Representation of Water Particle Velocity of Breaking Waves on Beaches by Dean's Stream Function

By

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(Received September 25, 1975)

Experimental values of water particle velocity of breaking waves on uniformly sloping beaches differ considerably from the theoretical values of Stokes waves of the third order and cnoidal waves of the second approximation. In this paper, Dean's stream functions are calculated by giving simultaneously measured time variations of the water level of the breaking waves. Vertical distributions of horizontal water particle velocity at the crest phase, calculated by using these stream functions, are compared with experimental distributions in order to discuss the applicability of Dean's stream function method. These theoretical distributions can explain the experimental results well, although this stream function method assumes permanent waves in uniform depth. Dean's stream function can express an asymmetric wave profile as that of breaking waves on sloping beaches, while the Stokes wave theory has a symmetric wave profile. It is suggested that the breaking wave profile dominates the water particle velocity field of breaking waves.

1. Introduction

The experimental results of water particle velocity field of breaking waves on uniformly sloping beaches were already presented by the authors in 1972¹⁾ and 1974²⁾. In these papers, the vertical distributions of horizontal water particle velocity at the crest phase were found to differ considerably from those by the Stokes wave theory of the third order³⁾ and the cnoidal wave theory of the second approximation⁴⁾, calculated by giving the experimental values of breaker depth h_b, wave period T and breaker height H_b. The vertical distributions of cnoidal waves were too steep compared with the experimental distributions. The slopes of distribution of Stokes waves resembled those of experimental distributions, but the values of velocities themselves were considerably large compared with the experimental ones. Stokes waves by the second definition of wave celerity⁵⁾ provided little improvement on the difference between the theoretical and experimental values. It was furthermore shown that the

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dimensionless expression of the vertical distributions of horizontal water particle velocity at the crest phase can be classified by the values of beach slope i and deepwater wave steepness H_0/L_0 which are predominant factors for classificaiton of the types of breakers.

Dean[6] proposed a stream function representing ocean waves. The coefficients of the series of this stream function are determined by the method of least squares, so as to minimize the deviation of the calculated wave profile from the measured wave profile and the variation of the Bernoulli constant at the water surface. He presented, using this stream function, the graphs of wave forces and moments acting on a vertical cylinder and showed that the ratio of maximum wave height to water depth is equal to 1.0[7),8)]. This stream function can represent an asymmetric wave profile, while it assumes the permanent waves propagating in uniform depth. Wave breaking phenomenon on a beach is the final stage in wave transformation process, and the profile of breaking waves on the beach is obviously asymmetric. In this investigation, it is discussed, from the view point of wave profile asymmetry, whether Dean's stream function can explain the vertical distribution of horizontal water particle velocity at the crest phase of breaking waves on the beach or not.

The experiments to measure the water particle velocity were conducted in a wave tank at Kyoto University. The experimental apparatus and procedure were described in the previous paper[2].

2. Dean's Stream Function[6]

A brief introduction of Dean's stream function method will be given. Permanent waves propagating in uniform depth h at a constant celerity L/T (L: wave length, T: wave period) are described with the following stream function ψ in the reference x-z moving with the waves (x: abscissa and positive in the direction of wave propagation, z: ordinate and positive in the upward direction, and $z=0$ at the still water level):

$$\psi=\frac{L}{T}z+\sum_{n=4,6,8\cdots}^{N-1}\sinh\frac{(n-2)\pi(h+z)}{L}\times\left\{X_n\cos\frac{(n-2)\pi x}{L}+X_{n+1}\sin\frac{(n-2)\pi x}{L}\right\}\tag{1}$$

Sine terms in Eq. (1) are introduced so as to represent an asymmetric wave profile. At the water surface $z=\eta$, the value of the stream function $\psi(x, \eta)$ becomes constant ($\equiv X_3$). The wave profile η is therefore expressed as follows:

$$\eta=\frac{T}{L}X_3-\frac{T}{L}\cdot\sum_{n=4,6,8\cdots}^{N-1}\sinh\frac{(n-2)\pi(h+\eta)}{L}\times\left\{X_n\cos\frac{(n-2)\pi x}{L}+X_{n+1}\sin\frac{(n-2)\pi x}{L}\right\}\tag{2}$$

