Continuum Mechanics in a Space of Any Dimension II. Isotropic Elastic Materials

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Abstract

The behavior of isotropic elastic materials in a space of any dimension is investigated. The linear approximations of the constitutive equation with respect to the strain and to the principal stretches are presented, and the Hookean solid is defined. In the case of space having more than one dimension, the material has two elastic constants. In one-dimensional space the material has an elastic constant. Young's modulus, Poisson's ratio and the bulk modulus of the Hookean solid depend not only on the elastic constants but also on the dimension of space. The wave propagation is analyzed and there are a longitudinal wave and (n-1) transverse waves in n-dimensional space. Simple shear deformation is investigated and there occurs a normal stress effect. It is proved that simple shear deformation is equivalent to a pure shear deformation and the principal stretches are determined by the total amount of shear.

1. Introduction

In the preceding paper[1)], the basic concepts of *continuum mechanics in a space of any dimension* were presented. Three fundamental laws, i.e., the conservations of mass, of linear momentum and of moment of momentum were assumed, and except for the case of one-dimensional space, the existence of the symmetric stress tensor was proved. From the three principles, i.e., the determinism, the local action and the frame-indifference, the constitutive equation of continuum was represented, and the simple material was defined.

Also, there was presented the constitutive equation of the isotropic elastic material, that is,

$$\boldsymbol{T}=\boldsymbol{K}(\boldsymbol{B})\,, \tag{1.1}$$

where $\boldsymbol{T}$ is the stress tensor, $\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy-Green tensor, which represents a measure of deformation, and the response function must satisfy identically

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$$\boldsymbol{K}(\boldsymbol{QBQ}^T) = \boldsymbol{QK}(\boldsymbol{B})\boldsymbol{Q}^T \tag{1,.2}$$

for all symmetric tensors $\boldsymbol{B}$ and all orthogonal tensors $\boldsymbol{Q}$.

In this paper, the isotropic elastic materials in a space of any dimension are investigated theoretically. The linear approximation of the constitutive equation is proposed and the Hookean solid is defined. The wave propagation in a Hookean solid is analyzed. The simple shear deformation of the non-linear materials is investigated.

2. Isotropic Linear Elasticity

Now we define the strain tensor

$$\boldsymbol{E} \equiv \frac{1}{2}(\boldsymbol{B}-\boldsymbol{1}), \tag{2.1}$$

which has the vector $\boldsymbol{0}$ in the reference state. The constitutive equation (1.1), then, is expressed as

$$\boldsymbol{T} = \boldsymbol{K}(\boldsymbol{1}+2\boldsymbol{E}) \equiv \boldsymbol{L}(\boldsymbol{E}) \tag{2.2}$$

and the identity (1.2) reduces to

$$\boldsymbol{L}(\boldsymbol{QEQ}^T) = \boldsymbol{QL}(\boldsymbol{E})\boldsymbol{Q}^T . \tag{2.3}$$

From the representation theorem of the isotropic function[2] we can obtain the expression

$$\boldsymbol{T} = \phi_0\boldsymbol{1}+\phi_1\boldsymbol{E}+\phi_2\boldsymbol{E}^2+\cdots+\phi_{n-1}\boldsymbol{E}^{n-1}, \tag{2.4}$$

where the coefficients $\phi_0, \phi_1, \phi_2, \cdots, \phi_{n-1}$ are scalar functions of the invariants of $\boldsymbol{E}$, which are given by

$$\det(\lambda\boldsymbol{1}+\boldsymbol{E}) = \lambda^n+I_1(\boldsymbol{E})\lambda^{n-1}+I_2(\boldsymbol{E})\lambda^{n-2}+\cdots+I_n(\boldsymbol{E}) . \tag{2.5}$$

If the principal axes of $\boldsymbol{E}$ are taken as the coordinate axes, we have

$$\det(\lambda\boldsymbol{1}+\boldsymbol{E}) = \begin{vmatrix} \lambda+E_1 & & 0 \\ & \lambda+E_2 & \\ & & \ddots \\ 0 & & \lambda+E_n \end{vmatrix} = (\lambda+E_1)(\lambda+E_2)\cdots(\lambda+E_n) . \tag{2.6}$$

