Preventive Maintenance of a Two-Unit Standby Redundant System with a Good State

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Abstract

A preventive maintenance policy is proposed for a two-unit standby redundant system, each of which has good, degraded and failed states. The maen time to first system down is derived by the theory of Semi-Markov process. Further, the condition under which the policy is effective is obtained.

1. Introduction

It is a major problem to operate a system in a specified long time interval without system down. In order to improve the system reliability, the following two means are to be considered, i.e., to use the redundant system and to make preventive maintenance. A two-unit redundant system is basic and important in the theory of reliability. Srinivasan [1], Gaver [2], Liebowitz [3] and others have made the reliability analysis of such a system. For a preventive maintenance, Barlow and Proschan [4] have introduced some policies, i.e., an age maintenance, a block maintenance and an inspection and maintenance policy. Mine, Osaki and Asakura [5] have proposed a preventive maintenance policy for a two unit standby redundant system, and have made the system reliability analysis. In all the above contributions, it is assumed that the unit has only two states, i.e., an operating state and a failed state. For such a unit, the age maintenance policy is shown to be effective when the failure rate of the unit is an increasing function of time (see [4, pp. 88]).

In this paper, we treat a unit which has three states: good (S_0), where the failure rate is constant, degraded (S_1), where it is increasing and failed (S_2). In this unit, it is assumed that the transition rate from S_0 to S_1 (degradation rate) is constant. For such a unit, it is natural to make a preventive maintenance only in the degraded state since in the good state the unit is as good as a new one. Con-

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sidering the above fact, we shall introduce a maintenance policy for a two-unit standby redundant system in which the units with the three states, S_0, S_1, S_2 have the same statistical properties, and we shall derive the mean time to first system down under the maintenance policy. We shall further discuss the effect of the policy.

2. Basic Notations

In this section, we shall give some basic notations and the transition probability of a unit with no maintenance among the states S_0, S_1, S_2.

α failure rate in S_0.

β degradation rate in S_0.

$\lambda \equiv \alpha + \beta$.

$G(t)$ Cumulative distribution function (*Cdf*) of the time to failure in S_1, where the time is measured from the beginning of S_1.

$g(t) \equiv dG(t)/dt$, where it is assumed that $G(t)$ has a density.

— indicates complement of a probability, e.g., $\bar{G}(t) = 1 - G(t)$.

$z(t) \equiv g(t)/\bar{G}(t)$, which is the failure rate function in S_1.

$\mu_1 \equiv \int_0^\infty \bar{G}(t)dt$, which is the mean sojourn time in S_1.

$$U(t) \equiv \begin{cases} 1 & \text{for } t \geqq 0, \\ 0 & \text{othersise.} \end{cases}$$

The following transition probabilities are defined with respect to the case where no maintenances are made.

$W_{ij}(t)$ $Pr\{$the unit is in S_j at time t given that it reached S_i at time $0\}$, $i=0, 1, j=0,1,2$.

By an elementary probabilistic consideration, we have

$$W_{00}(t) = \exp(-\lambda t) , \tag{1}$$

$$W_{01}(t) = \int_0^t \alpha \bar{G}(t-x) \exp(-\lambda x) dx , \tag{2}$$

$$W_{02}(t) = 1 - W_{00}(t) - W_{01}(t) , \tag{3}$$

$$W_{11}(t) = \bar{G}(t) , \tag{4}$$

$$W_{12}(t) = G(t) . \tag{5}$$

$\mu_0 \equiv \int_0^\infty \bar{W}_{02}(t)dt$, which is the mean time to failure of a unit when the unit begins to operate from $S._0$

From (1)–(3) and the definition of μ_1, we have

$$\mu_0 = (1 + \alpha\mu_1)/\lambda .$$

The following quantities are defined with respect to maintenance. It is noted that the maintenance for the unit which has not yet failed is called preventive maintenance (PM), and the one for the failed unit is called corrective maintenance (CM).

$R_1(t)$ *Cdf* of the tmie for PM,
$R_2(t)$ *Cdf* of the time for CM,
$r_i(t) \equiv dR_i(t)/dt,\ i=1,2.$

3. Assumption

(i) A failure of each unit is detected as soon as it occurs.

