Traveling Salesman Problems with a Capacity Constraint of the Delivery Truck

By

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Abstract

This paper considers the following delivery route problem. A truck delivers r_i unit production resources to cities $i=2, 3, \cdots, n$ in some order starting from city 1, and receives w_i unit production wastes at cities $i=2, 3, \cdots n$. Let c_{ij} $(1\leq i\leq n,\ 1\leq j\leq n,\ i\neq j)$ be the time the delivery truck requires from city i to city j. At city i $(i\neq 1)$, the production starts upon receiving the production resources and takes t_i (≥ 0) unit time until completion. Moreover, the delivery truck has the carrying capacity Δ and starts from city 1 with resources of $\sum_{i=2}^{n} r_i$ units. At each city i, the total of remaining resources and the collected wastes can not exceed Δ.

The problem is to find a delivery route that visits each city i exactly once, and minimizes the completion time of production at all cities $i=2, 3, \cdots, n$.

This paper shows that the well known dynamic programming approach for the traveling salesman problem can be generalized to incorporate the capacity constraint.

1. Introduction

This paper considers the following delivery route problem.[4),5),6)] A delivery truck delivers r_i unit production resources to cities $i=2, 3, \cdots, n$ in some order starting from city 1, and receives w_i unit production wastes at cities $i=2, 3, \cdots, n$. Let c_{ij} $(1\leq i\leq n,$ $1\leq j\leq n,\ i\neq j)$ be the time the delivery truck requires from city i to city j. At city i $(i\neq 1)$, the production is started upon receiving the production resources and takes t_i (≥ 0) unit time until completion.

The problem is to find a delivery route that visits each of cities $1, 2, \cdots, n$ exactly once, and minimizes the completion time of production at all cities $i=2, 3, \cdots, n$.

It is known that the above problem can be formulated as the mininax type traveling salesman problem.[4),5),6)] (The detailed description will be given in Section 3.) In

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particular, if $t_i=0$, $i=2, 3, \cdots, n$ (or $c_{ij}+t_j-t_i\geq 0$ for all i, j), the resulting problem is very similar to the well known traveling salesman problem.[1),2),3)] It differs in that the final city can be one of the cities $2, 3, \cdots, n$.

In this paper, the capacity of the delivery truck is also taken into account. The truck has the capacity $\Delta(\geq \sum_{i=2}^{n} r_i)$ and starts from city 1 with resources of $\sum_{i=2}^{n} r_i$ units. At each city i, the total of remaining resources and the collected wastes can not exceed Δ.

Most of the algorithms considered for the ordinary traveling salesman problem,[1),2),3)] and for the minimax type traveling salesman problem,[4),5),6)] cannot deal with this additional constraint in a direct manner. It is shown, however, that the dynamic programming approach by Held and Karp[3)] and Bellman[1)] for the ordinary traveling salesman problem, and a modification of the algorithm given in references (4), (5), (6) for the minimax type traveling salesman problem, can naturally incorporate this constraint.

Section 2 discusses how the constraint is incorporated in the Held-Karp-Bellman algorithm, and Sections 3–6 treat the minimax type traveling salesman problem.

2. Modification of the Held-Karp-Bellman algorithm to the traveling salesman problem with the capacity constraint

The problem given in Section 1 can be reduced to the following traveling salesman problem if $t_i=0$, $i=2, 3, \cdots, n$. Let $G=(V, E)$ be the complete directed graph with the vertex set $V=\{1, 2, \cdots, n\}$ and the arc set $E=\{(i, j) \mid i\in V, j\in V, i\neq j\}$. Let c_{ij} be the length associated with each arc (i, j). The hamiltonian path is a path starting from vertex 1 and containing every vertex exactly once. The length of a hamiltonian path is the sum of the lengths of the arcs in it. With each vertex $i=2, 3, \cdots, n$, r_i (≥ 0) and w_i (≥ 0) are associated as stated in Section 1. It is required that for a hamiltonian path $\pi=(\pi(1)\ (=1), \pi(2), \pi(3), \cdots, \pi(n))$,

$$\sum_{i=2}^{k} w_{\pi(i)}+\sum_{i=k+1}^{n} r_{\pi(i)}\leq \Delta \tag{2.1}$$

holds for $k=2, 3, \cdots, n$. Find a hamiltonian path that has the minimum length among those satisfying constraint (2·1).