Then we can obtain that

$$
\left.\begin{aligned}
I_1(\boldsymbol{E}) &= E_1+E_2+\cdots+E_n = \sum_{\alpha=1}^{n} E_\alpha = \operatorname{tr} \boldsymbol{E}\,, \\
I_2(\boldsymbol{E}) &= E_1E_2+E_2E_3+\cdots = \sum_{\alpha\neq\beta} E_\alpha E_\beta\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
I_n(\boldsymbol{E}) &= E_1E_2\cdots E_n = \det \boldsymbol{E}\,,
\end{aligned}\right\} \tag{2.7}
$$

where E_Γ $(\Gamma=1, 2, \cdots, n)$ are the proper numbers of strain.

We assume that the coefficients of (2.4) are analytic functions of the invariants and can be expanded by them. Then

$$
\phi_\alpha = \mu(a_{\alpha 0}+a_{\alpha 1}I_1(\boldsymbol{E})+a_{\alpha 2}I_1(\boldsymbol{E})^2+a_{\alpha 3}I_2(\boldsymbol{E})+\cdots) \tag{2.8}
$$

where μ is the elastic constant with stress dimension and $a_{\alpha 0}$, $a_{\alpha 1}$, $a_{\alpha 2}$, $\cdots$ are non-dimensional material constants.

Let us consider the linear elasticity. If the second and higher orders with respect to the proper numbers of strain are neglected, we have the linear elasticity. Substituting (2.8) into (2.4), neglecting $\boldsymbol{E}^2, \cdots, I_1(\boldsymbol{E})^2, I_2(\boldsymbol{E}), \cdots$, we have the constitutive equation of the *isotropic linear elastic material*

$$
\boldsymbol{T} = \mu(a_{00}+a_{01}I_1(\boldsymbol{E}))\,\mathbf{1}+\mu a_{10}\boldsymbol{E}. \tag{2.9}
$$

If the stress vanishes at the reference state, i.e., the *natural state,* we have $a_{00}=0$. According to the definition of the elastic constants, a constant can take any value, then we choose $a_{10}=2$ and take $a_{01}=\lambda/\mu$. We have

$$
\boldsymbol{T} = \lambda(\operatorname{tr}\boldsymbol{E})\,\mathbf{1}+2\mu\boldsymbol{E}\,, \tag{2.10}
$$

$$
T_\Gamma = \lambda(E_1+E_2+\cdots+E_n)+2\mu E_\Gamma \quad (\Gamma = 1, 2, \cdots, n)\,, \tag{2.11}
$$

where T_Γ $(\Gamma=1, 2, \cdots, n)$ are the proper numbers of stress, and λ and μ are called *Lame's elastic constants.* In a space of one-dimension we have $T=\mu(a_{00}+a_{01}I_1(E))$, then

$$
T_1 = \lambda E_1\,, \tag{2.12}
$$

where λ is a single elastic constant. Then we can say that the isotropic linear elastic material in a space having more than one-dimension has *two* elastic constants, while the linear elastic material in a space of one-dimension has *one* elastic constant.

The proper numbers of $\boldsymbol{B}$ are v_Γ^2 $(\Gamma=1, 2, \cdots, n)$ and v_Γ are the principal stretches. The relation (2.1) gives

$$
E_\Gamma = \frac{1}{2}(v_\Gamma^2-1) \quad (\Gamma = 1, 2, \cdots, n)\,, \tag{2.13}
$$

then the equation (2.10) is the linear approximation with respect to (v_Γ^2-1).

Let us now consider the linear approximation with respect to $(v_\Gamma-1)$. The displacement gradient is given by

$$\boldsymbol{H}=\boldsymbol{F}-\mathbf{1} \tag{2.14}$$

and

$$\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^T=\mathbf{1}+2\tilde{\boldsymbol{E}}+\boldsymbol{H}\boldsymbol{H}^T, \quad \boldsymbol{E}=\tilde{\boldsymbol{E}}+\frac{1}{2}\boldsymbol{H}\boldsymbol{H}^T, \tag{2.15}$$

where

$$\tilde{\boldsymbol{E}}\equiv\frac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^T) \tag{2.16}$$

