Analog Simulation of Problem of Heat Conduction Using Electric Resistive Circuits and Analog Memories

By

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Abstract

A method of analog simulation of problems of heat conduction using electric resistive circuits and analog memories is proposed. The principle of this method is to derive the difference equations corresponding to given problems and to simulate these difference equations by using resistive circuits accompanied with analog memories. Discussions of the practical set-up procedures for constructing the resistive circuits are given according to the various problems of heat conduction. By using this method, the approximate solutions of typical problems of heat conduction are easily obtained with considerable accuracy.

1. Introduction

Certain problems, such as finding the solution of a partial differential equation under given initial and boundary conditions, are extremely difficult to deal with by purely mathematical means. From a viewpoint of engineering, rather than an accurate mathematical solution, an approximate but swift and easily comprehensible solution is often preferred. Hence, the approximate solution applying the digital or analog computers is useful. However, the digital computer method employs very many memory devices and has the risk of divergence from the numerical solution[1)], while the analog computer method requires many integrating amplifiers.

From this point of view, the authors have developed a particular computing device suitable for the solution of various problems of heat conduction.

The aim of this paper is to explain the principle of this device and to discuss several methods or procedures of its application.

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2. Approximation of the Fundamental Equation of Heat Conduction by the Difference Equation

For an example, consider the following equation of one-dimensional heat conduction and discuss the method of approximation by the difference equation.

$$\left.\begin{array}{ll} \dfrac{\partial\theta}{\partial t}=k\dfrac{\partial^2\theta}{\partial x^2} & (0<x<L), \\ \theta(0,t)=\theta_0(t), & \theta(L,t)=\theta_L(t), \\ \theta(x,0)=f(x), & \end{array}\right\} \qquad (1)$$

where $\theta(x, t)$: temperature rise (°C)

t : time (sec.)

x : displacement in the direction of thermal conduction

k : coefficient of thermal diffusion (m^2/sec.)

L : length of thermal conductor (m)

$f(x)$: initial temperature distribution

θ_0, θ_L : boundary conditions.

Solution θ is clearly a function of x and t. Setting the x-, y- and $\theta(x, t)$-axes as shown in Fig. 1, solution $\theta(x, t)$ is represented by a curved surface as shown.

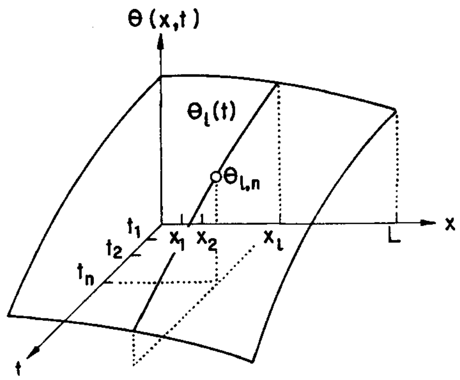

Fig. 1. Solution surface of Eq. (1).

Now, supposing that the x-axis is divided from the origin into a number of minute divisions of lengths δx_i ($i>1, 2, \ldots, l, \ldots$) at the points x_i ($=\sum_i \delta x_i$), and that the solution $\theta(x_l, t)$ at the arbitrary point x_l is written simply as θ_l, then θ_l is a function of t only (x being fixed at x_l). Then θ_l is represented by a curved line as shown in Fig. 1. Let us write $\left.\dfrac{\partial\theta}{\partial x}\right|_{x=x_l}$ and $\left.\dfrac{\partial^2\theta}{\partial x^2}\right|_{x=x_l}$ as $\dfrac{\partial\theta_l}{\partial x}$ and $\dfrac{\partial^2\theta_l}{\partial x^2}$, respectively.

According to the three cases concerning the division intervals δx, we discuss the method of approximation of Eq. (1) by the difference equation.

2.1 Case of $\delta x_i = \delta x =$ constant (equal division)

Assuming δx is sufficiently small, at any particular time t we can write

$$\frac{\partial \theta_l}{\partial x} = \frac{\theta_{l+1} - \theta_l}{\delta x} \quad \text{or} \quad \frac{\theta_l - \theta_{l-1}}{\delta x} \tag{2}$$

Differentiating partially with respect to x again, we get

$$\frac{\partial^2 \theta_l}{\partial x^2} = \frac{1}{\delta x}\left(\frac{\partial \theta_l}{\partial x} - \frac{\partial \theta_{l-1}}{\partial x}\right). \tag{3}$$

Substituting $\frac{\partial \theta_l}{\partial x}$, $\frac{\partial \theta_{l-1}}{\partial x}$ into Eq. (3), using Eqs. (2), we derive

$$\frac{\partial^2 \theta_l}{\partial x^2} = \frac{1}{\delta x}\left(\frac{\theta_{l+1} - \theta_l}{\delta x} - \frac{\theta_l - \theta_{l-1}}{\delta x}\right),$$

i. e.

$$\frac{\partial^2 \theta_l}{\partial x^2} = \frac{1}{(\delta x)^2}\ (\theta_{l+1} - 2\theta_l + \theta_{l-1}). \tag{4}$$

Substituting Eq. (4) into Eq. (1), we obtain

$$\frac{d\theta_l}{dt} = \frac{k}{(\delta x)^2}\ (\theta_{l+1} - 2\theta_l + \theta_{l-1}). \tag{5}$$

Similar to the case of x-axis, dividing the t-axis from the origin into a number of equal minute divisions of length δt at the points t_j ($j=1, 2, \ldots, n, \ldots$), and writing $\theta(x_l, t_n)$ as $\theta_{l,n}$, $\theta_{l,n}$ represents one point on the curved surface shown in Fig. 1, and $\left.\frac{d\theta_l}{dt}\right|_{t=t_n}$ may be written as $\frac{d\theta_{l,n}}{dt}$.