The wave period determined from the record of water level is adopted as the initial value of wave period $T(\equiv X_2)$. The wave length of small amplitude waves calculated by using the initial value of the wave period and the breaker depth is adopted as the initial value of wave length $L(\equiv X_1)$. The quantity indicating the deviation of the wave profile calculated from the measured profile is defined as follows:

$$E_2=\frac{1}{I}\sum_{i=1}^{I}(\eta_{mi}-\eta_{pi})^2 \tag{3}$$

where η_m and η_p are the measured and calculated wave profiles respectively, and $\eta_i=\eta(i\Delta t)$, $i=1, 2, \ldots\ldots, I$ $(I\Delta t=T)$. The initial values of other coefficients $X_3\sim X_N$ are determined so as to minimize the quantity E_2; $\partial E_2/\partial X_n=0$ $(n=3\sim N)$.

The stream function of Eq. (1) obviously satisfies the Laplace equation, the boundary condition on the sea bed and the kinematic condition at the water surface. The dynamic condition at the water surface is expressed in the moving reference as follows ($p=0$ at the water surface):

$$\eta+\left\{\left(u-\frac{L}{T}\right)^2+w^2\right\}\Big/2g=Q \tag{4}$$

where u and w are x and z components of water particle velocity in the fixed reference respectively, and Q is the Bernoulli constant. The quantity indicating the variation of Q is defined as follows, in order to minimize the variation of the Bernoulli constant during one wave period:

$$E_1=\frac{1}{I}\sum_{i=1}^{I}(Q_i-\bar{Q})^2,\quad \bar{Q}=\frac{1}{I}\sum_{i=1}^{I}Q_i \tag{5}$$

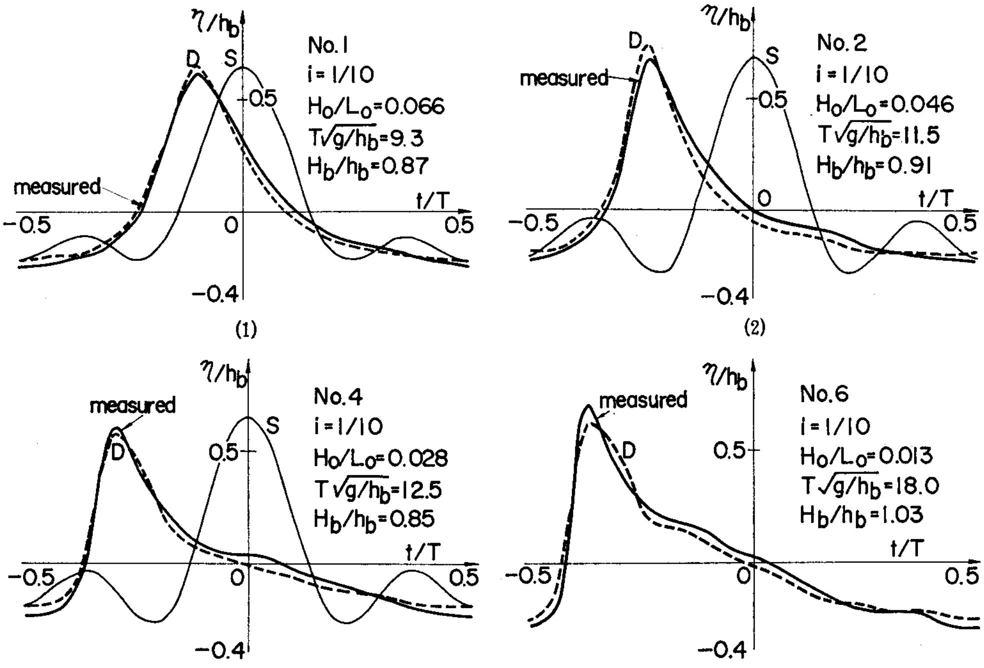
If the initial values of $X_n(n=1\sim N)$ increase by $X_n'(n=1\sim N)$, E_T which is equal to E_1+E_2 is approximately expressed as follows:

$$\begin{aligned}E_T=&\frac{1}{I}\sum_{i=1}^{I}\left[\left\{Q_i+\sum_{n=1}^{N}\frac{\partial Q_i}{\partial X_n}X_n'\right\}-\left\{\bar{Q}+\sum_{n=1}^{N}\frac{\partial\bar{Q}}{\partial X_n}X_n'\right\}\right]^2\\&+\frac{1}{I}\sum_{i=1}^{I}\left[\left\{\eta_{pi}+\sum_{n=1}^{N}\frac{\partial\eta_{pi}}{\partial X_n}X_n'\right\}-\eta_{mi}\right]^2\end{aligned} \tag{6}$$

The equations $\partial E_T/\partial X_j'=0(j=1\sim N)$ to minimize E_T become the N coupled equations in the N unknowns X_j'. If the solutions X_j' are not sufficiently small, the same procedure is repeated using X_j+X_j' as the new X_j.