is called the *infinitesimal strain*. From the relation $\boldsymbol{B}=\boldsymbol{V}^2$, where $\boldsymbol{V}$ is the left stretch tensor, we have

$$\boldsymbol{V}=\mathbf{1}+\tilde{\boldsymbol{E}} \tag{2.17}$$

in the first order approximation of the displacement. Then the proper numbers of $\tilde{\boldsymbol{E}}$ are given by

$$\tilde{E}_\Gamma=v_\Gamma-1\,. \tag{2.18}$$

Thus, in the first order approximation with respect to $(v_\Gamma-1)$, we can replace $\boldsymbol{E}$ in (2.10) by $\tilde{\boldsymbol{E}}$, and E_Γ in (2.11) by $\tilde{E}_\Gamma$. Then we have

$$\boldsymbol{T}=\lambda(\mathrm{tr}\,\tilde{\boldsymbol{E}})\mathbf{1}+2\mu\tilde{\boldsymbol{E}}, \tag{2.19}$$

$$T_\Gamma=\lambda(\tilde{E}_1+\tilde{E}_2+\cdots+\tilde{E}_n)+2\mu\tilde{E}_\Gamma \quad (\Gamma=1,2,\cdots,n)\,. \tag{2.20}$$

The material, which has (2.19) as the constitutive equation, is called the *Hookean solid.*

3. Special Deformations of Hookean Solid

Now we consider some special deformations.

Uniaxial Tension

Let us consider the elongation of a uniform bar by a force along it. When the x_1-axis is chosen along the bar we can put

$$T_2=T_3=\cdots=T_n=0\,, \tag{3.1}$$

$$\tilde{E}_2=\tilde{E}_3=\cdots=\tilde{E}_n\,, \tag{3.2}$$

Then we have

$$T_1 = (\lambda+2\mu)\tilde{E}_1+(n-1)\lambda\tilde{E}_2\,,$$
$$0 = \lambda\tilde{E}_1+\{(n-1)\lambda+2\mu\}\tilde{E}_2\,,$$

which yield the relations

$$T_1 = E\tilde{E}_1\,,\quad \tilde{E}_2 = -\sigma\tilde{E}_1\,, \tag{3.3}$$

where

$$E \equiv \frac{2\mu(n\lambda+2\mu)}{(n-1)\lambda+2\mu}\,,\quad \sigma \equiv \frac{\lambda}{(n-1)\lambda+2\mu} \tag{3.4}$$

are called, respectively, *Young's modulus* and *Poisson's ratio.*

Simple Extension

In the elongation of a uniform bar, if the lateral deformation is prevented, the deformation is called the simple extension. In this case we can put

$$\tilde{E}_2 = \tilde{E}_3 = \cdots = \tilde{E}_n = 0\,, \tag{3.5}$$
$$T_2 = T_3 = \cdots = T_n\,. \tag{3.6}$$

Then we have

$$T_1 = (\lambda+2\mu)\tilde{E}_1\,,\quad T_2 = \lambda\tilde{E}_1\,, \tag{3.7}$$

which can be written by

$$T_1 = E'\tilde{E}_1\,,\quad T_2 = \sigma' T_1\,, \tag{3.8}$$

where

$$E' \equiv \lambda+2\mu\,,\quad \sigma' \equiv \frac{\lambda}{\lambda+2\mu} \tag{3.9}$$

are called, respectively, *pseudo Young's modulus* and *pseudo Poisson's ratio,* and they are independent of the dimension of space.

Uniform Expansion

In the linear approximation, the jacobian is given as

$$\begin{aligned} J &= \sqrt{I_n(\boldsymbol{B})} = \sqrt{(1+2E_1)(1+2E_2)\cdots(1+2E_n)} \\ &= 1+I_1(\boldsymbol{E})+\cdots \cong 1+I_1(\tilde{\boldsymbol{E}})\,. \end{aligned} \tag{3.10}$$

Then

$$I_1(\tilde{\boldsymbol{E}}) \cong J-1 = \frac{\Delta V}{V} \tag{3.11}$$

indicates the volume change per unit volume. In this case we can put

$$T_1 = T_2 = \cdots = T_n = -p\,, \tag{3.12}$$

$$\tilde{E}_1 = \tilde{E}_2 = \cdots = \tilde{E}_n = \frac{\Delta V}{nV}\,, \tag{3.13}$$

where p denotes the pressure. Therefore, we have

$$p = -(n\lambda+2\mu)\,E_1 = -K\frac{\Delta V}{V}\,, \tag{3.14}$$

where

$$K \equiv \lambda+\frac{2}{n}\mu \tag{3.15}$$

is called the *bulk modulus*.