As before, we can write

$$\frac{d\theta_{l,n}}{dt} = \frac{\theta_{l,n} - \theta_{l,n-1}}{\delta t} \tag{6}$$

Substituting Eq. (6) into Eq. (5) we finally obtain

$$\theta_{l,n} - \theta_{l,n-1} = \frac{k\delta t}{(\delta x)^2}(\theta_{l+1,n} - 2\theta_{l,n} + \theta_{l-1,n}). \tag{7}$$

2.2 Case of $\delta x_i \neq \delta x_{i-1}$ (unequal division)[2]

This case can be illustrated as shown in Fig. 2, where $\delta x_{l+1} \neq \delta x_l$, H and K are the mid-points of AB, and BC respectively. As before, considering δx_{l+1} and δx_l to be sufficiently small, we can write

$$\left.\begin{aligned} \frac{\partial \theta_{l,n}}{\partial x} &= \frac{\theta_{l,n} - \theta_{l-1,n}}{\delta x_l}, \\ \frac{\partial \theta_{l+1,n}}{\partial x} &= \frac{\theta_{l+1,n} - \theta_{l,n}}{\delta x_{l+1}}. \end{aligned}\right\} \tag{8}$$

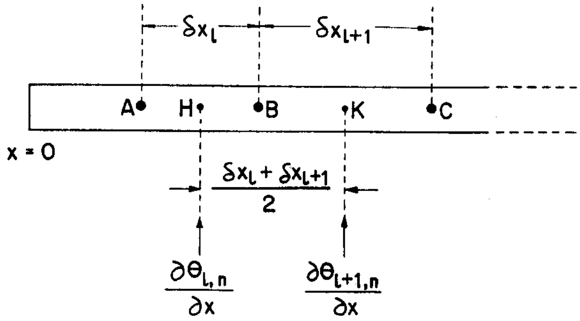

Fig. 2. Case of unequal division.

Now, $\frac{\partial\theta_{l,n}}{\partial x}$ and $\frac{\partial\theta_{l+1,n}}{\partial x}$ can be considered to be the temperature gradients at mid-points H and K respectively, which are $(\delta x_l+\delta x_{l+1})/2$ apart from each other. In accordance with the previous description,

$$\frac{\partial^2\theta_l}{\partial x^2}=\frac{\frac{\partial\theta_{l+1,n}}{\partial x}-\frac{\partial\theta_{l,n}}{\partial x}}{\frac{\delta x_l+\delta x_{l+1}}{2}}=\frac{\frac{\theta_{l+1,n}-\theta_{l,n}}{\delta x_{l+1}}-\frac{\theta_{l,n}-\theta_{l-1,n}}{\delta x_l}}{\frac{\delta x_l+\delta x_{l+1}}{2}}. \tag{9}$$

Dividing the t-axis into equal divisions as before, and hence substituting Eqs. (6) and (9) into Eq. (1), we obtain the following difference equation:

$$\theta_{l,n}-\theta_{l,n-1}=\frac{2k\delta t}{\delta x_l(\delta x_l+\delta x_{l+1})}\left[\frac{\delta x_l}{\delta x_{l+1}}\theta_{l+1,n}-\left(1+\frac{\delta x_l}{\delta x_{l+1}}\right)\theta_{l,n}+\theta_{l-1,n}\right]. \tag{10}$$

2.3 Case of $\delta x_i=\alpha\ \delta x_{i-1}$ (α : a suitable constant greater than 1)[3]

This is a special case of Section 2.2. Substituting $\delta x_{l+1}=\alpha\,\delta x_l$ into Eq. (10), we obtain

$$\theta_{l,n}-\theta_{l,n-1}=\frac{2k\delta t}{(\delta x_l)^2(1+\alpha)}\left[\frac{1}{\alpha}\theta_{l+1,n}-\left(1+\frac{1}{\alpha}\right)\theta_{l,n}+\theta_{l-1,n}\right]. \tag{11}$$

3. Simulation of the Difference Equation Using a Resistive Circuit Accompanied with the Analog Memories

The fundamental procedures for deriving the difference equation have been discussed in the former chapter. The main idea of the simulation of the difference equation using a resistive circuit accompanied with the analog memories is discussed in this chapter.

