Continuum Mechanics in a Space of Any Dimension IV. Viscometric Flows of General Fluids

By

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(Received June 22, 1977)

Abstract

The viscometric flows of the incompressible general fluids in n-dimensional space are investigated, where n takes 2, 3, 4 and 5, and the determinate stress of the fluids is determined by the relative deformation history. It is shown that the characteristic matrices of the viscometric flow have one rate of shear for $n=2$ and 3, and two for $n=4$ and 5. The general fluids in the flow are represented by the viscometric functions. There are one shear stress function for $n=2$, 3, 4 and 5, one normal stress difference function for $n=2$, and two for $n=3$, 4 and 5. The approximation forms of the viscometric functions are obtained in the case of a small magnitude of rate of shear. The generalized steady curvilinear flow is also investigated, and it is proved that the flow is a special case of the viscometric flow.

1. Introduction

The *simple material* defined by Noll[1] has the constitutive equation in which the stress is determined by the total history of the deformation gradient. If the deformation history of a material particle is constant when it is observed by an appropriately rotating time-dependent coordinate system, the motion was called the *substantially, stagnant motion* by Coleman[2]. Noll[3] called it the *motion with constant stretch history,* and he demonstrated that the motion is governed by one constant tensor. Further, Noll[3] defined the *viscometric flow* as a special case of the motion with constant stretch history, and he gave the representation theorem of its characteristic tensor.

The preceding three papers in this series[4),5),6)] concern the basic and simple investigations of the continuum mechanics in a space of any dimension. In this paper, the viscometric flows in two-, four-and five-dimensional space will be investigated. The obtained results may be regarded as a generalization of the results obtained by Noll[1),2)].

In the next section, the motion with constant stretch history is defined, and Noll's

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theorem is given which holds in any dimensional space. Also, the viscometric flow is defined there. In section 3, the matrices which characterize the viscometric flows are presented explicitly, and in section 4 the viscometric functions of the incompressible fluids are obtained. In the last section, the general steady curvilinear flow is analyzed and it is proved that it is a special case of the viscometric flow.

2. Motion with Constant Stretch History and Viscometric Flow

In the first paper[4] of this series we gave the constitutive equation of the simple material

$$\bar{\boldsymbol{T}}=\mathfrak{G}(\bar{\boldsymbol{C}}_t^t;\boldsymbol{C}), \qquad \cdots\cdots\cdots\cdots (2.1)$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy-Green tensor, $\boldsymbol{F}$ is the deformation gradient.

$$\boldsymbol{C}_t^t(s)=\boldsymbol{F}_t^t(s)^T\boldsymbol{F}_t^t(s) \qquad \cdots\cdots\cdots\cdots (2.2)$$

is the relative right Cauchy-Green tensor history.

$$\boldsymbol{F}_t^t(s)=\boldsymbol{F}_t(t-s)=\boldsymbol{F}(t-s)\boldsymbol{F}(t)^{-1} \qquad \cdots\cdots\cdots\cdots (2.3)$$

is the relative deformation history.

$$\bar{\boldsymbol{T}}=\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R}, \qquad \bar{\boldsymbol{C}}_t^t=\boldsymbol{R}^T\boldsymbol{C}_t^t\boldsymbol{R} \qquad \cdots\cdots\cdots\cdots (2.4)$$

are the rotated tensors by the orthogonal tensor $\boldsymbol{R}$ which is given by the polar decomposition of

$$\boldsymbol{F}=\boldsymbol{R}\boldsymbol{U}=\boldsymbol{V}\boldsymbol{R}, \qquad \cdots\cdots\cdots\cdots (2.5)$$

$\boldsymbol{U}$ and $\boldsymbol{V}$ are, respectively, the right and the left stretch tensor, t and s denote, respectively, the present and the past time, and $\mathfrak{G}$ is the response functional.

