On Relaxation of Infeasibility of Linear Programming Constraints

By

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(Received June 30, 1975)

Abstract

In an optimization of a real system planning or design, when a prescribed set of constraints is found infeasible, it needs to be modified somehow. This paper presents a method for making an infeasible set of linear programming constraints feasible. Infeasibility is detected in phase I of the two-phase simplex procedure. Numerical data in the final simplex tableau of phase I are used for relaxing infeasibility. The relaxation is carried out by modifying some upper or lower bounds of the constraints. A sufficient condition for the relaxation is derived. The method allows a wide variety of modifications of the constraints. Thus, it could effectively be used in a practical linear programming design.

1. Introduction

When an optimization problem of a real system is mathematically formulated in the form of a linear programming problem, it sometimes turns out to be infeasible due to, *e.g.*, too heavy requirements on the system. If that happens, the constrains of the problem need to be modified somehow in order to make a usable sense. In the case of a small scale problem, such a modification may be possible only by inspection based on some practical insight. However, in many real system cases of a large scale, an intuitive means does not work effectively. The usual packaged computer-codes for linear programming problems seem to give no special regard to this point.[1)]

The purpose of this paper is to present a method for making infeasible linear programming constraints feasible. Infeasibility is detected in phase I of the two-phase simplex, or revised simplex method. The final simplex tableau of phase I gives us information for relaxing infeasibility. The relaxation is done by relieving some of the upper and/or the lower bounds of the constraints. A sufficient condition for the relaxa-

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tion is derived. It yields a wider variety of modifications of the bounds than conventional ones.

2. Simplex Method with Two Phases and Infeasibility of Constraints

The constraints of a linear programming problem are generally given by the following form:

$$A_1x\geq b_1, \quad A_2x=b_2, \quad A_3x\leq b_3 \tag{1}$$

$$x\geq 0, \qquad b_i\geq 0 \quad (i=1, 2, 3) \tag{2}$$

where x is the n-variable vector, b_i the m_i-vector, and A_i the $m_i\times n$-matrix.* Assume that the right side of each constraint (1) initially has the value b_i^0, $i=1, 2, 3$.

In order to obtain a basic feasible solution, we usually use a two-phase algorithm. The phase-I problem is the following linear programming problem containing artificial variable vectors w_1 and w_2:

minimize the objective function

$$z=e'_{m_1}w_1+e'_{m_2}w_2 \tag{3}$$

subject to the constraints

$$\begin{array}{lllll} A_1x & -s_1 & & +w_1 & =b_1 \\ A_2x & & & & +w_2=b_2 \\ A_3x & & +s_3 & & =b_3 \\ x, & s_1, & s_3, & w_1, & w_2\geq 0 \end{array} \tag{4}$$

where s_i is the m_i-slack variable vector for making the constraints (1) equality, and w_i the m_i-vector; e_k is the k-vector with all the components of unity. A prime denotes the transposition of a vector or a matrix.

By choosing an initial set of basic feasible solutions as

$$w_i=b_i \ (i=1, 2), \quad s_3=b_3, \quad x=s_1=0 \tag{5}$$

the linear programming problem (3) and (4) can be solved by a simplex, or a revised simplex method. Assume that, after suitable transformations of the simplex tableau, Eqs. (3) and (4) have been changed into the following form:

$$\bar{c}'_1x_N+ \ \bar{c}'_2w_N=z-z^* \tag{6}$$

$$\begin{array}{r} x_B \quad +\bar{A}_{11}x_N+\bar{A}_{12}w_N=\bar{b}_1 \\ w_B+\bar{A}_{21}x_N+\bar{A}_{22}w_N=\bar{b}_2 \end{array} \tag{7}$$

where

* Throughout the paper, vectors are in column form; and, for k-vectors a and b, the expression $a\geq b$ means that the inequality sign holds component-wise.

$$c_1 \triangleq -\bar{A}'_{21}e_{l_2} \geq 0, \quad c_2 \triangleq e_{m_1+m_2-l_2} - \bar{A}'_{22}e_{l_2} \geq 0$$

$$z^* \triangleq e'_{l_2}\bar{b}_2, \quad \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} \triangleq \bar{b} \triangleq B^{-1}b \geq 0, \quad b \triangleq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{8}$$

$(x'_B, w'_B)'$ and $(x'_N, w'_N)'$ are the basic and nonbasic variables, respectively, in the present tableau; x_B and w_B are the l_1- and l_2-vectors composed of l_1 and l_2 components in $\tilde{x} \triangleq (x', s'_1, s'_3)'$ and $w \triangleq (w'_1, w'_2)'$, respectively. B is the $m \times m$-basis matrix for the present tableau, m being $m_1+m_2+m_3$.