3.1 Case of $\delta x_i = \delta x = $constant

Consider the purely-resistive T-shaped circuit shown in Fig. 3. Using the notations as shown in the figure and according to Kirchhoff's law, we obtain

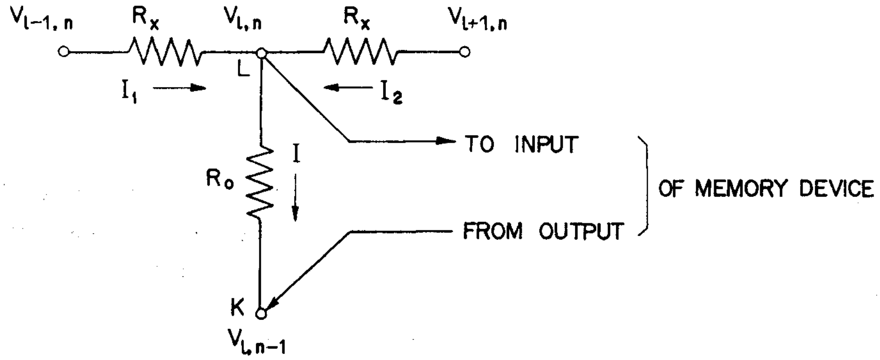

Fig. 3. T-shaped resistive circuit (equal division).

$$I_1 + I_2 = I,$$

$$\text{i.e. } \frac{1}{R_x}(V_{l-1,n} - V_{l,n}) + \frac{1}{R_x}(V_{l+1,n} - V_{l,n}) = \frac{1}{R_0}(V_{l,n} - V_{l,n-1}).$$

Arranging this equation, we have

$$V_{l,n} - V_{l,n-1} = \frac{R_0}{R_x}(V_{l+1,n} - 2V_{l,n} + V_{l-1,n}). \tag{12}$$

Comparing Eq. (12) with Eq. (7), it is clear that if the values of R_x and R_0 are chosen such that

$$\frac{R_0}{R_x} = \frac{k\delta t}{(\delta x)^2}, \tag{13}$$

then the voltage V in Eq. (12) behaves exactly as the temperature rise θ in Eq. (7).

Obviously, if one-dimensional conductor of length L is divided into p divisions, i. e. $i_{max} = p$ and $\delta x = L/p$, the number of resistors R_x and R_0 required are p and $(p-1)$ respectively. Now, $V_{0,n}$ and $V_{p,n}$ represent the boundary conditions (voltages corresponding to the temperature rises at the ends of the conductor), and $V_{l,n}$ $(0 < l < p)$ represents the voltage corresponding to the temperature rise $\theta_{l,n}$ at the point $x_l = Ll/p$ and time $t = t_n$.

With the application of this simulator to solve Eq. (1), the following procedures are adopted.

(1) The simulator, consisting of p R_x-resistors and $(p-1)$ R_0-resistors, is set up as shown in Fig. 4.

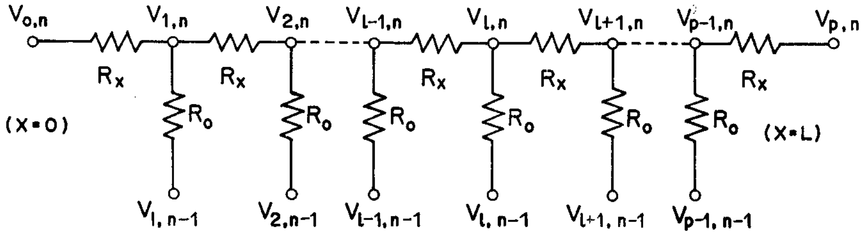

Fig. 4. Simulator circuit for equal division (finite length).

(2) To the two end-terminals of the simulator (corresponding to $x=0$ and $x=L$), the voltages $V_{0,n}$, $V_{p,n}$ (corresponding to the boundary values) are applied, while to the lower ends of the R_0-resistors, the voltages $V_{1,0}$, $V_{2,0},\ldots,$ $V_{p-1,0}$ (corresponding to the initial conditions of the points x_1, $x_2,\ldots,$ x_{p-1} respectively) are applied.

(3) When operation (2) is completed, the upper ends of the R_0-resistors indicate the voltages $V_{l,1}$ $(l=1,\ 2,\ldots,\ p-1)$ which correspond to the temperature rises $\theta_{l,1}$ at the time $t=\delta t$ after the initial state $(t=0)$. These voltages are memorized by the analog memory devices.

(4) The voltages $V_{l,1}$ memorized in the operation (3) are applied to the lower ends of the R_0-resistors; and then the voltages $V_{l,2}$ are caused at the upper ends, which are memorized, and so on.

In order to perform the above operations, a memory device which reads out one value while simultaneously memorizing the next value is required. This will be described in Chapter 5.

3.2 Case of $\delta x_l \neq \delta x_{l-1}$

Consider the resistive circuit shown in Fig. 5. Using the notations as shown in the figure (note that $R_{xl}\neq R_{xl+1}$), we have

$$V_{l,n}-V_{l,n-1}=\frac{R_{0l}}{R_{xl}}\left[\frac{R_{xl}}{R_{xl+1}}V_{l+1,n}-\left(1+\frac{R_{xl}}{R_{xl+1}}\right)V_{l,n}+V_{l-1,n}\right]. \qquad (14)$$

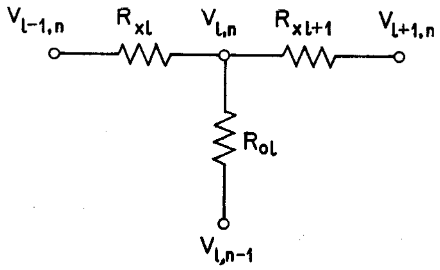

Fig. 5. T-shaped resistive circuit (unequal division).