Letting

$$\Delta_i=r_i-w_i, \quad i=2, 3, \cdots, n$$

$$\Delta_1=\Delta-\sum_{i=2}^{n} r_i,$$

(2·1) is transformed into

$$\min_{k=2,3,\cdots,n} \sum_{i=1}^{k} \Delta_{\pi(i)}\geq 0. \tag{2.2}$$

(Note that $\sum_{i=1}^{k} \Delta_{\pi(i)}$ denotes the margin of the truck capacity at city $\pi(k)$).

Now let S_n denote the set of permutations on $V=\{1, 2, \cdots, n\}$. $\pi \in S_n$ with $\pi(1)=1$ represents a hamiltonian path, where $\pi(i)$ stands for the i-th visited vertex. Let

$$\bar{S}_n=\{\pi \in S_n | \pi(1)=1 \text{ and } \pi \text{ satifies (2·2)}\}.$$

For $Q \subset V-\{1\}$ and $\alpha, \beta \in V-Q$, $S(\alpha, Q, \beta)$ denotes the subset of $\bar{S}_n$ such that

(a) $\pi(i)=\alpha$

(b) $\pi(i+j) \in Q, \quad j=2, 3, \cdots, |Q|$†

(c) $\pi(i+|Q|+1)=\beta$

for some i.

With these preparations, $f(1, Q, \beta)$ and $\Delta(\alpha, Q, \beta)$ are defined by

$$f(1, Q, \beta)=\min_{\pi}\{\sum_{i=1}^{|Q|+1} c_{\pi(i)\pi(i+1)} | \pi \in S(1, Q, \beta)\}$$

$$\Delta(\alpha, Q, \beta)=\Delta_\alpha+\sum_{i \in Q} \Delta_i+\Delta_\beta.$$

Obviously, the present problem is solved if $f(1, V-\{1, \beta\}, \beta)$ are obtained for all $\beta \in V-\{1\}$. The optimal value is given by

$$\min_{\beta} \{f(1, V-\{1, \beta\}, \beta) | \beta \in V-\{1\}\}.$$

Each $f(1, V-\{1, \beta\}, \beta)$ is calculated by the following recursion:

(a) $(|Q|=0)$; For $\beta \in V-\{1\}$, let

$$\Delta(1, \phi, \beta)=\Delta_1+\Delta_\beta$$

and

$$f(1, \phi, \beta)=\begin{cases} c_{1\beta} & \text{if } \Delta(1, \phi, \beta) \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

(b) $(|Q| \geq 1)$; Let

$$\Delta(1, Q, \beta)=\Delta(1, Q-\{\alpha\}, \alpha)+\Delta_\beta \text{ for any } \alpha \in Q,$$

and

$$f(1, Q, \beta)=\begin{cases} \min_{\alpha \in Q} (f(1, Q-\{\alpha\}, \alpha)+c_{\alpha\beta}) & \text{if } \Delta(1, Q, \beta) \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Step (b) is executed for all $Q \subset V-\{1\}$ in the non-decreasing order of $|Q|$. Upon completion of Step (b), $f(1, V-\{1, \beta\}, \beta)$ are calculated for all $\beta \in V-\{1\}$.

The validity of this algorithm may be proved by the principle of optimality in the theory of dynamic programming in a manner similar to 3). The difference from 3) is that the above recursion takes into account $\Delta(1, Q, \beta)$ (the margin of the truck capacity when it gets to vertex β), and whenever the capacity is exceeded (i.e., $\Delta(1, Q, \beta)<0$), the corresponding path is abandoned ($f(1, Q, \beta)$ is set to ∞). This is possible since all paths in $S(1, Q, \beta)$ have the same margin $\Delta(1, Q, \beta)$ at vertex β.

† $|Q|$ denotes the cardinality of Q.

3. Formulation of the minimax traveling salesman problem with a capacity constraint

This section treats the general case with $t_i \geq 0$, $i=2, 3, \cdots, n$, discussed in Section 1.

Rearranging the city number i ($i=2, 3, \cdots, n$) and setting $t_1=0$, we can assume $t_2 \geq \cdots \geq t_n \geq 0$ without loss of generality.

Given $\pi \in \bar{S}_n$, the completion time of production at vertex $\pi(k)$ is

$$\sum_{i=1}^{k-1} c_{\pi(i)\pi(i+1)} + t_{\pi(k)}$$

and the completion time at all vertices is

$$f^\pi = \max \{ \sum_{i=1}^{k-1} c_{\pi(i)\pi(i+1)} + t_{\pi(k)} \,|\, k=2, 3, \cdots, n \}.$$

The above problem is first formulated as the following problem $\bar{A}$.

