Continuum Mechanics in a Space of Any Dimension III. Stokes Fluids

By

Tatsuo TOKUOKA*

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Abstract

The behavior of the Stokes fluid in a space of any dimension is investigated. The linear approximation of the constitutive equations with respect to the stretching is obtained, and compressible and incompressible Navier-Stokes fluids are defined. In a space of more than one-dimension the former has two viscosity coefficients and the latter has one. In a space of one-dimension the former has a viscosity coefficient but the latter reduces to a rigid body. The general curvilinear flow is studied. In a space of dimension of more than two, the Stokes fluid has two viscometric functions, i.e., the shear-stress function and the normal-stress difference function. In a space of two-dimension the fluid has only the shear-stress function. It is proved that the curvilinear flow in a space of any dimension is equivalent to a pure shearing flow, if the motion is observed by an appropriately rotated coordinate system. The simple shearing flow with a rate of shear is also analyzed.

1. Introduction

This series concern the basic and simple concepts of continuum mechanics in a space of any dimension. In the first paper[1], three fundamental laws and three principles were depicted. There were presented the *constitutive equations* of the *simple material* and some special materials, i.e., the *incompressible material*, the *isotropic elastic material* and *Stokes fluid*. In the second paper[2], we studied the basic characteristics of the isotropic elastic material. The *Hookean solid* was defined and it was shown that Young's modulus, Poisson's ratio and the bulk modulus depend not only upon the Lamé elastic constants λ and μ but also upon the space dimension n. Also in the second paper, the wave propagation and the simple shear deformation were analyzed.

In this paper, we study the behavior of the Stokes fluid. At first the *Navier-Stokes fluid* is defined. Next the *curvilinear flow* of the Stokes fluid is analyzed and

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* Department of Aeronautical Engineering

it is proved that the flow is equivalent to a *pure shearing flow*. Furthermore, the simple shearing flow with a rate of shear of an incompressible Stokes fluid is studied.

2. Linear Viscosity

The *Stokes fluid* has the constitutive equation[1)]

$$\boldsymbol{T} = \boldsymbol{K}(\boldsymbol{D}; \rho) , \tag{2.1}$$

where $\boldsymbol{T}$ is the stress tensor, $\boldsymbol{D}$ is the stretching, which is the symmetric part of the velocity gradient, and ρ is the mass density. The *principle of frame-indifference* demands that

$$\boldsymbol{K}(\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T; \rho) = \boldsymbol{Q}\boldsymbol{K}(\boldsymbol{D}; \rho)\boldsymbol{Q}^T \tag{2.2}$$

holds identically for all symmetric tensors $\boldsymbol{D}$, all orthogonal tensors $\boldsymbol{Q}$ and all scalars ρ. Then we have its representation[3)] in the n-dimensional space

$$\boldsymbol{T} = \psi_0\mathbf{1} + \psi_1\boldsymbol{D} + \psi_2\boldsymbol{D}^2 + \cdots + \psi_{n-1}\boldsymbol{D}^{n-1} , \tag{2.3}$$

where the material functions $\psi_0, \psi_1, \psi_2, \cdots, \psi_{n-1}$ are scalar functions of the density and the invariants of the stretching, which are defined by the identity

$$\det(\lambda\mathbf{1} + \boldsymbol{D}) = \lambda^n + I_1(\boldsymbol{D})\lambda^{n-1} + I_2(\boldsymbol{D})\lambda^{n-2} + \cdots + I_n(\boldsymbol{D}) . \tag{2.4}$$

The invariants $I_\alpha(\boldsymbol{D})$ ($\alpha = 1, 2, \cdots, n$) have the magnitudes of the α-th order with respect to the proper numbers of $\boldsymbol{D}$.

In the case of the *incompressible Stokes fluid* the density remains a constant value and the isochoric motion demands that

$$I_1(\boldsymbol{D}) = \operatorname{tr} \boldsymbol{D} = 0 . \tag{2.5}$$

The constitutive equation is given by

$$\boldsymbol{T} = -p\mathbf{1} + \boldsymbol{K}(\boldsymbol{D}) , \tag{2.6}$$

where p is the indeterminate pressure. Its representation in an n-dimensional space is

$$\boldsymbol{T} = -p\mathbf{1} + \psi_1\boldsymbol{D} + \psi_2\boldsymbol{D}^2 + \cdots + \psi_{n-1}\boldsymbol{D}^{n-1} , \tag{2.7}$$

where the material functions $\psi_1, \psi_2, \cdots, \psi_{n-1}$ are scalar functions for $I_\alpha(\boldsymbol{D})$ ($\alpha = 2, 3, \cdots, n$).