Comparing Eq. (14) with Eq. (10), it is clear that if R_{xl}, R_{xl+1} and R_0 are chosen as

$$\left.\begin{aligned}\frac{R_{xl}}{R_{xl+1}}&=\frac{\delta x_l}{\delta x_{l+1}},\\ \frac{R_{0l}}{R_{xl}}&=\frac{2k\delta t}{\delta x_l(\delta x_l+\delta x_{l+1})},\end{aligned}\right\} \qquad (15)$$

then the voltage V in Eq. (14) behaves exactly as the temperature rise θ in Eq. (10). Note that Eqs. (15) decide only the ratios between the resistances R_{xl}, R_{xl+1} and R_{0l}. Hence, choosing R_{xl}, then others are decided from Eqs. (15).

3.3 Case of $\delta x_i = \alpha\ \delta x_{i-1}$

Substituting $\delta x_{l+1} = \alpha\, \delta x_l$ into Eqs. (15), we obtain

$$\left.\begin{aligned} \frac{R_{xl}}{R_{xl+1}} &= \frac{1}{\alpha}, \\ \frac{R_{0l}}{R_{xl}} &= \frac{2k\delta t}{(\delta x_l)^2(1+\alpha)}. \end{aligned}\right\} \tag{16}$$

Presuming to start from the origin, we first choose suitable values of α and R_{x1}. Then from Eqs. (16), other resistances R_{x2}, R_{x3}, ... are decided as follows:

$$R_{x2} = \alpha\, R_{x1},\ R_{x3} = \alpha\, R_{x2} = \alpha^2\, R_{x1},\ R_{x4} = \alpha^3\, R_{x1}, \cdots. \tag{17}$$

Further, since δx_1 and δt are determined by the requirements of the problem, R_{01} and R_{02} are fixed according to Eqs. (16) as follows:

$$\frac{R_{01}}{R_{x1}} = \frac{2k\delta t}{(\delta x_1)^2(1+\alpha)}, \tag{18}$$

$$\frac{R_{02}}{R_{x2}} = \frac{2k\delta t}{(\delta x_2)^2(1+\alpha)},\quad i.e.\ \frac{R_{02}}{\alpha R_{x1}} = \frac{2k\delta t}{(\alpha\, \delta x_1)^2(1+\alpha)}. \tag{19}$$

Dividing Eqs. (19) by Eq. (18), we have $R_{02} = R_{01}/\alpha$. Similarly, we can derive the relations between R_{0i} and R_{0i+1} ($i=1, 2, \ldots$) as follows:

$$R_{02} = \frac{R_{01}}{\alpha},\ \ R_{03} = \frac{R_{01}}{\alpha^2},\ \ R_{04} = \frac{R_{01}}{\alpha^3}, \cdots. \tag{20}$$

4. Simulation of Several Problems of One-Dimensional Heat Conduction

The fundamental problem of heat conduction described by Eq. (1) may be simulated by the method mentioned in the former chapter. However, we often come upon various problems of heat conduction whose system equations are different from Eq. (1). The methods of simulation of such problems are presented in this chapter.

4.1 Simulation of the Problem of Heat Conduction with Heat Radiation

We consider the problem of one-dimensional heat conduction with heat radiation:

$$\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2} - h\theta, \tag{21}$$

where h is a proportional constant corresponding to the heat radiation.

Eq. (21) can be approximated by the difference equation:

$$\theta_{l,n} - \theta_{l,n-1} = \frac{k\delta t}{(\delta x)^2}(\theta_{l+1,n} - 2\theta_{l,n} + \theta_{l-1,n}) - h\delta t\theta_{l,n}. \tag{22}$$

On the other hand, the resistive circuit shown in Fig. 6 derives the circuit equation which has the same form as Eq. (22). Hence, if the resistances are chosen to satisfy the relations:

$$\frac{R_0}{R_x}=\frac{k\delta t}{(\delta x)^2},\quad \frac{R_0}{R_h}=h\delta t, \tag{23}$$

we can simulate the difference equation (22) by the circuit shown in Fig. 6.

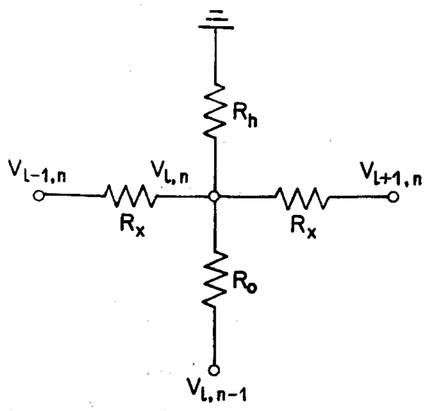

Fig. 6. Simulator circuit for Eq. (22).