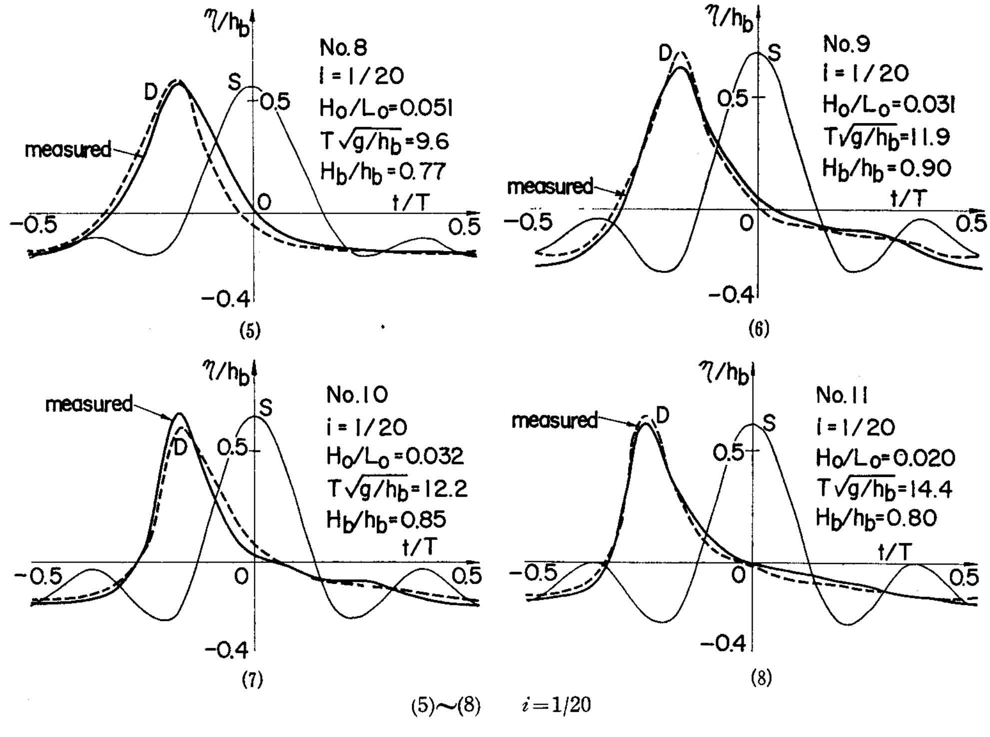
3. Calculation of Dean's Stream Functions

The stream functions were obtained for twenty cases among the experiments (twenty three cases) presented in the previous paper[2]. For three cases (Nos. 13~15), the stream functions could not be obtained because of a lack of the records of simulta-



(1) (2) (3) (4)

(1)~(4) $i=1/10$



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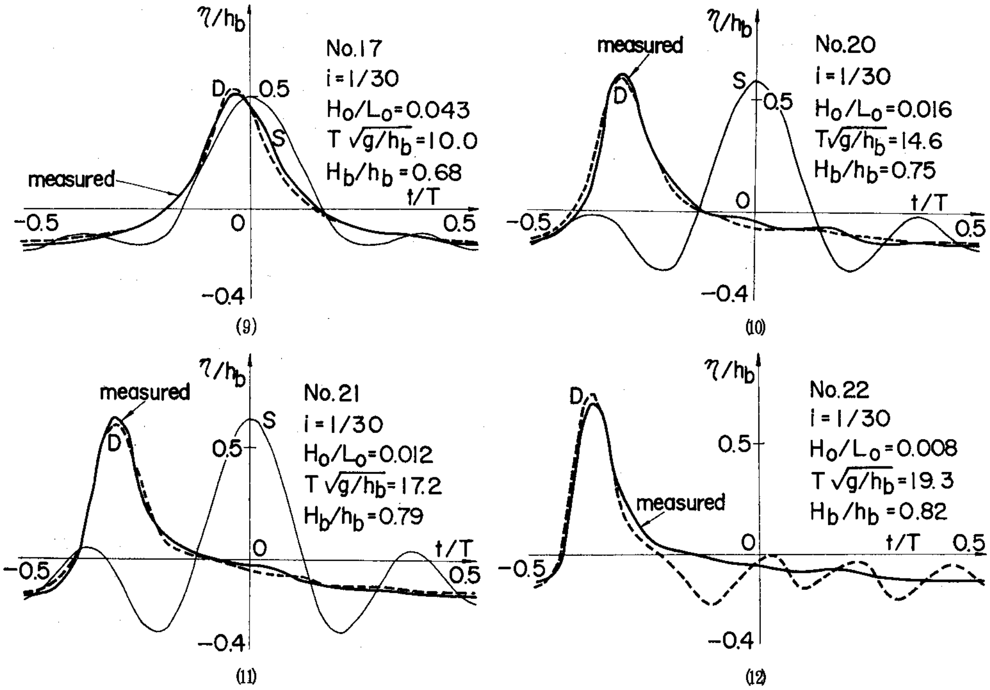