Now we assume that the coefficients of (2.3) and (2.7) are analytic functions of the invariants and can be expanded by them. Then

$$\psi_\alpha = b_{\alpha 0} + b_{\alpha 1}I_1(\boldsymbol{D}) + b_{\alpha 2}I_1(\boldsymbol{D})^2 + b_{\alpha 3}I_2(\boldsymbol{D}) + \cdots \quad (\alpha = 0, 1, 2, \cdots) , \tag{2.8}$$

where $b_{\alpha 0}$, $b_{\alpha 1}$, $\cdots$ $(\alpha=0, 1, 2, \cdots)$ are material functions of the density for compressible fluid, and $b_{\alpha 0}$, $b_{\alpha 1}$, $\cdots$ $(\alpha=1, 2, \cdots)$ are material constants for incompressible fluid.

Let us consider the linear viscosity. If the second and higher order terms with respect to the proper numbers of the stretching are neglected, we have the *linear viscosity.* Substituting (2.8) into (2.3) and (2.7), and neglecting the second and higher order terms, we have the constitutive equation of the *compressible linear viscous fluid*

$$\boldsymbol{T} = (-p(\rho)+\zeta(\rho)I_1(\boldsymbol{D}))\,\mathbf{1}+2\eta(\rho)\,\boldsymbol{D}\,, \tag{2.9}$$

$$T_\Gamma = -p(\rho)+\zeta(\rho)(D_1+D_2+\cdots+D_n)+2\eta(\rho)\,D_\Gamma \quad (\Gamma = 1, 2, \cdots, n)\,, \tag{2.10}$$

and that of the *incompressible linear viscous fluid*

$$\boldsymbol{T} = -p\mathbf{1}+2\eta\boldsymbol{D}\,, \tag{2.11}$$

$$T_\Gamma = -p+2\eta D_\Gamma \qquad (\Gamma = 1, 2, \cdots, n)\,, \tag{2.12}$$

where T_Γ and D_Γ $(\Gamma=1, 2, \cdots, n)$ are the proper numbers of the stress and the stretching, respectively, and we put

$$p(\rho)\equiv -b_{00}(\rho)\,,\quad \zeta(\rho)\equiv b_{01}(\rho)\,,\quad \eta(\rho)\equiv\frac{1}{2}b_{10}(\rho)\,,\quad \eta\equiv\frac{1}{2}b_{10}\,. \tag{2.13}$$

In general, the fluid having the linear viscosity is called the *Newtonian fluid.* The materials which have the constitutive equations (2.9) and (2.11) are called, respectively, the *Navier-Stokes fluid* and the *incompressible Navier-Stokes fluid.* In a space of more than one-dimension, the former has two viscosity coefficients ζ and η which are functions of the density, and the latter has a constant viscosity coefficient ζ.

In one-dimensional space all of the quantities reduce to scalars. The material functions $\psi_1, \psi_2, \cdots, \psi_{n-1}$ in (2.3) and (2.7) disappear and the linear constitutive equations (2.10) and (2.12) reduce, respectively, to

$$T_1 = -p(\rho)+\zeta(\rho)\,D_1\,, \tag{2.14}$$

$$T_1 = -p\,. \tag{2.15}$$

Then the incompressible fluid in one-dimensional space reduces to a *rigid body* and its stress is completely indeterminate.

3. Curvilinear Flow

Let us consider an *orthogonal curvilinear coordinate system* in an n-dimensional

Euclidean space. A set $(x^1, x^2, \cdots, x^n)$ corresponds to a point in the space and it is called the *curvilinear coordinates.* The line element is given by

$$ds^2 = \sum_{k=1}^{n} (g_k)^2 (dx^k)^2 , \tag{3.1}$$

where g_k $(k=1, 2, \cdots, n)$ are the magnitudes of the base vectors $\boldsymbol{g}_k$.