4.2 Simulation of the Boundary Condition of Thermal Insulation

Consider the case where one end of the conductor ($x=0$) is thermally insulated as shown in Fig. 7 (a). The condition of thermal insulation means that there exists no temperature gradient at $x=0$. Hence, supposing that $\theta_{-1,n}$ represents the temperature rise at an image point ($x=-\delta x$), as shown in Fig. 7 (b), $\theta_{-1,n}=\theta_{1,n}$ should be satisfied. Using this relation, we can derive the difference equation at the point x=0 as follows:

$$\theta_{0,n}-\theta_{0,n-1}=\frac{k\delta t}{(\delta x)^2}(2\theta_{1,n}-2\theta_{0,n}). \tag{24}$$

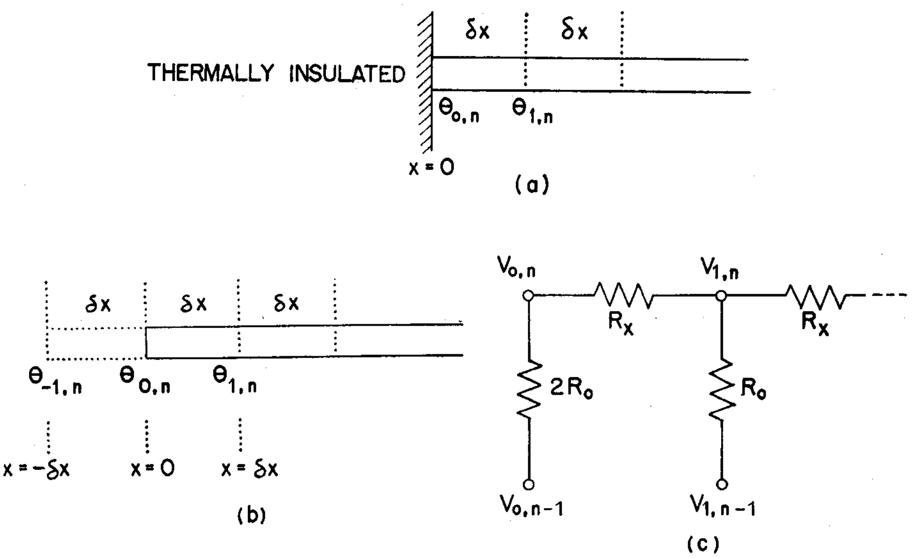

Fig. 7. Simulation of the boundary condition of thermal insulation.

This equation can be simulated by the circuit shown in Fig. 7 (c) if $R_0/Rx=k\ \delta t/(\delta x)^2$. In other words, when we simulate the boundary condition of thermal insulation, it is necessary to choose twice the value of resistance R_0 corresponding to the end point which is thermally insulated.

4.3 Simulation of the Boundary Condition of Heat Flow

In the case of the boundary condition of heat flow as shown in Fig. 8 (a), the fundamental equation is described as follows:

$$\frac{\partial\theta}{\partial t}=k\frac{\partial^2\theta}{\partial x^2},\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=-F, \tag{25}$$

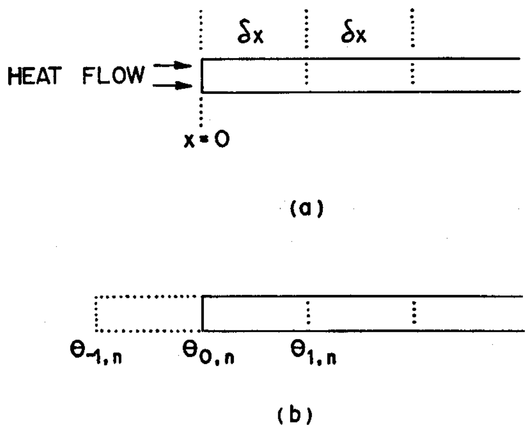

Fig. 8. Simulation of the boundary condition of heat flow.

where F represents a quantity corresponding to the given heat flow. Considering the image point $(x=-\delta x)$ as shown in Fig. 8 (b), we obtain from Eqs. (25)

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=\frac{1}{2\delta x}(\theta_{1,n}-\theta_{-1,n})=-F. \tag{26}$$

Substituting Eq. (26) into the difference equation derived at the point $x=0$, we have

$$\theta_{0,n}-\left(\theta_{0,n-1}+2kF\frac{\delta t}{\delta x}\right)=\frac{k\delta t}{(\delta x)^2}(2\theta_{1,n}-2\theta_{0,n}). \tag{27}$$

Comparing Eq. (27) with (24), this problem may be simulated by the circuit shown in Fig. 7 (c), only if the bias voltage corresponding to $2kF\delta t/\delta x$ is added in each operation cycle of analog memory situated at the point x=0.

4.4 Simulation of the Problem of Heat Conduction with Heat Generation

We consider the problem of one-dimensional heat conduction with heat generation:

$$\frac{\partial \theta}{\partial t}=k\frac{\partial^2 \theta}{\partial x^2}+\frac{k}{K}A, \tag{28}$$

where A represents the heat generation per unit time, and K is the thermal conductivity[4].