$\bar{A}$: Find $\pi \in \bar{S}_n$ minimizing f^π.

For the complete directed graph $G=(V, E)$ introduced in Section 2, adjoin length c'_{ij} to each arc (i, j) and capacity Δ_β to each vertex β where c'_{ij} is defined by

$$c'_{ij} = c_{ij} + t_j - t_i \; (1 \leq i \leq n, \; 1 \leq j \leq n, \; i \neq j).$$

Then, for a hamiltonian path $\pi \in \bar{S}_n$,

$$f^\pi_{\pi(k)} = \sum_{i=1}^{k-1} c'_{\pi(i)\pi(i+1)}$$

denotes the length from vertex 1 $(=\pi(1))$ to vertex $\pi(k)$. Since

$$\begin{aligned} &\sum_{i=1}^{k-1} c_{\pi(i)\pi(i+1)} + t_{\pi(k)} \\ &= \sum_{i=1}^{k-1} (c_{\pi(i)\pi(i+1)} + t_{\pi(i+1)} - t_{\pi(i)}) + t_{\pi(1)} \; (t_{\pi(1)} = t_1 = 0) \\ &= \sum_{i=1}^{k-1} c'_{\pi(i)\pi(i+1)} = f^\pi_{\pi(k)} \quad , \end{aligned}$$

problem $\bar{A}$ is equivalent to the problem $\bar{B}$ below.

$\bar{B}$: Find a hamiltonian path $\pi \in \bar{S}_n$ minimizing

$$\max_k \{ f^\pi_{\pi(k)} \,|\, k=2, 3, \cdots, n \}.$$

$\bar{B}$ is the same minimax type traveling salesman problem as the one discussed in references (4), (5), (6), except that the present problem has the capacity constraint.

4. Decomposition of $\bar{\text{B}}$ into subproblems

For $\pi \in S(\alpha, Q, \beta)$ (defined in Section 2), $f^\pi(\alpha, Q, \beta)$ denotes the length of the portion π from vertex α to vertex β of path π, *i.e.*,

$$f^{\pi}(a, Q, \beta)=\sum_{k=1}^{i+|Q|} c'_{\pi(k)\pi(k+1)},$$

where $\pi(i)=a$. $k^*(\pi)$ denotes the smallest k^* satisfying

$$\max_k \{f^{\pi}_{\pi(k)} \mid k=2, 3, \cdots, n\} = f^{\pi}_{\pi(k^*)}.$$

Define $S(l)$ by

$$S(l)=\{\pi \in \bar{S}_n \mid k^*(\pi)=l\}$$

(*i.e.*, $\pi \in S(l)$ satisfies $f^{\pi}_{\pi(l)} \geq f^{\pi}_{\pi(k)}$ for $k=2, 3, \cdots, n$).

Problem $\bar{B}$ is now decomposed into the following $(n-1)$ problems $T_2, \cdots, T_n$. T_l: Find a hamiltonian path $\pi=\pi_l \in S(l)$ minimizing $f^{\pi}_{\pi(l)}$. ($f^{\pi_l}_{\pi_l(l)}$ is hereafter denoted as f_l).

Theorem 1.

Let $\pi_2, \pi_3, \cdots, \pi_n$ be optimal solutions of $T_2, T_3, \cdots, T_n$ respectively. An optimal solution of $\bar{B}$ is π_{l*}, where

$$f_{l*}=\min \{f_l,\ l=2, 3, \cdots, n\}.$$

Proof.

$\bar{B}$ asks to calculate

$$\min_{\pi \in S_n} \max\{f^{\pi}_{\pi(k)} \mid k=2, 3, \cdots, n\}$$

$$= \min_{2 \leq l \leq n} \min_{\pi \in S(l)} \max\{f^{\pi}_{\pi(k)} \mid k=2, 3, \cdots, n\}$$

(since $\bar{S}_n=\bigcup_{l=2}^{n} S(l)$)

$$= \min_{2 \leq l \leq n} \min_{\pi \in S(l)} f^{\pi}_{\pi(l)} = \min_{2 \leq l \leq n} f_l.$$

f_l is obtained by solving T_l.