For an orthogonal curvilinear coordinate system the components of a tensor have no direct physical meaning. The components which are measured by a local orthogonal cartesian coordinate system are called the *physical components.* Those of a second-order tensor $\boldsymbol{A}$, are given by

$$A\langle kl\rangle = \frac{g_k}{g_l} A^k{}_l = \frac{g_l}{g_k} A_k{}^l \quad (k,\ l = 1,\ 2, \cdots, n;\ \text{not summed})\ . \tag{3.2}$$

If a material particle has the velocity components

$$\dot{x}^1 = 0\ , \quad \dot{x}^i = w_i(x_1) \qquad (i = 2,\ 3, \cdots, n) \tag{3.3}$$

in an orthogonal curvilinear coordinate system with

$$g_k = g_k(x_1) \qquad (k = 1,\ 2, \cdots, n)\ , \tag{3.4}$$

the motion is called the *curvilinear flow*[4),5)]. For this flow, a particle moves within the $(n-1)$-dimensional space and its velocity depends only on the x^1-coordinate.

The velocity gradient and the stretching are given, respectively, by

$$[\boldsymbol{L}] = \begin{bmatrix} 0 & [\boldsymbol{0}] \\ [w_i']^T & [\boldsymbol{0}] \end{bmatrix}, \quad [\boldsymbol{D}] = \frac{1}{2}\begin{bmatrix} 0 & [w_i'] \\ [w_i']^T & [\boldsymbol{0}] \end{bmatrix}, \tag{3.5}$$

where $w_i' \equiv dw_i/dx_i$ $(i=2, 3, \cdots, n)$ and

$$[w_i'] \equiv [w_2' w_3' \cdots w_n'] \tag{3.6}$$

is a $1\times(n-1)$ matrix. From the formula (3.2) the physical components of the stretching are given by

$$[\tilde{\boldsymbol{D}}] \equiv [D\langle kl\rangle] = \frac{1}{2}\begin{bmatrix} 0 & [\kappa_i] \\ [\kappa_i]^T & [\boldsymbol{0}] \end{bmatrix}, \tag{3.7}$$

where $[\tilde{\boldsymbol{D}}]$ denotes the matrix with the physical components of $\boldsymbol{D}$,

$$[\kappa_i] \equiv [\kappa_2 \kappa_3 \cdots \kappa_n]\ , \tag{3.8}$$

$$\kappa_i \equiv \frac{g_i}{g_1} w_i' \qquad (i = 2,\ 3, \cdots, n;\ \text{not summed}) \tag{3.9}$$

are called the *rates of shear.* We must remark that the stretching (3.7) is *not* identical with the strain matrix in the simple shear deformation obtained in the preceding paper[2].

According to the formula (2.4) we will obtain the invariants of the stretching (3.7). By the routine process of determinant calculation, we have

$$\det(\lambda\mathbf{1}+\tilde{\boldsymbol{D}}) = \frac{1}{2^n}\begin{vmatrix} 2\lambda & \kappa_2 & \kappa_3 & \cdots & \kappa_n \\ \kappa_2 & 2\lambda & 0 & \cdots & 0 \\ \kappa_3 & 0 & 2\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_n & 0 & 0 & \cdots & 2\lambda \end{vmatrix} = \lambda^n - \frac{\kappa^2}{4}\lambda^{n-2}, \tag{3.10}$$

where

$$\kappa \equiv \sqrt{\sum_{i=2}^{n} \kappa_i^2} \tag{3.11}$$

is called the *total rate of shear*. Then we have the invariants

$$I_2(\tilde{\boldsymbol{D}}) = -\frac{\kappa^2}{4}, \quad I_1(\tilde{\boldsymbol{D}}) = I_3(\tilde{\boldsymbol{D}}) = \cdots = I_n(\tilde{\boldsymbol{D}}) = 0\,. \tag{3.12}$$

Therefore, the material functions $\psi_0, \psi_1, \psi_2, \cdots, \psi_{n-1}$ are even functions of the total rate of shear.