Eq. (28) can be approximated by the difference equation:

$$\theta_{l,n}-\left(\theta_{l,n-1}+\frac{k\delta t}{K}A\right)=\frac{k\delta t}{(\delta x)^2}(\theta_{l+1,n}-2\theta_{l,n}+\theta_{l-1,n}). \tag{29}$$

Comparing Eq. (29) with Eq. (7), it is clear that the same resistive circuit shown in Fig. 3 is available to simulate Eq. (29), only if the bias voltage corresponding to $k\delta t/K$ is added in each operation cycle of analog memories[5].

5. Analog Memory Device

The principles of the simulator circuits for the various problems of heat conduction have been discussed in the above chapters. As mentioned before, these simulator circuits should be combined with the analog memory devices which are described in this chapter.

The set up of the memory device is shown in Fig. 9. It consists essentially of two condensers C_1, and C_2, which are controlled by the dual electronic switch SW 1.

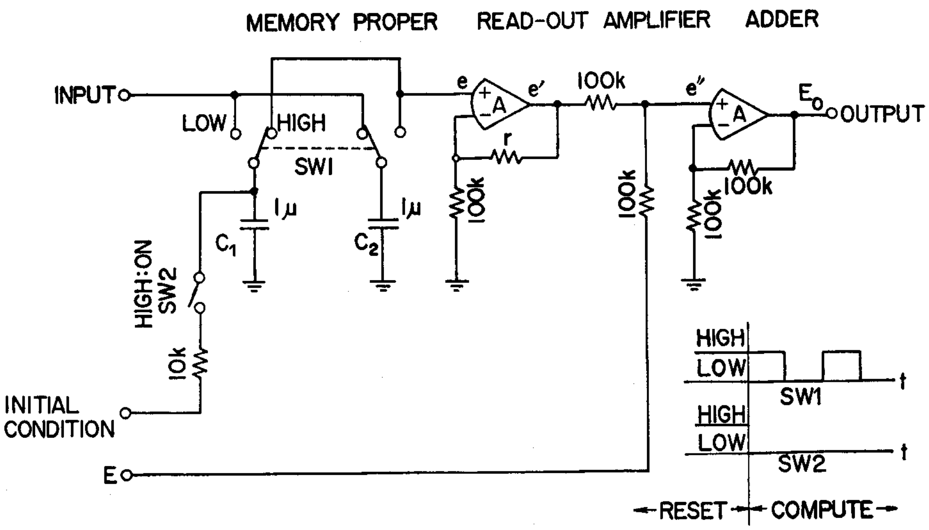

Fig. 9. Analog memory device.

Each condenser works in turn as the reading out and the memorizing condenser according to the sequence shown in Fig. 9.

In the RESET condition, both SW 1 and SW 2 are energized (HIGH position), and the voltage of the initial condition is given to C_1, while this voltage is read out at the output terminal. At the same time, the input voltage which should be memorized is given to C_2.

When the instruction COMPUTE is given to the device, SW 2 is de-energized at once and the initial condition is taken off. After some interval, SW 1 is de-energized and the voltage across C_2 is read out, while C_1 memorizes the next input value.

Resistor r used with the read-out amplifier compensates the error voltage caused by the leakage current of the condenser circuit. An adding device in Fig. 9 calculates the sum of the output voltage of the read-out amplifier and the bias voltage given at E terminal. As mentioned in Sections 4. 3 and 4. 4. it is necessary to add some bias voltage corresponding to the heat generation or forced heat flow.

6. Example of Application

In order to examine the accuracy of approximate solutions obtained by the simulator, the authors set up the simulator circuit corresponding to the fundamental problem described by Eq. (1), and compared the mathematical solutions with the experimental results.

The specific example considered in this chapter is described as follows:

$$\left.\begin{aligned} &\frac{\partial\theta}{\partial t}=k\frac{\partial^2\theta}{\partial x^2}, \quad (0<x<2), \\ &\theta(0,\ t)=\theta(2,\ t)=0, \\ &\theta(x,\ 0)=10\sin\frac{\pi}{2}x. \end{aligned}\right\} \tag{30}$$

We adopted the coefficient of thermal diffusion k corresponding to the copper, i. e. $k=1.11$.

Referring to Sections 2. 1 and 3. 1, we divided the heat conductor into ten equal divisions. Then we set $\delta x=0.2$, and the initial temperature rises at the points x_l ($l=1,\ 2,\ \dots,\ 9$) are $\theta_{l,0}=10\sin(0.1\pi l)$. It was necessary to prepare ten R_x-resistors and nine R_0-resistors whose magnitudes, when δt is set at 0. 01, are given by

$$\frac{R_0}{R_x}=\frac{k\delta t}{(\delta x)^2}=\frac{1.11\times 0.01}{(0.2)^2}=0.28\text{(from Eq. (13))}.$$

According to this relation, we chose as $R_x=100\ k\Omega$ and $R_0=28\ k\Omega$.

We set up the simulator circuit accompanied with nine analog memories as shown in Fig. 4.

