douglas function.

Introducing an objective function, we first optimize the system with respect to another criterion which is a linear combination of the given criteria. For the special case where the systems have similar nonlinearities, the solution of the linear combination problem is obtained in an explicit manner in terms of the weighting factors in the linear combination functional. A search procedure is then used to determine the optimum values of these weighting factors for the specified objective function.

Appendix

Consider the following optimization problem. Find the non-negative functions

$c_i(t)=\hat{c}_i(t)$, $i=1, \cdots, n$, which maximize

$$J(c)=\sum_{i=1}^{n}\int_{t_1}^{t_2}[f_i(\tau)]^q[c_i(\tau)]^p d\tau$$

subject to the constraints

$$c_i(t)\in\Omega$$

where

$$\Omega=\left\{c_i(t)\,|\sum_{i=1}^{n}\int_{t_1}^{t_2}c_i(t)dt\leq C,\ c_i(t)\geq 0, \quad i=1, \cdots, n\right\}$$

$f_i(t)$ are given non-negative continuous functions

C is a given positive number

$q=1-p$

The solution of this optimization problem can be formulated in the following theorem.

Theorem A (Allocation Theorem)[8]

The unique optimum allocation strategy

$$\hat{c}_i(t)=f_i(t)\frac{F}{C} \qquad i=1, \cdots, n$$

exists, such that

$$\max_{c\in\Omega} J(c)=J(\hat{c})=F^{1-p}C^p$$

where

$$F=\sum_{i=1}^{n}\int_{t_1}^{t_2}f_i(t)dt$$

Proof

Applying the Hölder inequality for integrals, one obtains:

$$\int_{t_1}^{t_2}[f_i(\tau)]^q[c_i(\tau)]^p d\tau\leq\left\{\int_{t_1}^{t_2}|f_i(\tau)|d\tau\right\}^q\left\{\int_{t_1}^{t_2}|c_i(\tau)|d\tau\right\}^p \qquad \text{(A-1)}$$

where $q=1-p$

Denoting

$$\left\{\int_{t_1}^{t_2}|f_i(\tau)|d\tau\right\}^q=x_i, \quad \left\{\int_{t_1}^{t_2}|c_i(\tau)|d\tau\right\}^p=y_i \qquad i=1, \cdots, n$$

and applying the Hölder inequality for sums, one obtains:

$$\sum_{i=1}^{n}x_i y_i\leq\{\sum_{i=1}^{n}|x_i|^{1/q}\}^q\{\sum_{i=1}^{n}|y_i|^{1/p}\}^p \qquad \text{(A-2)}$$

It is well known that the equality sign appears in (A-1) if and only if

$$|c_i(\tau)| \cong a|f_i(\tau)|, \quad \text{almost everywhere,} \tag{A-3}$$

where a=constant, $i=1, \cdots, n$.
The equality sign in (A-2) appears if and only if

$$|x_i|^{1/q} = b|y_i|^{1/p}, \quad i=1, \cdots, n, \quad b=\text{constant} \tag{A-4}$$

We shall show that for $c_i(\tau)=\hat{c}_i(\tau)$, $i=1, \cdots, n$, the conditions (A-3), (A-4) hold.
Indeed,

$$|c_i(\tau)| = |f_i(\tau)\frac{C}{F}| = |f_i(\tau)|(C/F)$$

and

$$|y_i|^{1/p} = \int_{t_1}^{t_2} |c_i(\tau)| d\tau = \int_{t_1}^{t_2} |f_i(\tau)| d\tau (C/F)$$

$$|x_i|^{1/q} = \int_{t_1}^{t_2} |f_i(\tau)| d\tau$$

Then

$$J(c) \leq \{\sum_{i=1}^{n} |x_i|^{1/q}\}^q \{\sum_{i=1}^{n} |y_i|^{1/p}\}^p = F^{1-p} C^p \tag{A-5}$$

and the equality sign appears in (A-5), if and only if

$$c_i(\tau) = \hat{c}_i(\tau)$$

It should be observed that

$$\sum_{i=1}^{n} \int_{t_1}^{t_2} \hat{c}_i(\tau) d\tau = C$$

and $\hat{c}_i(\tau) \geq 0$, $i=1, \cdots, n$, $t \in [t_1, t_2]$, so that $\hat{c}_i(\tau) \in \Omega$. This completes the proof of the theorem.

Remark

It is also possible to formulate the analogical theorem for the case

$$\underset{x \in \Omega'}{\text{Min}}\, J(c)$$

where

$$\Omega' = \left\{c_i(t) | \sum_{i=1}^{n} \int_{t_1}^{t_2} c_i(t) dt \geq C, \quad c_i(t) \geq 0, \quad i=1, \cdots, n\right\}$$

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