We can easily obtain that

$$\left.\begin{aligned} \tilde{\boldsymbol{D}}^{2\alpha} &= \left(\frac{\kappa}{2}\right)^{2\alpha-2}\tilde{\boldsymbol{D}}^2 \qquad (\alpha = 1, 2, \cdots)\,, \\ \tilde{\boldsymbol{D}}^{2\alpha+1} &= \left(\frac{\kappa}{2}\right)^{2\alpha}\tilde{\boldsymbol{D}} \qquad (\alpha = 0, 1, 2, \cdots)\,, \end{aligned}\right\} \tag{3.13}$$

and

$$[\tilde{\boldsymbol{D}}^2] = \frac{1}{4}\begin{bmatrix} \kappa^2 & [\mathbf{0}] \\ [\mathbf{0}]^T & [\kappa_i\kappa_j] \end{bmatrix}. \tag{3.14}$$

Then the stress $\tilde{\boldsymbol{T}}$ with the physical components of the stress $\boldsymbol{T}$ is given by

$$\tilde{\boldsymbol{T}} = \phi_0\mathbf{1} + \sum_{\alpha=0}^{[(n-2)/2]}\left(\frac{\kappa}{2}\right)^{2\alpha}\phi_{2\alpha+1}\tilde{\boldsymbol{D}} + \sum_{\alpha=1}^{[(n-1)/2]}\left(\frac{\kappa}{2}\right)^{2\alpha-2}\phi_{2\alpha}\tilde{\boldsymbol{D}}^2\,. \tag{3.15}$$

If we put

$$\tau(\kappa) \equiv \sum_{\alpha=0}^{[(n-2)/2]}\left(\frac{\kappa}{2}\right)^{2\alpha+1}\phi_{2\alpha+1}(\kappa)\,, \quad \sigma(\kappa) \equiv \sum_{\alpha=1}^{[(n-1)/2]}\left(\frac{\kappa}{2}\right)^{2\alpha}\phi_{2\alpha}(\kappa)\,, \tag{3.16}$$

the stress is given by

$$\tilde{\boldsymbol{T}} = \phi_0\mathbf{1} + \frac{2}{\kappa}\tau(\kappa)\tilde{\boldsymbol{D}} + \left(\frac{2}{\kappa}\right)^2\sigma(\kappa)\tilde{\boldsymbol{D}}^2\,. \tag{3.17}$$

The material function $\tau(\kappa)$ is an odd function and $\sigma(\kappa)$ is an even function, i.e.,

$$\tau(-\kappa) = -\tau(\kappa)\,, \quad \sigma(-\kappa) = \sigma(\kappa)\,. \tag{3.18}$$

From (3.7), (3.14) and (3.17) the stress components are expressed by

$$\left.\begin{aligned} \tilde{T}_{11} &= \phi_0 + \sigma(\kappa)\,, \\ \tilde{T}_{ij} &= \phi_0\delta_{ij} + \frac{\kappa_i\kappa_j}{\kappa^2}\sigma(\kappa) \qquad (i, j = 2, 3, \cdots, n)\,, \\ \tilde{T}_{i1} &= \tilde{T}_{1i} = \frac{\kappa_i}{\kappa}\tau(\kappa) \qquad (i = 2, 3, \cdots, n)\,. \end{aligned}\right\} \tag{3.19}$$

The shear stress components $T_{i1} = T_{1i}$ $(i=2, 3, \cdots, n)$ are determined by $\tau(\kappa)$, and the difference of two normal stresses are given by

$$T_{11} - T_{ii} = \left(1 - \frac{\kappa_i^2}{\kappa^2}\right)\sigma(\kappa) \qquad (i = 2, 3, \cdots, n;\ \text{not summed}), \tag{3.20}$$

which are determined by $\sigma(\kappa)$. The two material functions are called the *viscometric functions,* and $\tau(\kappa)$ and $\sigma(\kappa)$ are called, respectively, the *shear-stress function* and the *normal-stress difference function.*

For a two-dimensional space there is no normal-stress difference function, and the stress is given by

$$\tilde{\boldsymbol{T}} = \phi_0\mathbf{1} + \frac{2}{\kappa}\tau(\kappa)\tilde{\boldsymbol{D}}, \tag{3.21}$$

where the shear-stress function is given by

$$\tau(\kappa) = \frac{\kappa}{2}\phi_1(\kappa)\,. \tag{3.22}$$

For three- and higher-dimensional spaces the stress is determined by $\phi_0(\kappa)$ and two viscometric functions $\tau(\kappa)$ and $\sigma(\kappa)$.