The constitutive equations of the fluid and the incompressible fluid were, respectively, given by

$$\boldsymbol{T}=\mathfrak{K}(\boldsymbol{C}_t^t;\rho), \qquad \cdots\cdots\cdots\cdots (2.6)$$

$$\boldsymbol{T}=-p\boldsymbol{1}+\mathfrak{K}(\boldsymbol{C}_t^t), \qquad \cdots\cdots\cdots\cdots (2.7)$$

where ρ is the mass density, p is an undeterminate pressure and $\mathfrak{K}$ is the response functional. The materials which have the constitutive equations (2.6) and (2.7) are called here the *general fluids*. The principle of frame-indifference demands that the identities

$$\left.\begin{aligned}&\mathfrak{K}(\boldsymbol{Q}\boldsymbol{C}_t^t\boldsymbol{Q}^T;\rho)=\boldsymbol{Q}\mathfrak{K}(\boldsymbol{C}_t^t;\rho)\boldsymbol{Q}^T,\\&\mathfrak{K}(\boldsymbol{Q}\boldsymbol{C}_t^t\boldsymbol{Q}^T)=\boldsymbol{Q}\mathfrak{K}(\boldsymbol{C}_t^t)\boldsymbol{Q}^T\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (2.8)$$

hold for all orthogonal tensor $\boldsymbol{Q}$.

The equations depicted above are identical with those given by Noll[1] in three-dimensional space.

A motion has constant stretch history if there is an orthogonal tensor function $\boldsymbol{Q}(t)$ such that the history $\boldsymbol{C}_t^t(s)$ $s \geqq 0$ is related with the history $\boldsymbol{C}_0^0(s)$ by

$$\boldsymbol{C}_t^t(s) = \boldsymbol{Q}(t)^T \boldsymbol{C}_0^0(s) \boldsymbol{Q}(t). \qquad \cdots\cdots\cdots\cdots(2.9)$$

Noll's representation theorem[3] holds in a space of any dimension, which says that the relative deformation gradient is given by

$$\boldsymbol{F}_0(\tau) = \boldsymbol{Q}(\tau)\exp(\tau\boldsymbol{M})\boldsymbol{Q}(0), \qquad \cdots\cdots\cdots\cdots(2.10)$$

if and only if the motion has a constant stretch history, where $\boldsymbol{M}$ is a constant tensor, $\boldsymbol{Q}(\tau)$ is an orthogonal tensor function, and the exponential $\exp(\tau\boldsymbol{M})$ is defined by the convergent power series

$$\exp(\tau\boldsymbol{M}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\tau\boldsymbol{M})^n \qquad \cdots\cdots\cdots\cdots(2.11)$$

Here and henceforth, the dimension of space n is assumed $n \geqq 2$.

The motion of constant stretch history can be characterized completely by the constant tensor $\boldsymbol{M} \neq \boldsymbol{0}$ of (2. 10). Then, the motion can be classified into the following n categories:

i) *viscometric flows* $\boldsymbol{M}^2 = \boldsymbol{0}$, $\cdots\cdots\cdots\cdots$(2.12)

ii) motions in which $\boldsymbol{M}^{k-1} \neq \boldsymbol{0}$, $\boldsymbol{M}^k = \boldsymbol{0}$ $(k = 3, 4, \cdots, n)$, $\cdots\cdots\cdots\cdots$(2.13)

iii) motions in which $\boldsymbol{M}$ is not nilpotent. $\cdots\cdots\cdots\cdots$(2.14)

3. The Representation Theorems in Viscometric Flows

Now, let us consider the presentation of the tensor $\boldsymbol{M}$ of the viscometric flow. From (2. 12), the space $\mathscr{E}$ which is spanned by $\boldsymbol{x}$ of $\boldsymbol{M}^2\boldsymbol{x} = \boldsymbol{0}$ is an n-dimensional space, and the space $\mathscr{F}$ of $\boldsymbol{M}\boldsymbol{x} = \boldsymbol{0}$ is assumed to be an $(n\text{-}r)$-dimensional space. The space $\mathscr{G} = \mathscr{E} - \mathscr{F}$ is, then an r-dimensional space. We may chose the orthonormal basis of $\mathscr{G}$ $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots. \boldsymbol{x}_r$. It is easily verified that $\boldsymbol{M}\boldsymbol{x}_1, \boldsymbol{M}\boldsymbol{x}_2, \cdots, \boldsymbol{M}\boldsymbol{x}_r$ are vectors in $\mathscr{F}$ and independent linearly. Then we can say $r \leqq n - r$. From $\boldsymbol{M}^2 = \boldsymbol{0}$ we have det $\boldsymbol{M} = 0$, so we can say $r \geqq 1$. Then, we have the permissible range of r

$$1 \leqq r \leqq \frac{n}{2}. \qquad \cdots\cdots\cdots\cdots(3.1)$$

3. 1 *Characteristic Matrix in Two-Dimensional Space*

For two-dimensional space, $r=1$ and we may chose $\boldsymbol{x}_1$ in $\mathscr{G}$ and $\boldsymbol{x}_2$ in $\mathscr{F}$, which hold the following:

$$\boldsymbol{M}\boldsymbol{x}_1=\kappa\boldsymbol{x}_2,\quad \boldsymbol{M}\boldsymbol{x}_2=\mathbf{0}\quad \kappa\neq 0. \qquad\cdots\cdots\cdots\cdots(3.2)$$

