# Transient Stability Region of Power Systems Using Series Expansion of Lyapunov Function

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(Received March 29, 1977)

**Abstract**

In this paper, a method is proposed for improving the accuracy of the stability criterion by enlarging the estimation of the stability region determined by means of Lyapunov's direct method using the series expansion, and by putting it closer to the actual stability region. The Lyapunov function by the energy integration is used as the first estimation. The method is applied to a single-machine infinite bus power system, and the stability domain in the $\delta-\omega$ plane is shown together with the actual stability domain obtained by numerical integration. It is shown that the application of the proposed method results in a considerable improvement of the stability boundary estimation over that given by the original Lyapunov function. Zubov's method, which obtains an accurate stability boundary using the series expansion, is related to the method proposed in this paper. The last section is, therefore, devoted to comparing the two methods and examining the possibility of their application to multi-machine power systems.

## 1. Introduction

The transient stability of an electric power system is assessed by analyzing a set of nonlinear ordinary differential equations called swing equations. The method which has been used widely for this purpose has been by means of simulation. This method consists of solving the swing equation numerically, getting the performance of the system after the clearance of the fault, and then judging whether the system is stable or not. This method has an advantage that the system can be modelled as minutely as the computer permits. The method, however, also has a weak point. It requires much computing time, because the differential equations must be solved until the system has been judged for its stability, so it is rather expensive. To avoid this disadvantage, the application of the direct method of Lypunov has been considered[1)-4)]. This is the method whereby a proper Lyapunov function is first constructed, and its critical value which gives stability

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boundary is obtained. Then, by comparison of that critical value with the value of the Lyapunov function when the fault is cleared, it can be judged whether the system is stable or unstable. But this method also has disadvantage, in that it gives only a sufficient condition for stability. The stability region determined by Lyapunov's method might be much less than the actual one, depending upon the constructed Lyapunov function and the system conditions.

In this paper, we propose a method of improving the accuracy of the stability criterion by enlarging the estimation of the stability region determined by means of Lyapunov's direct method using the series expansion and by putting it closer to the actual stability region. As the Lyapunov function used was that by energy integration, the effect of the damping is not considered in this Lyapunov function. Hence, the difference between the estimation and the actual stability boundary becomes large in accordance with the damping being large. As will be made clear later, however, the method proposed in this paper has the advantage whereby the larger the damping is, the wider is the stability boundary, expanded by the same amount of calculation. The principle of this method is described in Section 2. In Section 3, it is applied to a single-machine infinite bus system, and it is shown how the estimation of the stability region is enlarged and how the estimation of the critical fault clearing time is improved. In Section 4, the method proposed here is compared with Zubov's method which is related to our method, and the storage requirement of both methods as well as the possibility of applying them to multi-machine systems are examined.

## 2. Enlargement of the Estimation of the Asymptotically Stable Domain by means of Series Expansion

### 2-1 Principle of the Method

We consider the autonomous system described by the following differential equation.

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \cdots, x_n) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \cdots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \cdots, x_n) \end{aligned} \qquad (1)$$

$$f_i(0, 0, \cdots, 0) = 0 \qquad i = 1, 2, \cdots, n$$

or

$$\dot{x} = F(x) \qquad F(0) = 0 \tag{1'}$$

where the $f_i$'s are presumed to be differentiable according to the desired high order. The origin is assumed to be a stable equilibrium, and the stability region containing the origin is of interest. The criterion of the local asymptotic stability using the Lyapunov function is as follows. An initial state $x(0)=(x_{10}, x_{20}, \cdots, x_{n0})$ approaches the origin with time, if the following inequality is satisfied,

$$V\{x(0)\} \leqslant V_{cr} \tag{2}$$

where $V(x)$ is the Lyapunov function and $V_{cr}$ is the critical value of the Lyapunov function. If $V\{x(0)\} > V_{cr}$, this initial state can not be judged to be stable or unstable, since Lyapunov's stability theorem is only a sufficient condition for stability. Provided that this $x(0)$ is included in the domain of attraction, $V\{x(t_c)\}$ equals $V_{cr}$ at some time $t_c$. The method described below is to enlarge the stability region estimated by its inequality (2) so as to make it as close to the actual one as possible.