5. An algorithm for solving $\bar{\mathbf{B}}$

Now an algorithm based on Theorem 1 is given for solving $\bar{B}$. It solves $T_2, T_3, \cdots, T_n$ in this order. In the algorithm, d^* denotes the current best value of f^{π} (initially set to ∞). For $Q \subset V-\{1\}$, such that $|Q|<l-2$, $f^l(1, Q, \beta)$ is equal to the minimum length at vertex β for those paths which start from vertex 1, pass through all the vertices in Q and reach β, provided that it is smaller than d^*; otherwise it is set to ∞. $\pi(1, Q, \beta)$ denotes the path giving $f^l(1, Q, \beta)$ (*i.e.*, it starts from vertex 1, passes through the vertices in Q, and reaches vertex β). $M(1, Q, \beta)$ denotes the maximum length attained by a vertex in $(1, Q, \beta)$.

For Q with $|Q|=l-2$, $f^l(1, Q, \beta)$ $(\beta=\delta \leq l)$ has the same meaning as above, if the path corresponding to $f^l(1, Q, \beta)$ $(\beta=\delta)$ satisfies

$$f^{\pi}_{\pi(l)} > f^{\pi}_{\pi(k)} \qquad k=1, 2, \cdots, l-1$$

If otherwise, it is again set to ∞(since $\pi \notin S(l)$ necessarily holds for any $\pi \in S(1, Q, \delta)$). $\pi(1, Q, \delta)$ has the same meaning as above. $\pi(\delta, Q', \gamma)$ and $\pi(\delta, \bar{Q})$ in Step 2 of Phase II denote the path giving $f^l(\delta, Q', \gamma)$ and $f^l(\delta, Q)$ respectively where $\bar{Q} = V - \{1, \delta\} - Q$, $Q' \subset \bar{Q}$ and $\gamma \in Q - Q'$.

[Algorithm for solving $\bar{B}$]

Phase I:

Step 1: Let $l \leftarrow 2$, $d^* \leftarrow \infty$ and go to phase II.

Step 2: If $l=n$, terminate. π^* is the optimal solution of $\bar{B}$ and d^* is its value; otherwise, go to phase II after $l \leftarrow l+1$.

Phase II: (Solve T_l)

Step 1: (Calculate $f^l(1, Q, \beta)$ for $|Q| \leq l-2$ and $\beta \in V - \{1\} - Q$) If $l=2$, go to (1c); otherwise go to (1a).

(1a) For $Q = \phi$ and $\beta \in V - \{1\}$, let

$$\Delta(1, \phi, \delta) \leftarrow \Delta_1 + \Delta_\beta,$$

$$f^l(1, \phi, \beta) \leftarrow \begin{cases} c'_{1\beta} & (\text{if } \Delta(1, \phi, \beta) \geq 0 \text{ and } c'_{1\beta} < d^*) \\ \infty & (\text{otherwise}), \end{cases}$$

$$M(1, \phi, \beta) \leftarrow f^l(1, \phi, \beta),$$

and $\pi(1, \phi, \beta) \leftarrow (1, \beta)$ if $f^l(1, \phi, \beta) > \infty$.

(1b) For Q with $1 \leq |Q| \leq l-3$ and $\beta \in V - \{1\} - Q$, let (in the non-decreasing order of Q)

$$\Delta(1, Q, \beta) \leftarrow \Delta(1, Q - \{\alpha\}, \alpha) + \Delta_\beta \text{ for any } \alpha \in Q,$$

$$f^l(1, Q, \beta) \leftarrow \begin{cases} F_m & (\text{if } \Delta(1, Q, \beta) \geq 0 \text{ and } F_m < d) \\ \infty & (\text{otherwise}), \end{cases}$$

$$M(1, Q, \beta) \leftarrow \max\,(f^l(1, Q, \beta), M(1, Q - \{\alpha^*\}, \alpha^*)),$$

and $\pi(1, Q, \beta) \leftarrow (\pi(1, Q - \{\alpha^*\}, \alpha^*), \beta)$ if $f^l(1, Q, \beta) < \infty$,

where

$$F_m = \min_{\alpha \in Q} \{c'_{\alpha\beta} + f^l(1, Q - \{\alpha\}, \alpha)\}$$
$$= c'_{\alpha^*\beta} + f^l(1, Q - \{\alpha^*\}, \alpha^*).$$