(ii) A good state and degraded state can be identified by inspection only, which requires negligibly small time.

(iii) The failure rate of the unit at the beginning of S_1 is the same as the one in S_0, i.e.,

$$z(0) = \alpha\ . \tag{6}$$

(iv) The failure rate in S_1 monotonically increases to infinity.

(v) The mean sojourn time in S_0 is larger than the one in S_1, i.e.,

$$\lambda\mu_1 < 1\ . \tag{7}$$

(vi) Immediately after maintenance, a unit is as good as a new one.

(vii) The switchover times from failure to maintenance, from maintenance completion to standby state and from standby state to operation are all instantaneous.

(viii) The CM time is statistically larger than the PM time, i.e.,

$$R_2(t) < R_1(t)\ . \tag{8}$$

(ix) In a standby state, a unit does not fail and is free from degradation.

(x) Initially, one unit begins to operate and the other is in standby.

4. Maintenance Policy

The two-unit system is operated under the following policy.

If one unit has operated during T without failure and it is observed to be in S_1 by inspection, then it is preventively maintanied and the standby unit begins to operate. If it is observed to be in S_0, then it continues to operate. If one unit has operated during T without failure and the other is under maintenance, then no preventive maintenance is made since it yields the system down. That is, PM is made only if it is observed to be in S_1 by inspection and the other is in standby.

If one unit fails within T and the other is in standby, then CM immediately begins. If before maintenance completion of one unit the other fails, then the system down occurs.

Our concern is the mean time to first system down $M(T)$, which is a function of T, when the system is operated under the policy mentioned above. In the following, we shall first derive the Laplace-Stieltjes (LS) transform of the *Cdf* of the time to the first system down by the theory of the Semi-Markov process [6], and obtain $M(T)$ from the LS transformation. Secondly, we shall show the effect of our maintenance policy.

5. Derivation of $M(T)$

Considering the time instants of the inspection and the failure of a unit for the analysis of the system, we shall define the following four states which are all regeneration points. It is to be noted that we make use of the memory-less property of a negative exponential distribution.

E_0 : One unit begins to operate and the other is in standby.

E_1 : One unit begins to operate and PM of the other is begun.

E_2 : One unit begins to operate and CM of the other is begun.

E_3 : The system down occurs.

We shall consider the *Cdf* of the time for the system to reach E_3 for the first time when the system starts from E_0. Therefore, E_3 is an absorbing state. Since each time instant is a regeneration point, it constitutes a Semi-Markov process. In order to analyze the process, we shall define the following quantities.

$Q_{ij}(t)$ one step transition probability of the Semi-Markov process, $i=0,1,2$, $j=0,1,2,3$.

$q_{ij}(s)$ LS transform of $Q_{ij}(t)$, i.e.,

$$q_{ij}(s) = \int_0^\infty \exp(-st)dQ_{ij}(t) .$$

$\varphi_i(s)$ LS transform of the *Cdf* of the time to reach E_3 for the first time when the process starts from E_i, $i=0,1,2$.

In the following, we shall consider each transition from one state to another. If we say that PM time has come, then it implies that one unit has operated during T without failure and the state of the unit is identified.

From E_0, the following three cases are possible.

(i) PM time has come and the unit is in S_0. In this case, a transition to E_0 occurs and we have

$$Q_{00}(t) = \int_0^t W_{00}(x)dU(x-T) . \tag{9}$$

(ii) PM time has come and the unit its in S_1. In this case, a transition to E_1 occurs and we have

$$Q_{01}(t) = \int_0^t W_{01}(x)dU(x-T)\,. \tag{10}$$

(iii) One unit fails within T. In this case, a transition to E_2 occurs and we have

$$Q_{02}(t) = \int_0^t \bar{U}(x-T)dW_{02}(x)\,. \tag{11}$$

From $E_1(E_2)$, the following five cases are possible.