The Lyapunov function $V\{x(t)\}$ is expressed in the series expansion form as follows:

$$V\{x(t)\} = V\{x(0)\} + t\cdot\frac{dV}{dt}\bigg|_{x=x(0)} + \frac{t^2}{2!}\cdot\frac{d^2V}{dt^2}\bigg|_{x=x(0)} + \cdots + \frac{t^n}{n!}\cdot\frac{d^nV}{dt^n}\bigg|_{x=x(0)} + \cdots \tag{3}$$

Hence, the value of $V$ at time $t$ can be known from the values of $V$ and its time derivatives at $t=0$. The time derivatives of $V$ can be calculated successively using the following relations:

$$\begin{aligned} \frac{dV}{dt} &= \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}\cdot f_i \\ \frac{d^2V}{dt^2} &= \frac{d}{dt}\left(\frac{dV}{dt}\right) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{dV}{dt}\right)\cdot f_i \\ &\vdots \end{aligned} \tag{4}$$

When an initial state $x(0)$ belongs to the stable region, the following relation holds for some time $t_c$ (refer to Fig. 1):

$$V\{x(t_c)\} = \sum_{i=0}^{\infty}\frac{t_c^i}{i!}\cdot\frac{d^iV}{dt^i}\bigg|_{x=x(0)} = V_{cr} \tag{5}$$

Therefore, the actual stability boundary is represented by the surface determined by the following equation.

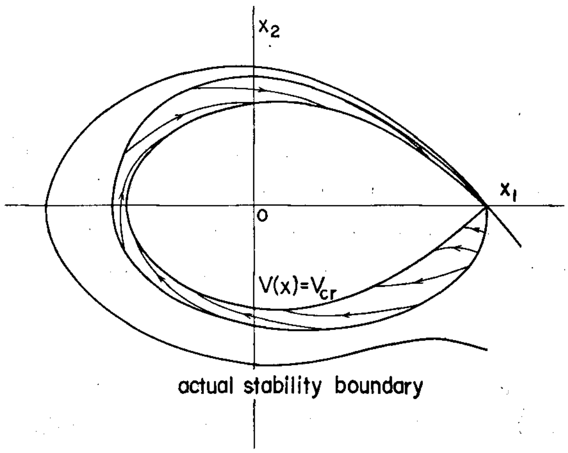


Fig. 1. Concept of the proposed method.

$$\lim_{t\to\infty} V\{x(t)\} = \lim_{t\to\infty} \sum_{i=0}^{\infty} \frac{t^i}{i!} \cdot \frac{d^i V}{dt^i}\bigg|_{x=x(0)} = V_{cr} \qquad (6)$$

However, because of the limited memory of the computer, it is impossible to calculate an infinite number of terms, and the series must be truncated at a finite $i$. In eq. (5), when $t_c$ is taken larger, more terms must be taken into account in order to approximate $V\{x(t_c)\}$ with a certain accuracy. When $m$, the number of the terms which can be calculated, is predetermined according to the memory of the computer, $t_m$, the time with which the value of $V\{x(t_m)\}$ can be computed with a certain accuracy using those terms, is determined. Then the stability domain initially estimated by inequality (2) is expanded to be

$$V'(x) \equiv V(x) + t_m \cdot \frac{dV}{dt} + \frac{t_m^2}{2!} \cdot \frac{d^2V}{dt^2} + \cdots + \frac{t_m^m}{m!} \cdot \frac{d^m V}{dt^m} = V_{cr} \qquad (7)$$

## 2-2 Application to the Stability Problem of Power Systems

The effectiveness of the method described in the preceding section is greatly affected by the asymptotic characteristics of the system. For systems with a large damping, the initial states belonging to the stability domain approach the stable equilibrium rapidly, and the estimation of the stability boundary is greatly enlarged for the fixed time $t_m$. On the other hand, when the damping of the system is small, the enlargement is also small.

For the transient stability problem of electric power systems, the most widely used Lyapunov function is the one obtained from the first integration (also called the energy integration) of the differential equations which describe the motion of the rotors of the generators neglecting the effect of the damping. This Lyapunov function gives a fairly accurate estimation of the stability domain, when the damping and the mutual conductances between the generators are negligible. When the system has a large damping, the estimation becomes conservative. Therefore, the Lyapunov function constructed from the energy integration can be said to be very suitable for the method proposed in this paper in order to enlarge the estimation of the stability region by means of a series expansion.