Now let us obtain the proper numbers and the principal axes of the stretching (3.7). The proper numbers of $\tilde{\boldsymbol{D}}$ are given by the roots of the equation

$$\det(\tilde{\boldsymbol{D}} - \lambda\mathbf{1}) = 0\,. \tag{3.23}$$

Replacing λ in (3.10) by $-\lambda$ we have the proper numbers

$$\tilde{D}_1, \tilde{D}_2 = \pm\frac{\kappa}{2}, \quad \tilde{D}_\Gamma = 0 \qquad (\Gamma = 3, 4, \cdots, n)\,. \tag{3.24}$$

The principal aexs $\boldsymbol{a}_\Gamma$ $(\Gamma = 1, 2, \cdots, n)$ are determined by the relations

$$(\tilde{\boldsymbol{D}} - \tilde{D}_\Gamma\mathbf{1})\boldsymbol{a}_\Gamma = \mathbf{0} \qquad (\Gamma = 1, 2, \cdots, n)\,. \tag{3.25}$$

From (3.7) and (3.24), then, there are for $\Gamma=1, 2$,

$$\left.\begin{aligned} &\mp\kappa a_{\Gamma 1}+\sum_{i=2}^{n}\kappa_i a_{\Gamma i}=0\,,\\ &\kappa_i a_{\Gamma 1}\mp\kappa a_{\Gamma i}=0 \qquad (i=2,3,\cdots,n)\,,\end{aligned}\right\} \tag{3.26}$$

and for $\Gamma=3, 4, \cdots, n$

$$\sum_{i=1}^{n}\kappa_i a_{\Gamma i}=0\,,\qquad \kappa_i a_{\Gamma 1}=0\,. \tag{3.27}$$

Then we have

$$a_\Gamma=(\pm\kappa,\kappa_2,\kappa_3,\cdots,\kappa_n)\qquad (\Gamma=1,2)\,, \tag{3.28}$$

$$a_\Gamma=(0,e_{\Gamma 2},e_{\Gamma 3},\cdots,e_{\Gamma n})\qquad (\Gamma=3,4,\cdots,n)\,, \tag{3.29}$$

where $\boldsymbol{e}_\Gamma$ ($\Gamma=3, 4, \cdots, n$) are mutually orthogonal vectors and they lie on the hyperplane

$$\sum_{i=2}^{n}\kappa_i x_i=0\,. \tag{3.30}$$

The n principal axes $\boldsymbol{a}_\Gamma$ ($\Gamma=1, 2, \cdots, n$) are clearly perpendicular to each other.

When the principal axes are adopted as coordinate axes, the stertching has the matrix representation

$$[\tilde{\boldsymbol{D}}]=\frac{\kappa}{2}\begin{pmatrix}1&0&0\cdots0\\0&-1&0\cdots0\\0&0&0\cdots0\\ \vdots&\vdots&\vdots\\0&0&0\cdots0\end{pmatrix}, \tag{3.31}$$

which indicates that the motion consists of an elongation along $\boldsymbol{a}_1$ and a contraction along $\boldsymbol{a}_2$, here the two directions are perpendicular to each other. Their velocity gradients, being equal to proper numbers, are related by

$$\tilde{D}_1+\tilde{D}_2=0\,, \tag{3.32}$$

which denotes an isochoric motion.

If the two coordinate axes $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are rotated into their two median lines,

$$[\tilde{\boldsymbol{Q}}]=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1&0\cdots0\\1&1&0\cdots0\\0&0&1\cdots0\\ \vdots&\vdots&\vdots\ \ddots\ \vdots\\0&0&0\cdots1\end{pmatrix}, \tag{3.33}$$

while other axes are fixed, the stretching is expressed by the matrix

$$[\tilde{Q}\tilde{D}\tilde{Q}^T] = \frac{\kappa}{2}\begin{pmatrix} 0 & 1 & 0 \cdots 0 \\ 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0 \cdots 0 \end{pmatrix}. \tag{3.34}$$

This shows a *pure shearing flow*. Then we can say that *any curvilinear flow in a space of any dimension is equivalent to a pure shearing flow, if the motion of a material particle is observed by an appropriately rotated coordinate system.*

4. Simple Shearing Flow with a Rate of Shear of Incompressible Stokes Fluid

In this section we will analyze a special case of the curvilinear flow of an incompressible Stokes fluid.