(9)~(12) $i=1/30$

Fig. 1. Comparisons between measured and Dean's theoretical wave profiles of breakers.

Table 1. Wave breaking conditions and comparisons between experimental and theoretical values.

No.	$T\sqrt{g/h_b}$	H_b/h_b	$(R.E.)_D$	$(R.E.)_S$	$\frac{\eta_{p0}-\eta_{m0}}{\eta_{m0}}$	$\left.\frac{u_D-u_E}{u_E}\right\|_{1.0}$	$\left.\frac{u_{S'}-u_E}{u_E}\right\|_{1.0}$	$\left.\frac{u_S-u_E}{u_E}\right\|_{1.0}$
1	9.3	0.87	0.00	0.00	0.07	0.27	0.78	1.05
2	11.5	0.91	−0.09	0.00	0.10	0.14	0.76	1.02
3	12.6	0.85	−0.03	0.01	0.05	−0.06	0.45	0.63
4	12.5	0.85	0.00	0.00	−0.04	0.15	0.79	1.03
5	14.0	0.98	0.06	0.03	−0.04	0.16	0.93	0.95
6	18.0	1.03	−0.06	0.28	−0.10	0.04		
7	7.7	0.66	0.01	0.00	−0.01	0.27	0.53	0.73
8	9.6	0.77	0.00	0.00	0.02	0.08	0.51	0.69
9	11.9	0.90	0.00	0.00	0.09	0.26	0.90	1.18
10	12.2	0.85	−0.02	0.00	−0.08	0.15	0.79	1.03
11	14.4	0.80	−0.07	0.02	0.06	0.02	0.60	0.79
12	18.7	0.92	−0.02	0.23	0.00	−0.07		
16	9.1	0.69	0.05	0.00	−0.10	−0.10	0.28	0.43
17	10.0	0.68	−0.03	0.00	0.02	0.14	0.38	0.54
18	11.8	0.66	−0.26					
19	13.1	0.86	0.02	0.01	0.01	0.14	0.61	0.84
20	14.6	0.75	−0.01	0.02	−0.02	0.10	0.50	0.67
21	17.2	0.79	−0.03	0.07	−0.05	0.12	0.56	0.78
22	19.3	0.82	−0.06	0.18	0.06	0.00		
23	23.9	0.89	−0.07	0.71	−0.02	−0.02		

neously measured wave profile. For each case, a wave η_m during one wave period was selected in the record of simultaneously measured time variation of water level at the breaking point. The number of division of one wave period I was set equal to 15. The number of terms in the series of Eq. (1) N was set equal to 13 corresponding to Stokes waves of the 5th order. The calculations to minimize E_T of Eq. (6) were repeated five times.