Simple Shear

In this case a component of shear stress does not vanish and

$$T_{12} = 2\mu\tilde{E}_{12}\,. \tag{3.16}$$

Then μ denotes the *modulus of transverse elasticity* or the *modulus of rigidity*. In section 5, we will investigate the simple shear deformation of general non-linear elastic material.

One-Dimensional Space

In a space of one-dimension from (2.12) we have

$$E = E' = K = \lambda\,, \tag{3.17}$$

which, by the condition $\mu\to 0$, coincides with (3.4), (3.9) and (3.15), where σ, σ' and μ lose their meanings.

4. Wave Propagation in Hookean Solid

In general, a continuum must satisfy Cauchy's first law of motion[1)]

$$\operatorname{div} \boldsymbol{T}+\rho\boldsymbol{b} = \rho\ddot{\boldsymbol{x}}\,, \tag{4.1}$$

where ρ is the mass density, $\boldsymbol{b}$ is the body force per unit mass and $\ddot{\boldsymbol{x}}$ is the acceleration vector.

The displacement vector is given by

$$\boldsymbol{u} = \boldsymbol{x}-\boldsymbol{X}\,, \tag{4.2}$$

where $\boldsymbol{x}$ and $\boldsymbol{X}$ are, respectively, the position vectors of a particle in the current and reference state. Then, the displacement tensor (2.14) is

$$\boldsymbol{H}=\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}=\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}+\cdots, \tag{4.3}$$

where dots mean the second and higher order terms with respect to the displacement gradient. In the first order approximation we can express the infinitesimal strain (2.16) as

$$\boldsymbol{E}=\frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}+\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^{T}\right), \tag{4.4}$$

$$\tilde{E}_{kl}=\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right) \qquad (k,\, l=1,\, 2,\, \cdots,\, n)\,, \tag{4.5}$$

where a cartesian coordinate system is assumed.

By substituting the Hookean equation (2.19) and the displacement vector (4.2) into the equation of motion (4.1), and by referring to (4.5) and neglecting the body force, we have

$$(\lambda+\mu)\frac{\partial^2 u_l}{\partial x_k \partial x_l}+\mu\frac{\partial^2 u_k}{\partial x_l^2}=\rho\frac{\partial^2 u_k}{\partial t^2} \qquad (k,\, l=1,\, 2,\, \cdots,\, n)\,. \tag{4.6}$$

Now let us consider the plane wave propagating along the x_1-axis,

$$u_k=u_k(x_1,\, t) \qquad (k=1,\, 2,\, \cdots,\, n)\,. \tag{4.7}$$

Then we have

$$(\lambda+2\mu)\frac{\partial^2 u_1}{\partial x_1^2}=\rho\frac{\partial^2 u_1}{\partial t^2}, \tag{4.8}$$

$$\mu\frac{\partial^2 u_k}{\partial x_1^2}=\rho\frac{\partial^2 u_k}{\partial t^2} \qquad (k=2,\, 3,\, \cdots,\, n)\,. \tag{4.9}$$

The wave equation (4.8) gives a *longitudinal wave* with the wave velocity

$$C_{\Gamma}\equiv\sqrt{\frac{\lambda+2\mu}{\rho}}, \tag{4.10}$$

and the wave equations (4.9) give $(n-1)$ *transverse waves* with a wave velocity

$$C_{\mathrm{T}}\equiv\sqrt{\frac{\mu}{\rho}}\,. \tag{4.11}$$

In a space of one-dimension there is a longitudinal wave and no transverse wave.

5. Simple Shear Deformation

Here, we will analyze the simple shear deformation of the isotropic elastic material with the constitutive equation (2.4).