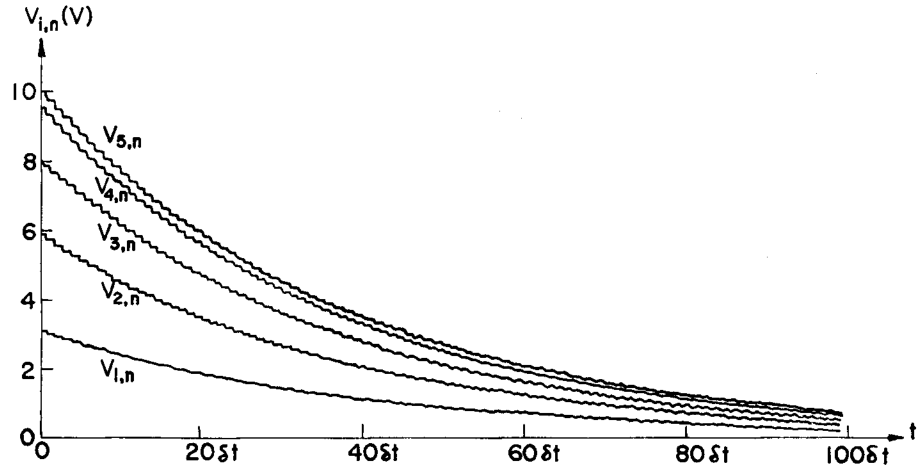

Fig. 10. Solution curves of Eq. (30) (experimental).

The examples of the experimental results are shown in Fig. 10. The curves shown in Fig. 10 illustrate the solutions obtained by the simulator.

The theoretical solution of Eq. (30) is found to be[4]:

$$\theta(x,\ t)=10\exp\left(-\frac{\pi^2}{4}kt\right)\sin\frac{\pi}{2}x. \tag{31}$$

Using this equation and $k=1.11$, the mathematical solutions are calculated.

Comparing the experimental results with the mathematical solutions, the errors are found to be sufficiently small (within 1% of full-scale voltage of the simulator (10 volts)).

Note that the solutions shown in Fig, 10 are obtained under the unity scale factor, i. e. 1 volt of the output of the simulator corresponds to 1 °C of the temperature rise. We may adopt the arbitrary scale factor between the output voltage V of the simulator and the temperature rise.

7. Simulation of the Problem of Two-Dimensional and Three-Dimensional Heat Conduction

We may derive the difference equation for the problem of two-dimensional and three-dimensional heat conduction by applying the same principles mentioned in Chapter 2.

7.1 Simulation of the Fundamental Problem of Two-Dimensional Heat Conduction

We consider the problem of two-dimensional heat conduction described by

$$\frac{\partial\theta}{\partial t}=k\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)\quad\begin{pmatrix}0<x<L\\0<y<M\end{pmatrix},\tag{32}$$

and divide the x-, y- and t-axes from the origin into a number of minute divisions of lengths δx, δy and δt respectively. Then we obtain the following difference equation corresponding to Eq. (32).

$$\begin{aligned}\theta_{l,m,n-1}=&\frac{k\delta t}{(\delta x)^2}\left(\theta_{l+1,m,n}-2\theta_{l,m,n}+\theta_{l-1,m,n}\right)\\&+\frac{k\delta t}{(\delta y)^2}\left(\theta_{l,m+1,n}-2\theta_{l,m,n}+\theta_{l,m-1,n}\right),\end{aligned}\tag{33}$$

where $\theta_{l,m,n}$ represents the temperature rise at the point corresponding to $x=l\ \delta x$, $y=m\ \delta y$ and $t>n\ \delta t$.

Eq. (33) can be simulated by the resistive circuit shown in Fig. 11. Using the notations as shown in the figure, we have

$$\begin{aligned}V_{l,m,n}-V_{l,m,n-1}=&\frac{R_0}{R_x}\left(V_{l+1,m,n}-2V_{l,m,n}+V_{l-1,m,n}\right)\\&+\frac{R_0}{R_y}\left(V_{l,m+1,n}-2V_{l,m,n}+V_{l,m-1,n}\right).\end{aligned}\tag{34}$$

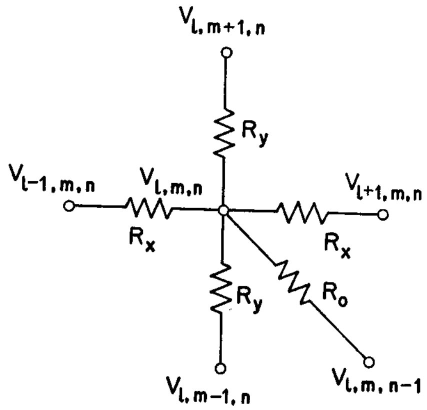

Fig. 11. Simulator circuit for two-dimensional problem (equal division).

Comparing Eq. (34) with Eq. (33), it is clear that the resistive circuit shown in Fig. 11 simulates the differential equation (33) if the resistances R_x, R_y and R_0 are chosen as follows:

$$\frac{R_0}{R_x}=\frac{k\delta t}{(\delta x)^2},\quad\frac{R_0}{R_y}=\frac{k\delta t}{(\delta y)^2}.\tag{35}$$

Note that Eqs. (35) decide only the ratios among three resistances, i. e. choosing one resistance, then others are decided.