## 3. Numerical Example

The method represented previously is applied to the single-machine infinite bus system[5] described by the following differential equation,

$$M\frac{d^2\delta}{dt^2}+D\frac{d\delta}{dt} = P_m-P_e\sin\delta \tag{8}$$

$$M = 0.0138 \qquad D = 0.057$$

$$P_m = 0.91 \qquad P_e = 3.02\,.$$

The Lyapunov function constructed from the energy integration is as follows:

$$V(\delta,\,\omega) = M\omega^2/2-P_m(\delta-\delta_0)-P_e(\cos\delta-\cos\delta_0) \tag{9}$$

where $\delta_0$ is the value of $\delta$ at the stable equilibrium point ($\delta_0=\sin^{-1}(P_m/P_e)$ $=0.30608$ rad). The critical value of the Lyapunov function is found to be

$$V_{cr} = V(\pi-\delta_0,\,0) = 3.4575\,. \tag{10}$$

From eqs. (9) and (10), the stability boundary in the $\delta-\omega$ plane in case of no damping is given as follows:

$$M\omega^2/2-P_m\delta-P_e\cos\delta = 0.29933$$

or

$$0.0069\omega^2-0.91\delta-3.02\cos\delta=0.29933\,. \tag{11}$$

From the application of the series expansion method to this system it resulted that, with $m=20$ in eq. (7), $V\{x(t_m)\}$ can be obtained until $t_m=0.08$ sec with the error less than, 0.001 and, with $m=10$, $t_m=0.05$ sec. Fig. 2 shows the actual stability domain obtained by the *R.K.G.* method, its estimation by the original Lyapunov function and the enlargements of the estimation. For this example, the value of the damping $D$ is large, hence every trajectory in the $\delta-\omega$ plane except the stability boundary approaches one of the stable equilibriums, which are located on the $\delta$-axis at the interval of $2\pi$ rad. As seen from the figure, the estimation of the stability region by the Lyapunov function is much less than the

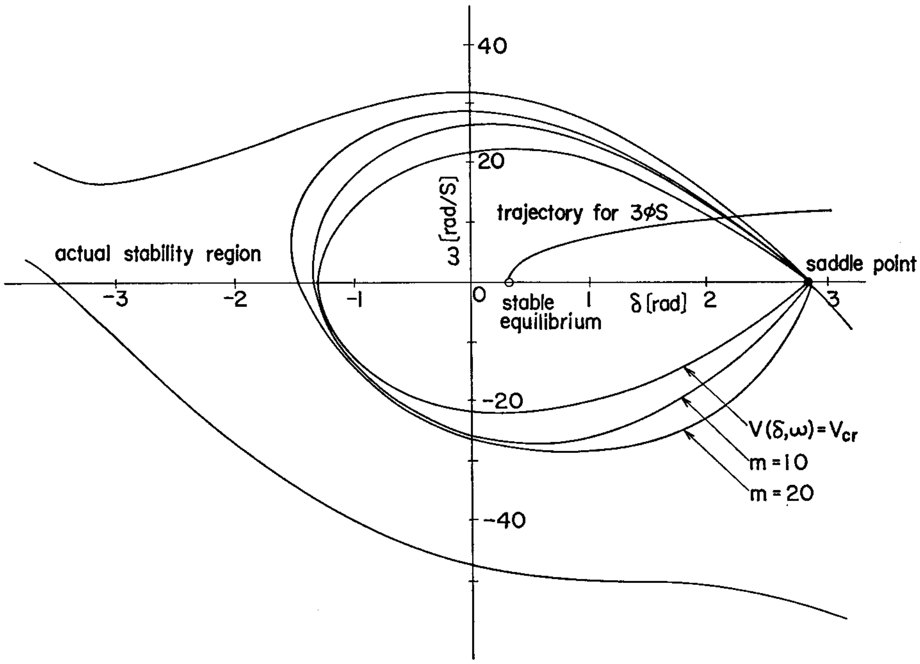


Fig. 2. Stability regions for the model system.

actual stability region because of the large value of $D$. The estimation is fairly extended using the method proposed in this paper. The difference between the estimation and the actual stability region is especially small in the first quadrant, where the step-out of the synchronous generator usually occurs. In the same