Equations (6) through (8) show that the constraints (1) and (2) are *infeasible* if and only if[2]

$$z^* > 0 \tag{9}$$

3. Relaxation of Infeasibility by Modifying the Constraints

When we have known the constraints to be infeasible, we have to make them feasible in one way or another. One of the methods is to subtract the values $\bar{b}_2$ from b_1^0 and/or b_2^0 corresponding to the nonnegative artificial variables w_B.* We may, however, not want to change the values of the components of b_1 or b_2 associated with these artificial variables. In this paper, we derive a sufficient condition for making infeasible constraints feasible, which yields a wide variety of possible modifications of the value of b.

Feasibility is attained, if the value of b is changed in such a way that the inequality (9) is violated by keeping the basic variables nonnegative, that is, if

$$z^* \leq 0, \quad \bar{b} \geq 0 \tag{10}$$

is satisfied. The conditions (10) are equivalent to

$$\bar{b}_1 \geq 0, \quad \bar{b}_2 = 0 \tag{11}$$

or

$$\tilde{B}_1 b \geq 0, \quad \tilde{B}_2 b = 0, \quad \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \triangleq B^{-1} \tag{12}$$

where $\tilde{B}_i$ is the $l_i \times$m-matrix. Hence, if we change b by Δb, *i.e.*,

$$b = b^0 - \Delta b, \quad b^0 \triangleq \begin{bmatrix} b_1^0 \\ b_2^0 \\ b_3^0 \end{bmatrix} \tag{13}$$

Δb must satisfy

$$\tilde{B}_1 \Delta b \leq \bar{b}_1^0, \quad \tilde{B}_2 \Delta b = \bar{b}_2^0, \quad \bar{b}_i^0 \triangleq \tilde{B}_i b^0 \quad (i=1, 2) \tag{14}$$

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* The computer-code for linear programming problems in the IBM Mathematical Programming System/360 utilizes this method.[1]

Note that, since c_i are independent of b, the nonnegativity of c_i of Eqs. (8) are always retained.

When we use the new value of b as given by Eqs. (13) and (14), the region S of x satisfying the constraints (1) and (2) has the following quality:

Theorem 1

Regardless of the value of Δb, S is an at most $(n-1)$-dimensional subspace in the n-dimensional space, particularly if $c_1>0$, S consists of only one point.

Proof

If $m_2\neq 0$ and $A_2\neq 0$, the first half of the theorem is obvious. Assume that $m_2=0$ or $A_2=0$. For proof of the first half, it is sufficient to show that, in Eqs. (4), s_i $(i=1, 3)$ can not be positive without w_i.

Due to the nonsingularity of B, the slack variables in Eqs. (4) containing l_2 components of w_B can not be basic variables. Hence, the vector x_N contains at least l_2 components of s_1. The columns of $\bar{A}_{21}$ corresponding to these components are of the form $(0, \cdots, 0, -1, 0, \cdots, 0)'$, and consequently the components of corresponding c_1 have the value of unity.

On the other hand, from Eqs. (11), the second of Eqs. (7) without the artificial variables becomes

$$\bar{A}_{21}x_N=0 \tag{15}$$

Equation (15) consists of l_2 equations. Summing up these equations and using the definition of c_1 yields

$$-c_1'x_N=0 \tag{16}$$

We conclude that, from the above, at least l_2 components of s_1 have to be zero.

In particular, when $c_1>0$, we have $x_N=0$ from Eq. (16). Therefore, the solution of Eqs. (7) without w_B and w_N is only

$$x_B=\bar{b}_1, \qquad x_N=0 \tag{17}$$

Q. E. D.

Theorem 2

For b_i given by

$$\begin{aligned} b_1&=b_1^0-\Delta b_1-\Delta b_1^+ \\ b_2&=b_2^0-\Delta b_2 \\ b_3&=b_3^0-\Delta b_3+\Delta b_3^+ \end{aligned} \tag{18}$$

where $\Delta b \triangleq (\Delta b_1', \Delta b_2', \Delta b_3')'$ is of any value satisfying Eqs. (14) *and Δb_i^+ is any m_i-nonnegative vector, the constraints* (1) *and* (2) *are feasible.*

Proof

From the assumption, there exists x satisfying

$$\begin{aligned} A_1x &\geq b_1^0-\Delta b_1 \\ A_2x &= b_2^0-\Delta b_2 \\ A_3x &\leq b_3^0-\Delta b_3 \end{aligned} \tag{19}$$

Since, by the nonnegativity of Δb_i^+,

$$\begin{aligned} A_1x+\Delta b_1^+ &\geq A_1x \\ A_3x-\Delta b_3^+ &\leq A_3x \end{aligned} \tag{20}$$

the theorem is true.

Q. E. D.

4. Infeasibility in the Dual Simplex Method

For simplicity, let $m_2=0$. We consider the following linear programming problem with slack variables:

minimize the objective function

$$z=d'x \tag{21}$$

subject to the constraints

$$\begin{aligned} -A_1x+s_1 \quad &= -b_1 \\ A_3x \quad +s_3 &= \ b_3 \\ x,\ s_1,\ s_3\leq 0,\quad & b_i\geq 0 \quad (i=1, 3) \end{aligned} \tag{22}$$

where d is any n-nonnegative vector.