With respect to a rectangular cartesian coordinate system the velocity components of a fluid particle are assumed to be

$$\dot{x}_2 = w(x_1)\,, \quad \dot{x}_1 = \dot{x}_3 = \cdots = \dot{x}_n = 0\,. \tag{4.1}$$

The rate of shear is given by

$$\kappa \equiv \frac{dw}{dx_1} \equiv w'\,, \tag{4.2}$$

and the stretching and its square are given, respectively, by

$$[D] = \frac{\kappa}{2}\begin{pmatrix} 0 & 1 & 0 \cdots 0 \\ 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0 \cdots 0 \end{pmatrix}, \quad [D^2] = \left(\frac{\kappa}{2}\right)^2\begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0 \cdots 0 \end{pmatrix}. \tag{4.3}$$

This motion is called the *simple shearing flow*[6]. The stretching is equivalent to that of a pure shearing flow (3.34).

The stress is expressed by the constitutive equation (2.7). Then, from the viscometric functions (3.16) and the stretching (4.3) we have

$$\left.\begin{aligned} &T_{11} = T_{22} = -p+\sigma(\kappa)\,, \\ &T_{12} = T_{21} = \tau(\kappa)\,, \\ &T_{ij} = -p\delta_{ij} \qquad (i, j = 3, 4, \cdots, n)\,, \\ &T_{i1} = T_{1i} = T_{i2} = T_{2i} = 0 \qquad (i = 3, 4, \cdots, n)\,. \end{aligned}\right\} \tag{4.4}$$

In general, all of the continuum must satisfy *Cauch's first law of motion*

$$\operatorname{div} \boldsymbol{T}+\rho\boldsymbol{b} = \rho\ddot{\boldsymbol{x}} . \tag{4.5}$$

In this case we have no acceleration. Substituting the stress components (4.4) and the conservative body force

$$\boldsymbol{b} = -\operatorname{grad} \psi , \tag{4.6}$$

where ψ is a potential function, to the equation (4.5), we have

$$\frac{\partial\sigma}{\partial x_1}-\rho\frac{\partial\phi}{\partial x_1} = 0 , \quad \frac{\partial\tau}{\partial x_1}-\rho\frac{\partial\phi}{\partial x_2} = 0 , \quad \frac{\partial\phi}{\partial x_i} = 0 \ (i = 3, 4, \cdots, n) , \tag{4.7}$$

where we put

$$\phi\equiv\psi+\frac{p}{\rho} . \tag{4.8}$$

Then ϕ is independent of x_i (i=3, 4, $\cdots$, n) and we can easily obtain the pressure

$$p = -ax_2+h(x_1)+k(t)-\rho\psi , \tag{4.9}$$

where a is a constant, called the *specific driving force,* and $h(x_1)$and $k(t)$ are arbitrary functions of x_1 and t, respectively, which are determined by given boundary conditions.

From the second equation of (4.7) we have

$$\tau(w'(x_1)) = -ax_1+b , \tag{4.10}$$

where b is a constant. This equation can be regarded as a differential equation to determine the velocity distribution $w(x_1)$.

For a two-dimensional space, there is no $\sigma(\kappa)$ and the stress is given by

$$\tilde{\boldsymbol{T}} = -p\mathbf{1}+\frac{2}{\kappa}\tau(\kappa)\tilde{\boldsymbol{D}} , \tag{4.11}$$

and the equations (4.7) reduce to

$$\frac{\partial\tau}{\partial x_1}-\rho\frac{\partial\phi}{\partial x_2} = 0 , \quad \frac{\partial\phi}{\partial x_1} = 0 . \tag{4.12}$$

Then we can easily obtain

$$p = -ax_2+k(t)-\rho\psi \tag{4.13}$$

and the velocity distribution may be determined by the equation (4.10).

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