(i) PM time has come and the unit is in S_0, when the other unit is in standby after PM (CM) completion. In this case a transition to E_0 occurs and we have

$$Q_{i0}(t) = \int_0^t R_i(x)W_{00}(x)dU(x-T),\ \ i=1,2\,. \tag{12}$$

(ii) PM time has come and the unit is in S_1, when the other unit is in standby after PM (CM) completion. In this case a transition to E_1 occurs and we have

$$Q_{i1}(t) = \int_0^t R_i(x)W_{01}(x)dU(x-T),\ \ i=1,2\,. \tag{13}$$

(iii) One unit fails within T, when the othre is in standby after PM (CM) completion.

(iv) Before PM (CM) completion, PM time has come and after PM (CM) completion, the unit fails. In cases (iii) and (iv) a transition to E_2 occurs and we have

$$Q_{i2}(t) = \int_0^t R_i(x)\bar{U}(x-T)dW_{02}(x)+\int_0^t\int_0^x U(y-T)dR_i(y)dW_{02}(x),\ \ i=1,2. \tag{14}$$

(v) Before PM (CM) completion one unit fails. In this case, the system down occurs and we have

$$Q_{i3}(t) = \int_0^t \bar{R}_i(x)dW_{02}(x),\ \ i=1,2\,. \tag{15}$$

Here, we have obtained all the one-step transition probabilities. By using the LS transformation, we can easily get the following equations with respect to $\varphi_i(s)$.

$$\varphi_i(s) = q_{i3}(s)+\sum_{j=0}^{3} q_{ij}(s)\varphi_j(s),\ \ i=0,1,2\,. \tag{16}$$

where $q_{03}(s)=0$.

From (9)–(16), we have

$$\varphi_0(s) = A(s)/B(s)\,,$$

,where

$$A(s) = q_{13}(s)[q_{01}(s)\{1-q_{22}(s)\}+q_{02}(s)q_{21}(s)]$$
$$+q_{23}(s)[q_{02}(s)\{1-q_{11}(s)+\}q_{01}(s)q_{12}(s)],$$
$$B(s) = \{1-q_{00}(s)\}\{1-q_{11}(s)\}\{1-q_{22}(s)\}-q_{10}(s)q_{02}(s)q_{21}(s)$$
$$-q_{01}(s)q_{12}(s)q_{20}(s)-q_{02}(s)q_{20}(s)\{1-q_{11}(s)\}$$
$$-q_{21}(s)q_{12}(s)\{1-q_{00}(s)\}-q_{10}(s)q_{01}(s)\{1-q_{22}(s)\}. \tag{17}$$

Here, we have obtained the LS transform of the *Cdf* of the time to the first system down.

The maen time can be derived from $\varphi_0(s)$, i.e.,

$$M(T) = -\frac{d\varphi_0(s)}{ds}\bigg|_{s=0}. \tag{18}$$

For simplicity of expression, we introduce the following notations:

$$q_{ij} \equiv q_{ij}(0),$$
$$h_i \equiv -\frac{d}{ds}\sum_{j=0}^{3} q_{ij}(s)\bigg|_{s=0}, \quad i = 0, 1, 2.$$
$$m_i \equiv \int_0^\infty \bar{R}_i(t)dW_{02}(t),$$
$$R_i \equiv R_i(T), \qquad i = 1, 2,$$
$$W_i \equiv W_{0i}(T),$$
$$w_i \equiv \frac{dW_i}{dT}, \qquad i = 0, 1, 2.$$

It is noted that q_{ij}, h_i, R_i, W_i and w_i are the functions of T. Then we have

$$q_{0i} = W_i, \qquad i = 0, 1, 2. \tag{19}$$
$$q_{ij} = W_j R_i, \quad i = 1, 2, \quad j = 0, 1. \tag{20}$$
$$q_{i2} = 1-\bar{W}_2 R_i - m_i, \tag{21}$$
$$q_{i3} = m_i, \qquad i = 1, 2. \tag{22}$$
$$h_0 = \int_0^T \bar{W}_{02}(t)dt. \tag{23}$$
$$h_i = h_0 R_i + \mu_0 \bar{R}_i, \quad i = 1, 2. \tag{24}$$

From (17)–(24), we have the following mean time to first system down under the maintenance policy mentioned in section 4,

$$M(T) = \frac{(m_2+R_2)h_0+[W_1(m_2\bar{R}_1-m_1\bar{R}_2+\bar{R}_1)+W_2\bar{R}_2]\mu_0}{W_1(m_1R_2+m_2\bar{R}_1)+W_2m_2}. \tag{25}$$

In particular, $M(\infty)$ denotes the mean time in the case where no PM is made.