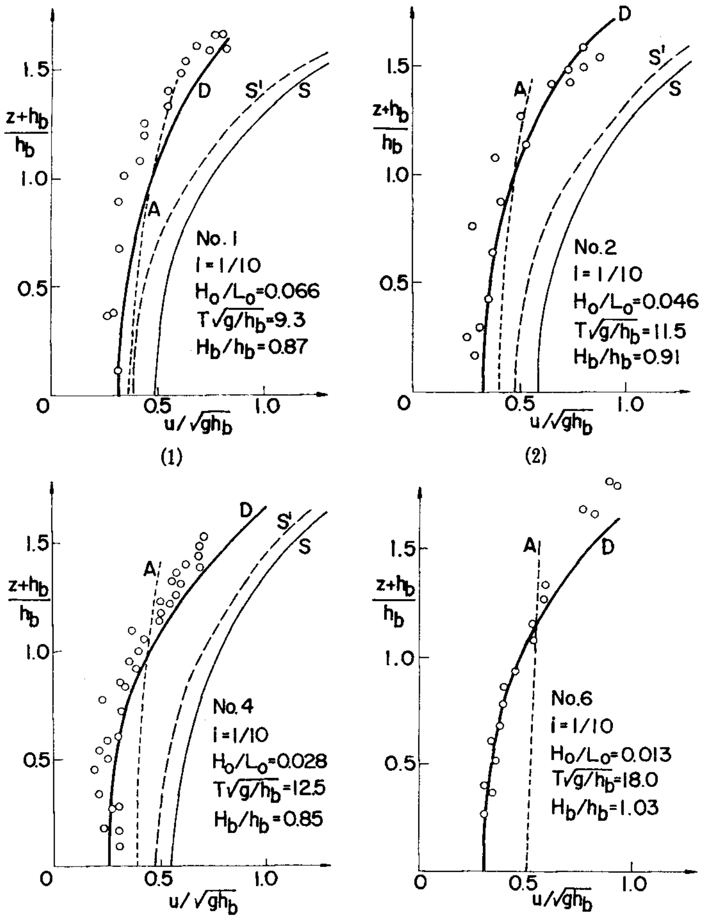
Figs. 1, (1)~(12) show the examples of comparisons between the theoretical wave profile η_p calculated from Dean's stream function (thick broken line indicated with "D") and the measured wave profile η_m (thick full line). The results of four cases in each beach slope i are selected and shown. In these figures, the values of the beach slope i, the deepwater wave steepness H_0/L_0, $T\sqrt{g/h_b}$ and H_b/h_b are also shown. In the previous paper, the breaker height obtained from the photographs taken with a 16 mm high speed cine camera was used as H_b. In this calculation, the breaker height obtained from the record of water level at the breaking point is used. Table 1 shows the values of $T\sqrt{g/h_b}$ and H_b/h_b for all cases. In this table, the relative errors $(R.E.)_D$ between the 4th and 5th values of the horizontal water particle velocity at the crest phase of the sea bed in repeated calculations are shown. These relative errors indicate the convergency of repeated calculations. The relative error of Case No. 18 is larger than 0.1.

In Fig. 1, the theoretical wave profile of Stokes waves of the 3rd order[3] calculated by giving the values of $T\sqrt{g/h_b}$ and H_b/h_b (thin full line indicated with "S") is also shown. In the calculations of the Stokes wave profile, the computations of the ratio of breaker depth to wave length h_b/L_b were repeated twenty times. As well as Dean's stream function, the relative errors $(R.E.)_S$ of the 19th and 20th calculated values of the velocity are also shown in Table 1 as the parameter of convergency. The Stokes wave profiles of Nos. 6 and 22 are not shown in Fig. 1, because the relative errors of these cases are larger than 0.1.

The horizontal water particle velocity u of Dean's stream function Eq. (1) is expressed as follows in the fixed reference:

$$u=-\sum_{n=4,6,8\cdots}^{N-1}\frac{(n-2)\pi}{L}\cosh(n-2)\pi\frac{h+z}{L}\times\left[X_n\cos(n-2)\pi\frac{x}{L}+X_{n+1}\sin(n-2)\pi\frac{x}{L}\right] \tag{7}$$