The deformation is given by, in a cartesian coordinate system,

$$\left.\begin{aligned} x_k &= X_k \qquad (k=1,2,\cdots,n-1)\,, \\ x_n &= X_n+K_1X_1+K_2X_2+\cdots+K_{n-1}X_{n-1}\,. \end{aligned}\right\} \tag{5.1}$$

This deformation is homogeneous and accelerationless. Then, the equation of motion (4.1) is satisfied in the case where there is no body force.

The deformation gradient, the strain tensor, and the left Cauchy-Green tensor are given by the matrices

$$[\boldsymbol{F}] = \begin{vmatrix} 1 & 0\cdots & 0 & 0 \\ 0 & 1\cdots & 0 & 0 \\ \vdots & \vdots\ \ddots & \vdots & \vdots \\ 0 & 0\cdots & 1 & 0 \\ K_1 & K_2\cdots & K_{n-1} & 1 \end{vmatrix}, \tag{5.2}$$

$$[\boldsymbol{B}] = \begin{bmatrix} [\mathbf{1}] & [K_i]^T \\ [K_i] & 1+K^2 \end{bmatrix}, \quad [\boldsymbol{E}] = \frac{1}{2}\begin{bmatrix} [\mathbf{0}] & [K_i]^T \\ [K_i] & K^2 \end{bmatrix}, \tag{5.3}$$

respectively, where

$$[K_i] = [K_1\ K_2\cdots K_{n-1}] \tag{5.4}$$

is a $1\times(n-1)$ matrix, K_i $(i=1, 2, \cdots, n-1)$ are called the *amounts of shear* and

$$K \equiv \sqrt{\sum_{i=1}^{n-1} K_i^2} \tag{5.5}$$

is called the *total amount of shear*.

By the routine process of determinant calculation, we have

$$\det(\lambda\mathbf{1}+\boldsymbol{E}) = \frac{1}{2^n}\begin{vmatrix} 2\lambda & 0\cdots & 0 & K_1 \\ 0 & 2\lambda\cdots & 0 & K_2 \\ \vdots & \vdots\ \ddots & \vdots & \vdots \\ 0 & 0\cdots & 2\lambda & K_{n-1} \\ K_1 & K_2\cdots & K_{n-1} & 2\lambda+K^2 \end{vmatrix}$$

$$= \frac{1}{2^n}\left(2\lambda\begin{vmatrix} 2\lambda\cdots & 0 & K_2 \\ \vdots\ \ddots & \vdots & \vdots \\ 0\ \cdots & 2\lambda & K_{n-1} \\ K_2\cdots & K_{n-1} & 2\lambda+K^2 \end{vmatrix} - K_1^2(2\lambda)^{n-2}\right)$$

$$= \lambda^{n-2}\left(\lambda^2+\frac{K^2}{2}\lambda-\frac{K^2}{4}\right). \tag{5.6}$$

Then by (2.5) we have the invariants

$$I_1(\boldsymbol{E}) = \frac{K^2}{2}, \quad I_2(\boldsymbol{E}) = -\frac{K^2}{4}; \quad I_\alpha(\boldsymbol{E}) = 0 \quad (\alpha = 3, 4, \cdots, n) . \tag{5.7}$$

If we assume

$$[\boldsymbol{E}^\alpha] = [\boldsymbol{E}]^\alpha = \frac{1}{2^\alpha}\begin{bmatrix} [C_{ij}{}^{(\alpha)}] & [C_i{}^{(\alpha)}]^T \\ [C_i{}^{(\alpha)}] & C^{(\alpha)} \end{bmatrix} \qquad (\alpha = 2, 3, \cdots) , \tag{5.8}$$

where $[C_{ij}{}^{(\alpha)}]$ is an $(n-1)\times(n-1)$ symmetric matrix, $[C_i{}^{(\alpha)}]$ is a $1\times(n-1)$ matrix, we have

$$[\boldsymbol{E}^{\alpha+1}] = [\boldsymbol{E}]^{\alpha+1} = \frac{1}{2^{\alpha+1}}\begin{bmatrix} [K_i C_j^{(\alpha)}] & [K_i C^{(\alpha)}]^T \\ [\sum_{j=1}^{n-1} K_j C_{ij}{}^{(\alpha)} + K^2 C_i{}^{(\alpha)}] & [\sum_{i=1}^{n-1} K_i C_i{}^{(\alpha)} + K^2 C^{(\alpha)}] \end{bmatrix} .$$