On the other hand, $M(0)$ corresponds to the case where PM is begun immediately after the operating unit has reached the degradation state. In the case of $0<T<\infty$, the degradation is tolerated to some degree.

From (25), we can easily obtain

$$M(0) = \frac{m_2+[\lambda+\beta(m_2-m_1)]\mu_0}{\lambda m_2}, \tag{26}$$

$$M(\infty) = \frac{(1+m_2)\mu_0}{m_2}. \tag{27}$$

6. The Effect of Maintenance Policy

In this section, we shall discuss the effect of the maintenance policy on the mean time to first system down. The following quantities are introduced to unclutter the equation.

$$a \equiv \int_0^\infty \bar{G}(t) \exp(\lambda t)dt,$$
$$b \equiv (1+\beta\mu_1)/(1-\alpha\mu_1),$$
$$c \equiv a(1+\beta\mu_1)/\mu_1(1+\beta a).$$

From the assumptions (iv) and (v), we can easily show

$$a<\infty, \qquad 1<c<b. \tag{28}$$

By comparing $M(T)$, $M(0)$ and $M(\infty)$, we can find the conditions under which the value of T exists such that

$$\max\{M(0), M(\infty)\} < M(T). \tag{29}$$

Theorem.

(i) If $m_2 \leqq cm_1$, then

$$M(T) < M(\infty) \qquad \text{for } 0 \leqq T < \infty.$$

(ii) Otherwise, there exists $T(0<T<\infty)$ such that

$$\max\{M(0), M(\infty)\} < M(T).$$

In case (ii), our PM policy is effective in the sense of mean time.

Proof.

$$M(\infty)-M(0) = \beta(1-\alpha\mu_1)(bm_1-m_2)/\lambda^2 m_2. \tag{30}$$

We consider the two cases, i.e., (i) $m_2<bm_1$ and (ii) $m_2 \geqq bm_1$.
Case (i). $m_2<bm_1$.

In this case, we have $M(0)<M(\infty)$ from (30), since $1-\alpha\mu_1$ is positive. Subtracting $M(\infty)$ from $M(T)$, we have

$$M(T)-M(\infty) = \frac{(m_2+R_2)[m_2h_0-(W_1m_1+W_2m_2)\mu_0]}{m_2[W_1(m_1R_2+m_2R_1)+m_2W_2]}\,. \tag{31}$$

It is evident that the denominator and the first factor of the numerator of the right hand side of the above equation are both positive. Let $H(T)$ denote the bracket of the numerator of (31), and we have

$$H(0)=0,\ \ H(\infty)=0\,. \tag{32}$$

Differentiating $H(T)$ with respect to T, we have

$$\frac{dH(T)}{dT} = \overline{W}_2K(T)\,, \tag{33}$$

where

$$K(T) = m_2-\mu_0[(m_2-m_1)w_2+\lambda m_1 \exp(-\lambda T)]/\overline{W}_2\,. \tag{34}$$

Moreover, we have

$$K(0) = \beta(1-\alpha\mu_1)(m_2-bm_1)/\lambda<0\,, \tag{35}$$

$$K(\infty) = \beta\mu_1(cm_1-m_2)\,. \tag{36}$$

Differentiating $K(T)$, we have

$$\frac{dK(T)}{dT} = \frac{-\mu_0\beta \exp(\lambda T)\bar{G}(T)}{[1+\beta\int_0^T \exp(\lambda t)\bar{G}(t)dt]^2}\,L(T)\,, \tag{37}$$

where

$$L(T) = (m_2-m_1)[z(T)-\alpha+\beta\int_0^T \exp(\lambda t)\bar{G}(t)\{z(T)-z(t)\}dt]-\lambda m_1\,. \tag{38}$$