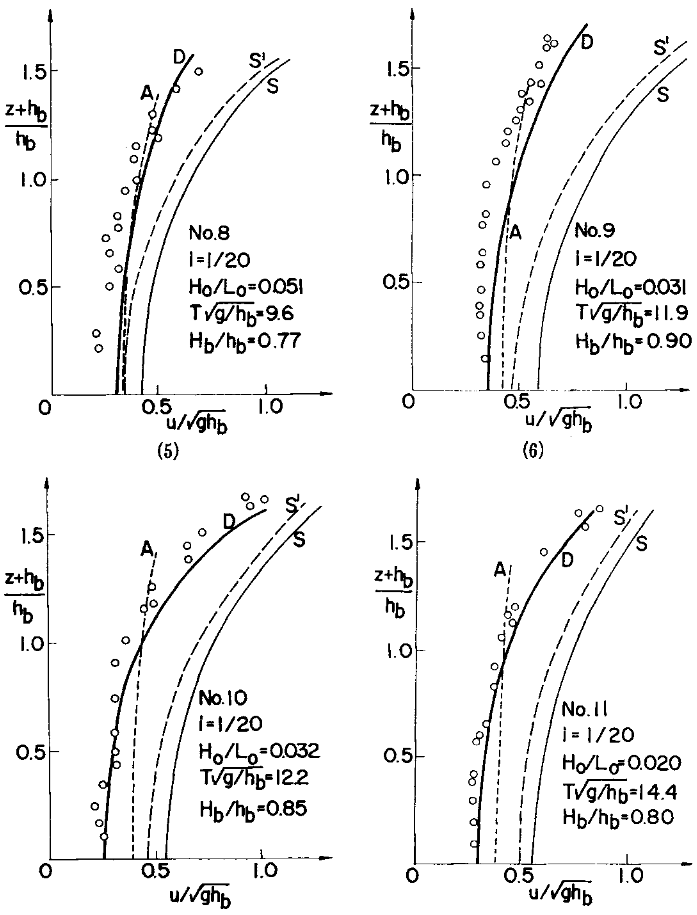
The horizontal water particle velocity at the crest phase is obtained by using x_i of the crest phase in Eq. (7). Figs. 2, (1)~(12) show the comparisons between the theoretical vertical distributions of u at the crest phase by Dean's stream function (thick full line indicated with "D") and experimental data of the same cases as shown in Fig. 1. The theoretical distributions of Stokes waves of the third order based



(1) (2)

(3) (4)

(1)～(4) $i=1/10$



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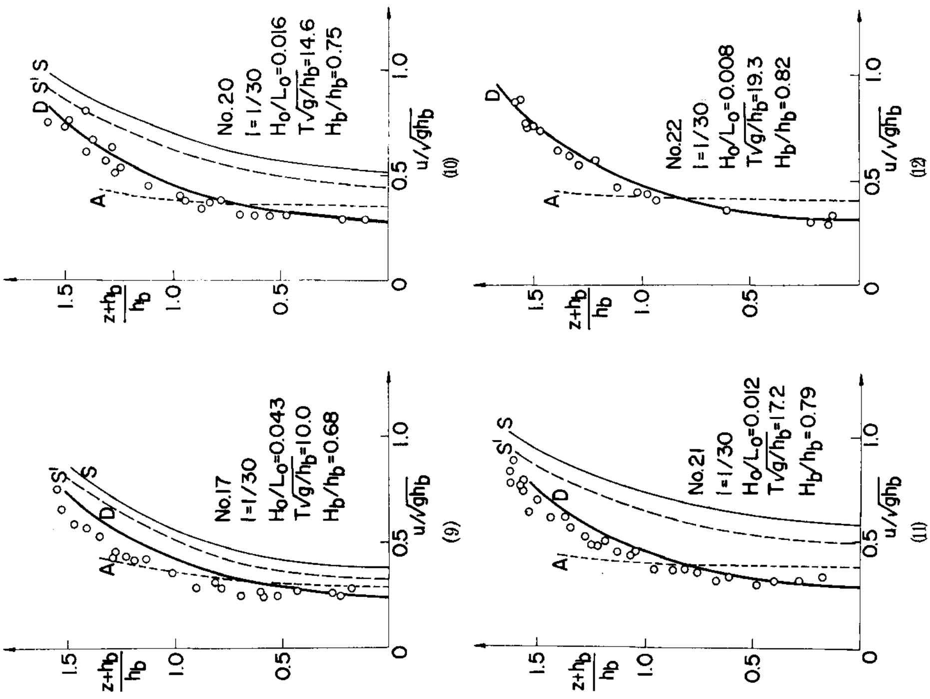

Fig. 2. Comparisons between measured and Dean's theoretical water particle velocities of breakers. (9)~(12) $i=1/30$

on the first and second definitions of wave celerity[5] (thin full line indicated with "S" and broken line with "S′" respectively), and small amplitude waves (thin dotted line indicated with "A") are also shown in these figures.