(1c)† For Q with $|Q| = l-2$ and $\delta \in V - \{1\} - Q$ with $\delta \leq l$, let

$$\Delta(1, Q, \delta) \leftarrow \Delta(1, Q - \{\alpha\}, \alpha) + \Delta_\delta \text{ for any } \alpha \in Q,$$

$$f^l(1, Q, \delta) \leftarrow \begin{cases} F_m & (\text{if } \Delta(1, Q, \delta) \geq 0 \text{ and } d^* > F_m > M(1, Q - \{\alpha^*\}, \alpha^*)) \\ \infty & (\text{otherwise}), \end{cases}$$

and $\pi(1, Q, \delta) \leftarrow (\pi(1, Q - \{\alpha^*\}, \alpha^*), \delta)$ if $f^l(1, Q, \delta) < \infty$,

† For $\delta > l$, $f^l(1, Q, \delta)$ cannot satisfy $f^l(\delta, \bar{Q}) \leq 0$. For details, see $V(\delta)$ defined in references (4), (6).

where

$$F_m = \min_{a \in Q} \{c'_{a\beta} + f^l(1, Q - \{a\}, a)\}$$
$$= c'_{a*\delta} + f^l(1, Q - \{a^*\}, a^*).$$

Step 2: (Test of the feasibility of each $f^l(1, Q, \delta)$)† For each $f^l(1, Q, \delta) < \infty$ obtained in (1c), let $\bar{Q} \leftarrow V - Q - \{1, \delta\}$. If $\bar{Q} = \phi$, go to Step 3; otherwise go to Steps (2a) (2b) (2c) and obtain $f^l(\delta, Q', \gamma)$ for each Q' and γ such that $Q' \subset \bar{Q}$ and $\gamma \in \bar{Q} - Q'$.

(2a) for $Q' = \phi$ and $\gamma \in \bar{Q} (= V - \{1, \delta\} - Q)$, let

$$\Delta(\delta, \phi, \gamma) \leftarrow \Delta_\delta + \Delta_\gamma,$$

$$f^l(\delta, \phi, \gamma) \leftarrow \begin{cases} c'_{\delta\gamma} \text{ (if } \Delta(1, Q, \delta) + \Delta(\delta, \phi, \gamma) - \Delta_\delta (= \Delta(1, Q \cup \{\delta\}, \gamma)) \geq 0 \text{ and } c'_{\delta\gamma} \leq 0) \\ \infty \quad \text{(otherwise)}, \end{cases}$$

and $\quad \pi(\delta, \phi, \gamma) \leftarrow (\gamma)$ if $f^l(\delta, \phi, \gamma) < \infty$.

(2b) For each $Q' \subset \bar{Q}$ with $Q' \neq \phi$ and $\gamma \in \bar{Q} - Q'$, let (in the non-decreasing order of $|Q'|$)

$$\Delta(\delta, Q', \gamma) \leftarrow \Delta(\delta, Q' - \{a\}, a) + \Delta_\gamma \text{ for any } a \in Q',$$

$$f^l(\delta, Q', \gamma) \leftarrow \begin{cases} G_m \text{ (if } \Delta(1, Q, \delta) + \Delta(\delta, Q', \gamma) - \Delta_\delta (= \Delta(1, Q \cup \{\delta\} \cup Q', \gamma)) \geq 0 \text{ and } G_m \leq 0) \\ \infty \quad \text{(otherwise)}, \end{cases}$$

and $\quad \pi(\delta, Q', \gamma) \leftarrow (\pi(\delta, Q' - \{a^*\}, a^*), \gamma)$ if $f(\delta, Q', \gamma) < \infty$,

where

$$G_m = \min_{a \in Q'} \{c'_{a\gamma} + f^l(\delta, Q' - \{a\}, a)\}$$
$$= c'_{a*\gamma} + f^l(\delta, Q' - \{a^*\}, a^*).$$

(2c) For Q' and γ with $Q' \cup \{\gamma\} = \bar{Q}$, let

$$f^l(\delta, \bar{Q}) \leftarrow G_m,$$

and $\quad \pi(\delta, \bar{Q}) \leftarrow \pi(\delta, \bar{Q} - \{\gamma^*\}, \gamma^*)$ if $f^l(\delta, \bar{Q}) < \infty$,

where

$$G_m = \min_{\gamma \in \bar{Q}} \{f^l(\delta, \bar{Q} - \{\gamma\}, \gamma)\}$$
$$= f^l(\delta, \bar{Q} - \{\gamma^*\}, \gamma^*).$$

Step 3:

$$\pi^* \leftarrow \begin{cases} \pi^* \text{ (if } f^l(1, Q^*, \delta^*) \geq d^*) \\ (\pi(1, Q^*, \delta^*), \ \pi(\delta^*, Q^*)) \quad \text{(otherwise)}, \end{cases}$$