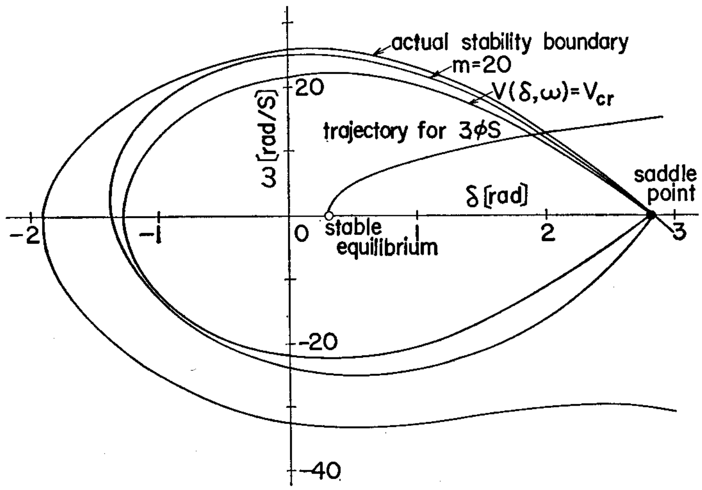


Fig. 3. Stability regions for the model system with reduced damping.

figure is also shown the trajectory starting from the initial steady state in the case of the vanishing electrical power output $P_e$. The critical clearing time (*c.c.t.*) for the three phase short circuit on the transmission line can be obtained from the time along this trajectory. It was found that the actual *c.c.t.* is 0.288 sec., that its estimation by the Lyapunov function is 0.272 sec. and by the proposed method 0.282 sec. reducing the error by more than half.

Fig. 3 shows the stability boundary and its estimations in the case of reducing the value of $D$ by half. The estimation of the stability region by the Lyapunov function is closer to the actual region than Fig. 2. The estimation is enlarged to some extent by the series expansion method. The actual *c.c.t.*, its estimation by the Lyapunov function and by the proposed method are 0.246 sec., 0.238 sec. and 0.242 sec., respectively.

## 4. Considerations

Zubov's method, which obtains an accurate stability boundary using the series expansion, is related to the method proposed here. Zubov's method is to construct a Lyapunov function by solving the following partial differential equation[6)-8)]:

$$\{\nabla V(x)\}^T F(x) = -U(x)\{1-V(x)\} \tag{12}$$

where $U(x)$ is an arbitrary positive definite function. The solution of eq. (12) can not usually be obtained in a closed-form, and a series form of solution is assumed as follows:

$$V(x) = \sum_{i=2}^{\infty} V_i(x) \tag{13}$$

where $V_i(x)$ is a homogeneous polynomial of $i$ th degree. The righthand side of eq. (1′) is also represented in a series form.

$$F(x) = \sum_{i=1}^{\infty} F_i(x) \tag{14}$$

where $F_i(x)$ is a homogeneous polynomial of $i$ th degree. From eqs. (12), (13) and (14), we get

$$\begin{aligned}
&\{\nabla V_2(x)\}^T F_1(x) = -U(x)\\
&\{\nabla V_3(x)\}^T F_1(x) + \{\nabla V_2(x)\}^T F_2(x) = 0\\
&\{\nabla V_4(x)\}^T F_1(x) + \{\nabla V_3(x)\}^T F_2(x) + \{\nabla V_2(x)\}^T F_3(x) = U(x)V_2(x)\\
&\vdots\\
&\{\nabla V_i(x)\}^T F_1(x) + \{\nabla V_{i-1}(x)\}^T F_2(x) + \cdots + \{\nabla V_2(x)\}^T F_{i-1}(x) = U(x)V_{i-2}(x)
\end{aligned} \tag{15}$$

Since $F_i$'s are known functions, the eqs. (15) can successively be solved downward from the first one for $V_2(x)$, $V_3(x)$, $\cdots$, $V_i(x)$, $\cdots$ and from eq. (13) the Lyapunov function $V(x)$ is obtained. Each of the equations (15) is a linear equation with the coefficients of each polynomial $V_i(x)$ as the unknowns. Therefore, a large coefficient matrix must be stored in the memory of the computer, if one desires to approximate $V(x)$ by polynomials of a high degree. For example, when the eqs. (15) are solved as high as the 20th degree, the order of the coefficient matrix is 21 for the 2nd-order system, 231 for the 3rd-order system and as large as 1771 for the 4th-order system. Those coefficient matrices are sparse matrices according to the property of $F_i$, and some methods have been proposed in order to reduce the required memory and computing time by making use of this sparsity. However, the required memory storage gets enormous anyhow, as the order of the system gets large.