Therefore, we have (5.9)

$$\left.\begin{aligned} &C_{ij}{}^{(\alpha+1)} = K_i C_j{}^{(\alpha)}, \; C_i^{(\alpha+1)} = K_i C^{(\alpha)} = \sum_{j=1}^{n-1} K_j C_{ij}{}^{(\alpha)} + K^2 C_i{}^{(\alpha)} , \\ &C^{(\alpha+1)} = K_i C_i{}^{(\alpha)} + K^2 C^{(\alpha)} , \end{aligned}\right\} \tag{5.10}$$

which give

$$\left.\begin{aligned} &C_{ij}{}^{(\alpha)} = K_i K_j C_i^{(\alpha-2)} , \quad C_i{}^{(\alpha)} = K_i C^{(\alpha-1)} , \\ &C^{(\alpha)} = K^2(C^{(\alpha-1)} + C^{(\alpha-2)}) . \end{aligned}\right\} \tag{5.11}$$

From (5.3) we have

$$C^{(-1)} = 0 , \quad C^{(0)} = 1 , \quad C^{(1)} = K^2 , \tag{5.12}$$

which satisfy the last relation of (5.11), so that

$$\left.\begin{aligned} &C^{(2)} = K^2(1+K^2), \quad C^{(3)} = K^4(2+K^2) \\ &C^{(4)} = K^4(1+3K^2+K^4) , \quad C^{(5)} = K^6(3+4K^2+K^4), \cdots. \end{aligned}\right\} \tag{5.13}$$

Then we obtain

$$\begin{aligned} [E^\alpha] = \frac{1}{2^\alpha}\begin{bmatrix} [K_i K_j] & [\boldsymbol{0}]^T \\ [\boldsymbol{0}] & 0 \end{bmatrix} C^{(\alpha-2)} + \frac{1}{2^\alpha}\begin{bmatrix} [\boldsymbol{0}] & [K_i]^T \\ [K_i] & 0 \end{bmatrix} C^{(\alpha-1)} \\ + \frac{1}{2^\alpha}\begin{bmatrix} [\boldsymbol{0}] & [\boldsymbol{0}]^T \\ [\boldsymbol{0}] & 1 \end{bmatrix} C^{(\alpha)} \qquad (\alpha = 1, 2, \cdots) . \end{aligned} \tag{5.14}$$

Substituting (5.14) into the constitutive equation (2.5) we have the stress tensor

$$\begin{aligned} [\boldsymbol{T}] = [\boldsymbol{1}]\phi_0 + \begin{bmatrix} [K_i K_j] & [\boldsymbol{0}]^T \\ [\boldsymbol{0}] & K^2 \end{bmatrix} \sum_{\alpha=2}^{n-1} \frac{1}{2^\alpha} \phi_\alpha C^{(\alpha-2)} \\ + \begin{bmatrix} [\boldsymbol{0}] & [K_i]^T \\ [K_i] & K^2 \end{bmatrix} \sum_{\alpha=1}^{n-1} \frac{1}{2^\alpha} \phi_\alpha C^{(\alpha-1)} . \end{aligned} \tag{5.15}$$

The components of the stress are given by

$$\left.\begin{aligned} T_{ij} &= \phi_0\delta_{ij}+K_iK_j\sum_{\alpha=2}^{n-1}\frac{1}{2^\alpha}\phi_\alpha C^{(\alpha-2)} \qquad (i, j = 1, 2, \cdots, n-1)\,, \\ T_{in} &= T_{ni} = K_i\sum_{\alpha=1}^{n-1}\frac{1}{2^\alpha}\phi_\alpha C^{(\alpha-1)}\,, \\ T_{nn} &= \phi_0+\sum_{\alpha=1}^{n-1}\frac{1}{2^\alpha}\phi_\alpha C^{(\alpha)}\,. \end{aligned}\right\} \tag{5.16}$$

Then there are the differences between the normal stresses

$$T_{nn}-T_{ii} = \sum_{\alpha=1}^{n-1}\frac{1}{2^\alpha}\phi_\alpha\{K^2C^{(\alpha-1)}+(K^2-K_i^2)C^{(\alpha-2)}\}$$
$$(i = 1, 2, \cdots, n-1;\ \text{not summed}), \tag{5.17}$$

which show the *normal stress effect*.