† This step tests whether the path corresponding to $f^l(1, Q, \delta)$ (i.e., $\pi(1, Q, \delta)$) obtained in Step (lc) can be completed by attaching the last portion (i.e., $\pi(\delta, \bar{Q})$) so that the capacity constraint is satisfied, and the resulting path still has the maximum at δ. This completion is possible if and only if $f^l(\delta, \bar{Q}) \leq 0$ holds.

and $\quad d^* \leftarrow \min\,[d^*, f^l(1, Q^*, \delta^*)]$,

where

$$f^l(1, Q^*, \delta^*)^\dagger = \min\,[f^l(1, Q, \delta)\,|\,\delta \leq l,\ |Q| = l-2,\ Q \subseteq V - \{1, \delta\}].$$

Return to Step 2 of Phase I.

Theorem 2.

The above algorithm terminates in a finite number of steps, and d^* upon termination is equal to the optimal value of $\bar{B}$.

Proof.

The finiteness directly follows from the finiteness of V. To prove that the optimal value is obtained, first note that $f^l(1, Q, \delta)$ calculated in Step 1 of Phase II, has the interpretation as mentioned prior to the algorithm description. (This is a direct application of the principle of optimality used in dynamic programming. It is similar to the method discussed in Section 2.) $f^l(\delta, \bar{Q})$ calculated in Step 2 of Phase II may be interpreted as follows: $f^l(\delta, \bar{Q}) \leq 0$ if there exists a subpath $q = (\delta_0 = \delta, \delta_1, \cdots, \delta_{n-l})$, that starts from δ and passes through all the vertices in $\bar{Q}$ satisfying

$$\max_k \{\sum_{i=1}^{k-1} c'_{\delta_i \delta_{i+1}} \,|\, k = 1, 2, \cdots, n\text{-}l\} \leq 0 \tag{5.1}$$

$$\sum_{\kappa \in Q \cup \{1, \delta\}} \Delta_\kappa + \min_k \{\sum_{i=1}^{k} \Delta_{\delta_i} \,|\, k = 1, \cdots, n\text{-}l\} \leq 0. \tag{5.2}$$

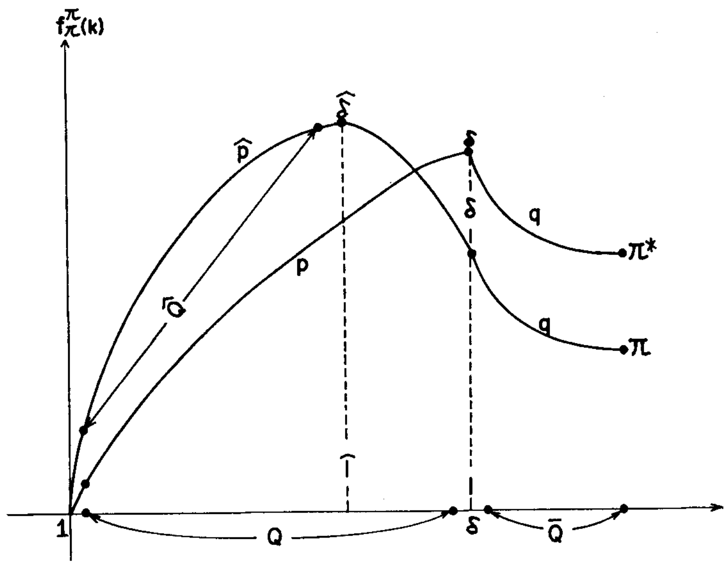

Fig. 1. The relation between π and π^*.

† Note that $f^l(1, Q^*, \delta^*)$ is set to ∞ if $f^l(1, Q, \delta) = \infty$ or $f^l(\delta, Q) = \infty$ holds for all $\delta \leq l$, $|Q| = l-2$.

(Note that $\sum_{\kappa \in Q \cup \{1, \delta\}} \Delta_\kappa$ is the margin of the truck capacity at δ.) Thus, the subpath corresponding to $f^l(1, Q, \delta)$ such that its length assumes the maximum at δ (in this portion) can be completed by subpath q corresponding to $f^l(\delta, \bar{Q})$. The resulting path still assumes its maximum length at δ (by (5·1)) and satisfies the capacity constraint (by (5·2)).

Next, we show that an optimal path π^* (see Fig. 1: π^* assumes its maximum at vertex δ) is in fact obtained by the above computation. For that, it is sufficient to prove that the π^*'s first portion p provides $f^l(1, Q, \delta)$ (i.e., $f^{\pi^*}_{\pi^*(l)} = f^l(1, Q, \delta)$) where $l = |Q| + 2$. This is proved below.