The difference equation (33) is derived in the case of equal divisions of x- and y-axes. It is necessary to modify the equation in the case of unequal divisions.

Supposing the division intervals $\delta x_l\neq\delta x_{l+1}$ and $\delta y_m\neq\delta y_{m+1}$ as shown in Fig. 12 (a), we derive the following modified difference equation:

$$\begin{aligned}\theta_{l,m,n}-\theta_{l,m,n-1}=&\frac{2k\delta t}{\delta x_l(\delta x_l+\delta x_{l+1})}\left[\frac{\delta x_l}{\delta x_{l+1}}\theta_{l+1,m,n}\right.\\&\left.-\left(1+\frac{\delta x_l}{\delta x_{l+1}}\right)\theta_{l,m,n}+\theta_{l-1,m,n}\right]\end{aligned}$$

$$+\frac{2k\delta t}{\delta y_m(\delta y_m+\delta y_{m+1})}\left[\frac{\delta y_m}{\delta y_{m+1}}\theta_{l,m+1,n}-\left(1+\frac{\delta y_m}{\delta y_{m+1}}\right)\theta_{l,m,n}+\theta_{l,m-1,n}\right]. \quad (36)$$

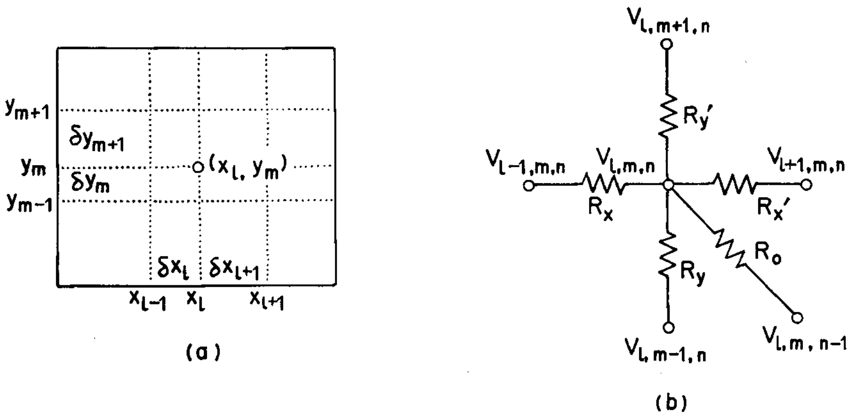

Fig. 12. Case of unequal division for two-dimensional problem.

This equation can be simulated by the resistive circuit shown in Fig. 12 (b) if the following relations are satisfied:

$$\left.\begin{aligned}&\frac{R_x}{R_x'}=\frac{\delta x_l}{\delta x_{l+1}},\quad \frac{R_y}{R_y'}=\frac{\delta y_m}{\delta y_{m+1}},\\ &\frac{R_0}{R_x}=\frac{2k\delta t}{\delta x_l(\delta x_l+\delta x_{l+1})},\\ &\frac{R_0}{R_y}=\frac{2k\delta t}{\delta y_m(\delta y_m+\delta y_{m+1})}.\end{aligned}\right\} \quad (37)$$

7.2 Simulation of the Problem of Three-Dimensional Heat Conduction

In order to simulate the problems of three-dimensional heat conduction, we may apply the same principles mentioned in Section 7. 1.

However, in the case of three-dimensional problems, it is necessary to use a great many analog memories.

For an example, we consider the case of ten equal divisions of x-, y- and z-axes. Then we have to prepare at least 9^3 analog memories.

Generally, the systems of three-dimensional heat conduction may be dealt with by applying the cylindrical coordinates. Using this principle, we may observe the temperature distribution of solids in the two-dimensional system. Hence, from the practical point of view, it is favorable to reduce the dimensions of the given problems by using such techniques.

8. Concluding Remarks

In this paper, the fundamental method of an approximate solution of the problems of heat conduction using resistive circuits and analog memories is discussed.

It is found that the simulator proposed is available for the various problems of heat conduction, and that the errors of the solutions are sufficiently small.

This simulator is applicable to the more complicated problems of heat conduction, though the scale of the simulator circuit is limited by the number of analog memory devices.

The results of the simulation of complicated problems will be reported in the next paper.

References

1) G. E. Forsythe, W. R. Wasow, "Finite-Difference Methods for Partial Differential Equations", John Wiley & Sons, Inc., 1960.

2) G. Liebmann, "A New Electrical Analog Method for the Solution of Transient Heat-Conduction Problems", Trans. ASME, April, 1956.

3) M. Abe, "Analog Simulation Techniques of the System Described by Partial Differential Equations and Its Application", Control Techniques, JAACE, Vol. 21, 1975.

4) H. S. Carslaw, J. G. Jaeger, "Conduction of Heat in Solids", Clarendon Press, 1960.

5) I. Ukita, M. Abe, "Analysis of Heat-Conduction Problems Using the Simulator Circuits", Monograph No. 3345, IEEJ, 1969.