When the shear directions are inversed, that is, $K_i \rightarrow -K_i$, from (5.5) and (5.7), the values of ϕ_α (α=0, 1, 2, $\cdots$, $n-1$) do not change. Then from (5.16) we can say that by the inversion of the shear directions the normal stresses do not change and the shear stresses change their signs.

In a special case when ($n-2$) amounts of shear vanish and an amount of shear K_i has non-vanishing value, we have $K=K_i$ and the *universal relation*

$$T_{nn}-T_{ii} = K_iT_{in} \qquad (i = 1, 2, \cdots, n-1;\ \text{not summed}) \tag{5.18}$$

holds.

Let us consider the principal stretches and the principal axes of $\boldsymbol{V}$, which concide with those of $\boldsymbol{B}$, of the simple shear deformation (5.1).

The proper numbers of $\boldsymbol{E}$ are the roots of the equation

$$\det(\boldsymbol{E}-\lambda\mathbf{1}) = 0\,. \tag{5.19}$$

From (5.6) we have

$$E_1, E_2 = \frac{1}{4}(K^2\pm K\sqrt{4+K^2})\,, \quad E_\Gamma = 0 \qquad (\Gamma = 3, 4, \cdots, n)\,. \tag{5.20}$$

Thus by the relation (2.13) we get

$$v_1, v_2 = \frac{1}{2}(\sqrt{4+K^2}\pm K)\,, \quad v_\Gamma = 1 \qquad (\Gamma = 3, 4, \cdots, n)\,, \tag{5.21}$$

which are determined completely by the total amount of shear.

The principal axes are determined by the relations

$$(\boldsymbol{B}-v_\Gamma{}^2\mathbf{1})\,\boldsymbol{b}_\Gamma = \mathbf{0} \qquad (\Gamma = 1, 2, \cdots, n)\,. \tag{5.22}$$

From (5.3) and (5.21), then, there are for Γ=1 and 2

$$\left.\begin{aligned}&\frac{K}{2}(K\pm\sqrt{4+K^2})\,b_{\Gamma k}-K_k b_{\Gamma n}=0 \qquad (k=1,2,\cdots,n-1)\\&\sum_{i=1}^{n-1}K_i b_{\Gamma i}+\frac{K}{2}(K\mp\sqrt{4+K^2})\,b_{\Gamma n}=0\,,\end{aligned}\right\} \tag{5.23}$$

and for $\Gamma=3, 4, \cdots, n$

$$\left.\begin{aligned}&K_k b_{\Gamma n}=0 \qquad (k=1,2,\cdots,n-1)\,,\\&\sum_{i=1}^{n-1}K_i b_{\Gamma i}+K^2 b_{\Gamma n}=0\,.\end{aligned}\right\} \tag{5.24}$$

Then we have

$$\boldsymbol{b}_\Gamma=\left(K_k,\ \frac{K}{2}(K\pm\sqrt{4+K^2})\right) \qquad (\Gamma=1,2)\,, \tag{5.25}$$

$$\boldsymbol{b}_\Gamma=(e_{\Gamma k},\ 0) \qquad (\Gamma=3,4,\cdots,n)\,, \tag{5.26}$$

where the vectors $\boldsymbol{e}_\Gamma$ lie on the hyperplane

$$\sum_{i=1}^{n-1}K_i x_i=0 \tag{5.27}$$

and they are perpendicular with each other.

Clearly, we have

$$v_1 v_2=1 \tag{5.28}$$

and all of the principal axes $\boldsymbol{b}_\Gamma$ ($\Gamma=1, 2, \cdots, n$) are perpendicular to each other.

The proper numbers (5.21) and the perpendicular property of the two principal axes $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ show that the simple shear deformation (5.1) is equivalent to a *pure shear deformation* except for a pure rotation. A pure shear deformation means that an elongation along a direction and a contraction along another direction perpendicular to the former direction and their stretches have a reciprocal relation (5.28).

References

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