Assume that $f^{\pi^*}_{\pi^*(l)} > f^l(1, Q, \delta)$, i.e., there exists a path $\hat{p}$ that starts from vertex 1, passes through the vertices in Q (in an order different from p) and reaches vertex δ, giving $f^l(1, Q, \delta)$ smaller than $f^{\pi^*}_{\pi^*(l)}$ (see Fig. 1). Then, $\pi = (\hat{p}, q)$ is also a path satisfying the capacity constraint and having its maximum at vertex $\delta \in Q$. This implies that π is found (or a similar argument may again be applied to π) when $T_{\hat{l}}$ is solved, where $\hat{l} = |\hat{Q}| + 2 < l$. As a result, we have $d^* \leq f^\pi$ when T_l is solved. Thus, $f^l(1, Q, \hat{\delta})$ is set to ∞ in Step (1b) of Phase II, there-by contradicting the notion that $f^l(1, Q, \delta)$ ($Q \supset \hat{Q} \cup \{\hat{\delta}\}$) is obtained corresponding to π. □

6. Example

Consider the five-city problem given in Table 1, 2, 3. The computation process is illustrated in Tables 4∼8. An optimal route obtained is (1, 5, 4, 2, 3) and its value is 21. This route is shown in Fig. 2.

Table 1.

$i \backslash j$	1	2	3	4	5
1		13	6	9	3
2	7		2	4	1
3	6	8		3	2
4	2	3	7		1
5	9	2	2	3	

Table 2.

i	t_i	w_i	r_i	Δ_i
1	0			7
2	12	10	13	3
3	9	9	3	−6
4	5	5	8	3
5	3	11	6	−5

Table 3.

$i \backslash j$	1	2	3	4	5
1		25	15	14	6
2	−5		−1	−3	−8
3	−3	11		−1	−4
4	−3	10	11		−1
5	6	11	8	5	

Table 1. c_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j)$

Table 2. t_i, r_i, w_i, Δ_i $(t_1 = 0,\ \Delta_i = r_i - w_i,\ \Delta = 37,\ \Delta_1 = \Delta - \sum_{i=2}^{5} r_i)$

Table 3. $c_{ij}' = c_{ij} + t_j - t_i$ $(1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j)$

Table 4. Computation Process of the example in Section 6.

T_l	STEP 1							STEP 2				STEP 3	
	(1*a*) (1*b*)				(1*c*)			(2*a*) (2*b*)	(2*c*)			π^*	d^*
	$(1, Q, \beta)$	$f^l(1, Q, \beta)$	$M(1, Q, \beta)$	$\pi(1, Q, \beta)$	$(1, Q, \delta)$	$f^l(1, Q, \delta)$	$\pi(1, Q, \delta)$		$(\delta, \bar{Q})$	$f^l(\delta, \bar{Q})$	$\pi(\delta, \bar{Q})$		
T_2	—	—	—	—	$(1, \phi, 2)$	25	(1, 2) →	TABLE 5	(2, {3,4,5})	−3	(3, 4, 5) →	(1,2,3,4,5)	25
T_3	$(1, \phi, 2)$	∞	—	—									
	$(1, \phi, 3)$	15	15	(1, 3)	Others	∞	—						
	$(1, \phi, 4)$	14	14	(1, 4) →	(1, {4}, 2)	24	(1, 4, 2) →	TABLE 6	(2, {3, 5})	−5	(3, 5) →	(1,4,2,3,5)	24
	$(1, \phi, 5)$	6	6	(1, 5) →	(1, {5}, 2)	17	(1, 5, 2) →	TABLE 7	(2, {3, 4})	∞	—		
T_4	$(1, \phi, 2)$	∞	—	—									
	$(1, \phi, 3)$	15	15	(1, 3)									
	$(1, \phi, 4)$	14	14	(1, 4)	Others	∞	—						
	$(1, \phi, 5)$	6	6	(1, 5)									
	(1, {3}, 4)	14	15	(1, 3, 4)									
	(1, {4}, 5)	13	14	(1, 4, 5) →	(1, {4,5}, 2)	21	(1, 5, 4, 2) →	TABLE 8	(2, {3})	−1	(3) →	(1,5,4,2,3)	21
	(1, {5}, 4)	11	11	(1, 5, 4) →									
	(1, {5}, 2)	17	17	(1, 5, 2)									
	Others	∞	—	—									
T_5	All	∞	—	—									

Table 5. Calculation of $f^l(\delta, \bar{Q})$ for $f^2(1, \phi, 2)$.