It was shown by Zubov that the critical value of $V$, which gives the stability boundary, is 1, when $V$ is obtained accurately, that is, the series in eq. (13) is calculated infinitely. In the case where the series is truncated at $i$, the point where the surface $\dot{V}(x)=0$ is tangent to a surface $V(x)=$const. must be obtained using a nonlinear programming method in order to determine the critical value of $V$[8]. It is known, moreover, that the estimation of the stability domain does not approach the actual one uniformly with an increasing number of terms in eq. (13), and that the choice of the function $U(x)$ also affects the estimation[7),8)].

In the method proposed in this paper, the critical value is calculated only once for the original Lyapunov function, and does not need to be recalculated regardless of the truncated number of the series. This method also has the merit whereby the estimation of the stability boundary is always enlarged with an increasing number of terms, since the trajectories never intersect each other.

Computer programs intended to calculate the partial derivatives in eq. (4) were developed. When the series is truncated at a higher order, the number of terms and hence the storage requirement increases. Unlike Zubov's method, the storage requirement depends upon the form of the original Lyapunov function and the right-hand side of eq. (1′). In general cases, it is difficult to compare the required storage between Zubov's method and the method proposed here. Comparison for the transient stability of electric power systems is tried subsequently. For an $n$-machine system, $2n$ state variables are taken to be $\delta_i$ and $\omega_i$, $i=1, \cdots, n$, and the series is assumed to be calculated up to the 20th term. In Zubov's method, the coefficient matrix of the linear equation to obtain the coefficients of the 20th term $V_{20}$ almost dominates the storage requirement, and if the sparsity of the matrix is not made use of, the required storage is at least

$$m = \{(i+2n-1)!/i!(2n-1)!\}^2$$

where $i$ is the order up to which the series is calculated (here $i=20$). On the other hand, in the method proposed in this paper, $4n$ elements, $\delta_i$, $\omega_i$, $\sin\delta_i$ and $\cos\delta_i$, $i=1, \cdots, n$ must be taken for an $n$-machine system. If it is assumed that the modified Lyapunov function is represented by the summation of the polynomials of up to 20th order of these elements, and that each term requires $(n+1)$ memories*, then the storage requirement is as follows:

$$m = \sum_{j=1}^{i} \{(i+4n-1)!/j!(4n-1)!\} \cdot (n+1) = \{(i+4n)!/i!4n!-1\} \cdot (n+1)$$

Table 1 shows the storage requirement for various values of $n$ for each method. It is seen that the method proposed here requires less storage than Zubov's method for systems with more than two machines.

Table 1. Comparison of the storage requirement.

| Number of machines | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Zubov's method | $4.41\times10^{2}$ | $3.14\times10^{6}$ | $2.82\times10^{9}$ | $7.89\times10^{11}$ | $1.00\times10^{14}$ | $7.17\times10^{15}$ |
| Proposed method | $2.13\times10^{4}$ | $9.32\times10^{6}$ | $9.03\times10^{8}$ | $3.65\times10^{10}$ | $8.27\times10^{11}$ | $1.23\times10^{13}$ |

## 5. Conclusions

In this paper, a series expansion method of enlarging the estimation of the asymptotic stability region determined by a Lyapunov function was proposed and applied to the transient stability problem of electric power systems. The Lyapunov function constructed from the energy integration was used as the original Lyapunov function. Numerical results for a single-machine infinite bus system show that the estimation of the stability region is considerably expanded by calculating the series expansion up to the 20th-order term. As for the estimation of the critical fault clearing time, by applying the proposed method, the estimation error in the case of using the Lyapunov function was reduced by more than half.

As described in Section 4, the method proposed in this paper has some advantages compared with Zubov's method, and it is straightfoward in its application to practical problems. Therefore, it seems that the method is worthy of being investigated further in order to reduce the storage requirement and enable it to be applied to higher order systems.

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* It was assumed that the indices of each four elements can be stored in one memory.

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