By using the assumption that $z(T)$ is an increasing function of T, we can show that $L(T)$ is an increasing function of T since m_2-m_1 is positive. Moreover, from the assumptions (iii) and (iv), we have

$$L(0)<0, \qquad L(\infty)>0\,. \tag{39}$$

The above consideration implies that $K(T)$ has only one maximum. Thus, $H(T)$ and $K(T)$ look like the curves in Figure 1 or 2.
Therefore, it is shown that
(i) if $cm_1<m_2<bm_1$, then

$$M(0)\leqq M(\infty)<M(T) \qquad \text{for } T>T_1,$$

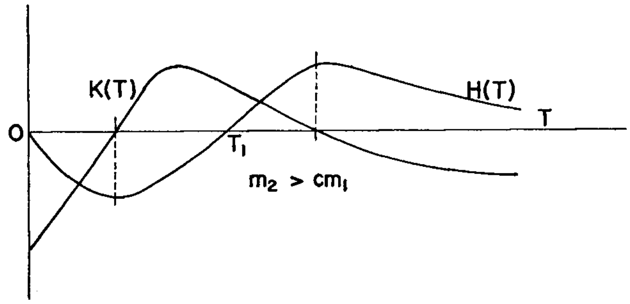

Fig. 1.

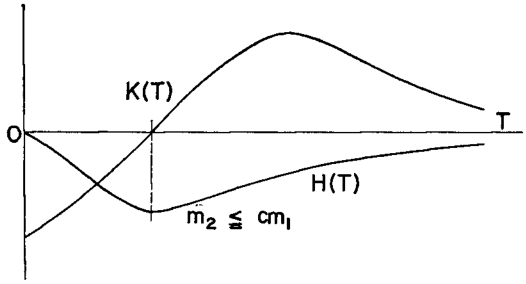

Fig. 2.

(ii) if $m_2 \leqq cm_1$, then

$$M(T) < M(\infty) \qquad \text{for } 0 \leqq T < \infty .$$

This implies that the theorem holds in the case (i).

Case (ii). $m_2 \geqq bm_1$.

In this case, we have $M(\infty) \leqq M(0)$.

Differentiating $M(T)$ with respect to T and putting $T=0$, we have

$$\left.\frac{dM(T)}{dT}\right|_{T=0} = M_1/M_2 ,$$

where

$$M_1 = 2\beta(1-\beta\mu_1)(m_2-bm_1)\left[\beta m_2\frac{dR_1(0)}{dt}+(\lambda-\beta m)_1\frac{dR_2(0)}{dt}\right]\Big/\lambda$$
$$+\beta\lambda m_2[(1-\alpha\mu_0)m_2+\alpha\mu_0 m_1] , \tag{40}$$

$$M_2 = 2m_2[d\{W_1(m_1R_2+m_2R_1)+m_2W_2\}/dT|_{T=0}]^2 > 0 . \tag{41}$$

Considering that $\lambda-\beta m_1$, $1-\alpha\mu_0$ and m_2-bm_1 are all non-negative, M_1 is easily seen to be positive, which implies that there exists positive and finite T such that $M(0) < M(T)$.

Noting that $m_2 \geqq bm_1$ implies $m_2 > cm_1$, the theorem is shown to hold.

7. Conclusion

We have considered a two-unit standby redundant system, where each unit has good, degraded and failed states. We have 1) proposed a preventive maintenance policy which is based on both age and state, 2) derived the mean time to first system down and 3) shown that the maintenance policy is effective under certain conditions.

In this model, the function $w_{02}(t)/\overline{W}_{02}(t)$ is shown to be a monotonically increasing function of time, which implies that the PM policy based on only age may be effective. However, in a good state the unit is regarded as a new one from the memory-less property of a negative exponential distribution. This means that the maintenance is not needed for the unit in a good state. That is, for such units, the maintenance policy should depend on both age and state.

Though it is difficult to obtain analytically the optimal time of PM, it can be obtained by the numerical computation of $M(T)$ for $T>T_1$.

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