Step (2a) (2b)			
	(δ, Q', γ)	$f^2(\delta, Q', \gamma)$	$\pi(\delta, Q', \gamma)$
$\lvert Q'\rvert=0$	$(2, \phi, 3)$	-1	(3)
	$(2, \phi, 4)$	-3	(4)
	Others	∞	—
$\lvert Q'\rvert=1$	$(2, \{3\}, 4)$	-2	(3, 4)
	$(2, \{5\}, 4)$	-3	(5, 4)
	$(2, \{4\}, 5)$	-4	(4, 5)
	Others	∞	—
$\lvert Q'\rvert=2$	$(2, \{3, 4\}, 5)$	-3	(3, 4, 5)
	Others	∞	—

Table 6. Calculation of $f^l(\delta, \bar{Q})$ for $f^3(1, \{4\}, 2)$.

Step (2*a*) (2*b*)			
	(δ, Q', γ)	$f^3(\delta, Q', \gamma)$	$\pi(\delta, Q', \gamma)$
$\lvert Q'\rvert=0$	$(2, \phi, 3)$	-1	(3)
	$(2, \phi, 5)$	-8	(5)
$\lvert Q'\rvert=1$	$(2, \{3\}, 5)$	-5	(3, 5)
	$(2, \{5\}, 3)$	0	(5, 3)

Table 7. Calculation of $f^l(\delta, \bar{Q})$ for $f^3(1, \{5\}, 2)$.

Step (2a) (2b)		
(δ, Q', γ)	$f^3(\delta, Q', \gamma)$	$\pi(\delta, Q', \gamma)$
$(2, \phi, 4)$	-3	(4)
Others	∞	—

Table 8. Calculation of $f^l(\delta, \bar{Q})$ for $f^4(1, \{4, 5\}, 2)$.

Step (2*a*) (2*b*)		
(δ, Q', γ)	$f^4(\delta, Q', \gamma)$	$\pi(\delta, Q', \gamma)$
$(2, \phi, 3)$	-1	(3)

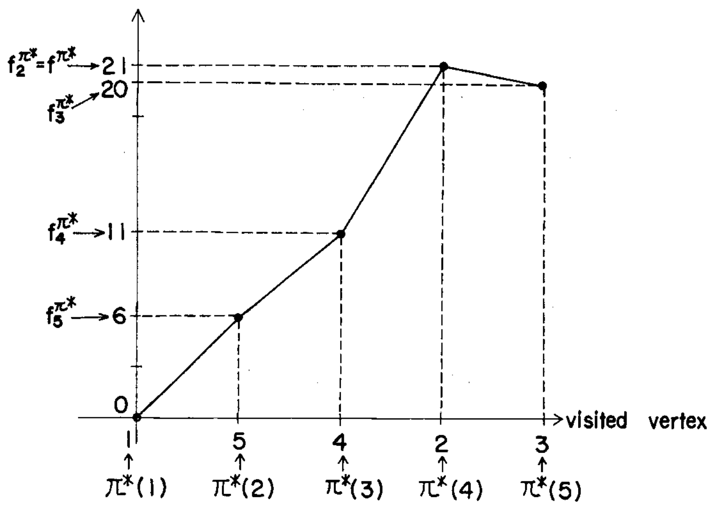

Fig. 2. An optimal solution π^* of the example.
(an optimal route; 1→5→4→2→3)

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References

1) R. Bellman; J. ACM. **9**, pp. 61–63 (1962).
2) M. Bellmore and G. L. Nemhauser; J. Opr. Res. **16**, pp. 538–558 (1968).
3) M. Held and R. M. Karp; SIAM J. Appl. Math. **10**, pp. 196–210 (1962).
4) H. Ishii, T. Ibaraki and H. Mine; Spring Convention Record of Operations Society of Japan, pp. 47–48 (1974).
5) H. Ishii, T. Ibaraki and H. Mine; Fall Convention Record of Operations Research Society of Japan, pp. 29–30 (1974).
6) H. Ishii, T. Ibaraki and H. Mine; Keiei Kagaku, **19**, pp. 112–122 (1975).
7) S. Mitsumori; Research Report of Automatic Control Group AC-74-13, Institute of Electrial Engineering of Japan, 1974.