4. Discussions

As seen in Fig. 1, the measured wave profile (time variation of water level) during one wave period is obviously asymmetric at the breaking point on the beach. The Stokes wave theory which has a symmetric wave profile can not explain this fact. The method of Dean's stream function in itself determines the values of coefficients X_n, so as to minimize the deviation of the calculated wave profile from the measured profile. The wave profile by Dean's stream function therefore resembles the measured profile well. The water level at the crest of the wave profile by Dean's stream function may have a direct effect on the water particle velocity at the crest phase. The differences between the water levels of the measured η_{m0} and Dean's theoretical profile η_{p0} at the crest phase are shown in Table 1. The differences are lower than 10% in all eighteen cases (except for No. 18) and exceed 5% in ten cases.

It is not easy to represent quantitatively the degree of agreement between the experimental and theoretical distributions of water particle velocity. It was tried, in the previous paper[1], to express the degree of agreement by two quantities of the mean value $\bar{u}$ in the vertical direction and the mean slope of distribution $\Delta(u/\sqrt{gh})/\Delta(z+h/h)$. In this paper, at first, the calculated results will be discussed qualitatively before such quantitative discussions.

As seen in Fig. 2, the theoretical value of Stokes waves of the first definition of wave celerity is largest. The theoretical value of Stokes waves of the second definition of wave celerity is not so large as the value of the first definition, but considerably large compared with the experimental value. The theoretical values of both Dean's stream function waves and small amplitude waves are smaller than the values of two kinds of Stokes waves. The theoretical distributions of Dean's stream function waves and small amplitude waves cross each other, and the distribution of small amplitude waves is more uniform than those of Dean's stream function waves. However, the smaller the ratio of breaker height to depth H_b/h_b becomes, the smaller the differences between the four kinds of theoretical distributions mentioned above. Also, all theoretical curves approach the experimental values. This is clear in the results of Nos. 8 and 17 in which the values of H_b/h_b are smaller than 0.7.

The theoretical value by Dean's stream function explains the experimental value best among the four kinds of theoretical distributions. The value of the water particle velocity of small amplitude waves also agrees roughly with the experimental values in average, although the non-linearity of waves is predominant in wave breaking on the beach. The velocity distribution of small amplitude waves, however, crosses the experimental distribution, and is more uniform than the experimental one. The estimated moment of wave force on a pile near the breaking point on the beach by using the small amplitude wave theory may be considerably smaller than the real moment.

In order to express the degree of agreement between the experimental and theoretical values quantitatively, the differences between the experimental value $u_E/\sqrt{gh_b}$ and the three kinds of theoretical values $u_D/\sqrt{gh_b}$ (Dean's stream function), $u_S/\sqrt{gh_b}$ (Stokes waves of the 1st definition) and $u_{S'}/\sqrt{gh_b}$ (Stokes waves of the 2nd definition) at the still water level ($(z+h)/h=1.0$) were calculated and shown in Table 1. The difference between Dean's theoretical values and experimental ones is, as seen in Table 1. much smaller than that between the theoretical values of Stokes waves and experimental ones. The differences between Dean's theoretical values and experimental ones, except for Nos. 1, 5, 7 and 9, are less than 15%. But in Nos. 1, 7 and 9, the differences are about 30%, and in No. 5 20%. Also, in general, the difference between Dean's theoretical value and experimental one for the horizontal water particle velocity is more than the difference for the wave crest height η_0.

The error of measurement of the horizontal water particle velocity in this experiment was about 9%. Considering this fact, it is concluded in general that the horizontal water particle velocity at the crest phase of breaking waves by Dean's stream function agrees roughly with the measured velocity, except for Nos. 1, 5, 7 and 9.

5. Conclusion

Dean's stream functions were calculated by giving the time variations of water level during one wave period at the breaking point. The theoretical values of horizontal water particle velocity at the crest phase of breaking waves by Dean's stream functions are smaller than the theoretical values of Stokes waves. Considering the error of measurement of water particle velocity, it is concluded that Dean's stream function can explain the experimental values well. The fact that Dean's stream function can explain the experimental results of water particle velocity, though it assumes permanent waves in uniform depth, suggests that the wave profile of breaking waves dominates the water particle velocity field of breaking waves on a beach. Therefore, theoretical and experimental investigations of wave profile of breaking waves on a beach are required.

6. Acknowledgement

The authors wish to express their gratitudes to Mr. K. Tsukioka for providing the experimental data of water particle velocity of breaking waves on beaches and Mr. Y. Yamada for providing the computer program of Dean's stream function. This work was partly supported by Scientific Research Funds from the Ministry of Education.

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