We can choose an initial set of dual feasible solutions as

$$s_1=-b_1, \quad s_3=b_3 \tag{23}$$

Assume that, by a dual simplex procedure, Eqs. (21) and (22) are transformed into

$$\bar{d}'x_N=z-z^* \tag{24}$$

$$\begin{aligned} x_{1B} \quad +\bar{A}_1x_N &= \ \bar{b}_1 \\ x_{2B}+\bar{A}_2x_N &= -\bar{b}_2 \end{aligned} \tag{25}$$

where

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} \triangleq \bar{b} \triangleq B^{-1}b, \quad \bar{b}_i\geq 0, \quad \bar{d}\geq 0 \tag{26}$$

x_{iB} is the l_i-vector composed of l_i components in x.

If the components of at least one row of the matrix $\bar{A}_2$ are all nonnegative, the constraints (1) and (2) are infeasible.[2] Thus, if we change b in such a way that

$$\bar{b}\geq 0 \tag{27}$$

holds, the constraints become feasible. For that purpose, Δb of Eqs. (13) needs to satisfy

$$\begin{aligned} &\tilde{B}_1 \Delta b \leq \bar{b}_1^0, \qquad \tilde{B}_2 \Delta b \geq \bar{b}_2^0 \\ &\bar{b}_1^0 \triangleq \tilde{B}_1 b^0, \qquad \bar{b}_2^0 \triangleq -\tilde{B}_2 b^0 \end{aligned} \tag{28}$$

Theorem 2 is true also in the present case; while Theorem 1 is not true.

5. Illustrative Example

Let us consider a simple two-variable problem:

$$3x_1 + 8x_2 \geq 24 \tag{29}$$

$$10x_1 + 3x_2 \geq 30 \tag{30}$$

$$x_1 + x_2 \leq 4 \tag{31}$$

$$x_1, \quad x_2 \geq 0 \tag{32}$$

Figure 1 shows the boundaries of the constraints. By inspection, we know of no existence of x_1 and x_2 satisfying all the constraints. By introducing slack and artificial variables and exchanging the basis several times, we obtain the tableau

$$\begin{array}{llr} & s_1 + \frac{4}{7} s_2 + \frac{103}{7} s_3 + \frac{3}{7} w_2 = z - \frac{34}{7} & \\ x_1 & \quad - \frac{1}{7} s_2 - \frac{3}{7} s_3 + \frac{1}{7} w_2 = \frac{18}{7} & \\ \quad x_2 & \quad + \frac{1}{7} s_2 + \frac{10}{7} s_3 - \frac{1}{7} w_2 = \frac{10}{7} & \\ & w_1 - s_1 - \frac{5}{7} s_2 - \frac{71}{7} s_3 + \frac{5}{7} w_2 = \frac{34}{7} & \end{array} \tag{33}$$

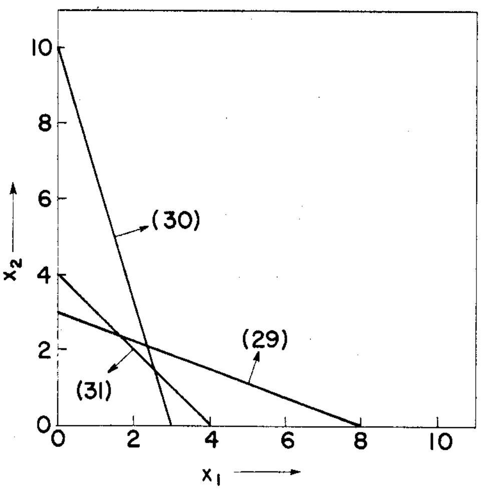

Fig. 1. The boundaries of the constraints (29) to (31).

Surely, the constraints are not feasible.

Now we try to change the values of b according to the previous discussion. From Eqs. (14), Δb_i $(i=1, 2, 3)$ need to satisfy

$$\begin{aligned} \Delta b_2 - 3\Delta b_3 &\leq 18 \\ -\Delta b_2 + 10\Delta b_3 &\leq 10 \\ 7\Delta b_1 + 5\Delta b_2 - 71\Delta b_3 &= 34 \end{aligned} \tag{34}$$

Equations (34) are satisfied, for instance, by three sets of Δb_i:

$$\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \begin{bmatrix} \frac{34}{7} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{34}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{34}{71} \end{bmatrix} \tag{35}$$

We readily see that S resulting from each of these sets is only one point.

6. Conclusion

A simple method has been presented for making infeasible linear programming constraints feasible. When infeasibility is found in phase I of the two-phase procedure, the way to relax infeasibility is established by Theorems 1 and 2. The method has been also applied to infeasible constraints found in a dual simplex procedure. As the present method yields a wide variety of modifications of constraints, it could effectively be used in a practical linear programming design.

Acknowledgment

The authors would like to acknowledge the helpful discussions of Dr. N. Sannomiya, Lecturer of Kyoto University.

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