The matrix $[\boldsymbol{M}]$ of the tensor $\boldsymbol{M}$ relative to the basis $\boldsymbol{x}_1, \boldsymbol{x}_2$ is then given by

$$[\boldsymbol{M}]=\begin{bmatrix} 0 & 0 \\ \kappa & 0 \end{bmatrix}. \qquad\cdots\cdots\cdots\cdots(3.3)$$

3.2 *Characteristic Matrix in Three-Dimensional Space*

For three-dimensional space, $r=1$ and we may choose the orthonormal basis $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$

$$\boldsymbol{M}\boldsymbol{x}_1=\kappa\boldsymbol{x}_2,\quad \boldsymbol{M}\boldsymbol{x}_2=\boldsymbol{M}\boldsymbol{x}_3=\mathbf{0}\quad \kappa\neq 0. \qquad\cdots\cdots\cdots\cdots(3.4)$$

We have the matrix proved by Noll[3]

$$[\boldsymbol{M}]=\begin{bmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad\cdots\cdots\cdots\cdots(3.5)$$

3.3 *Characteristic Matrices in Four-Dimensional Space*

For four-dimensional space, we have $r=1$ or $r=2$. In the case of $r=1$, we have $\boldsymbol{x}_1$ is in $\mathscr{G}$ and $\boldsymbol{x}_2, \boldsymbol{x}_3$ and $\boldsymbol{x}_4$ are in $\mathscr{F}$ such that

$$\boldsymbol{M}\boldsymbol{x}_1=\kappa\boldsymbol{x}_2,\quad \boldsymbol{M}\boldsymbol{x}_2=\boldsymbol{M}\boldsymbol{x}_3=\boldsymbol{M}\boldsymbol{x}_4=\mathbf{0}\quad \kappa\neq 0, \qquad\cdots\cdots\cdots\cdots(3.6)$$

$$[\boldsymbol{M}]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad\cdots\cdots\cdots\cdots(3.7)$$

In the case of $r=2$, we can choose the orthonormal basis, $\boldsymbol{x}_1$ and $\boldsymbol{x}_3$ are in $\mathscr{G}$ and $\boldsymbol{x}_2$ and $\boldsymbol{x}_4$ are in $\mathscr{F}$ such that

$$\boldsymbol{M}\boldsymbol{x}_1=\kappa\boldsymbol{x}_2,\quad \boldsymbol{M}\boldsymbol{x}_3=\kappa'\boldsymbol{x}_2+\kappa''\boldsymbol{x}_4,\quad \boldsymbol{M}\boldsymbol{x}_2=\boldsymbol{M}\boldsymbol{x}_4=\mathbf{0}\quad \kappa\neq 0,\ \kappa''\neq 0 \qquad\cdots\cdots\cdots\cdots(3.8)$$

However, by a rotation in the space spanned by $\boldsymbol{x}_1$ and $\boldsymbol{x}_3$

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_3 \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_3 \end{bmatrix} \qquad\cdots\cdots\cdots\cdots(3.9)$$

with

$$\tan 2\theta=\frac{2\kappa\kappa'}{\kappa^2-\kappa'^2-\kappa''^2}, \qquad\cdots\cdots\cdots\cdots(3.10)$$

the two vectors $\boldsymbol{M}\boldsymbol{y}_1$ and $\boldsymbol{M}\boldsymbol{y}_3$ are reduced so as to become perpendicular with each other. Then, we have the orthonormal basis which hold the following:

$$\boldsymbol{M}\boldsymbol{y}_1=\kappa_1\boldsymbol{y}_2,\quad \boldsymbol{M}\boldsymbol{y}_3=\kappa_2\boldsymbol{y}_4,\quad \boldsymbol{M}\boldsymbol{y}_2=\boldsymbol{M}\boldsymbol{y}_4=\mathbf{0},\quad \kappa_1\neq 0,\quad \kappa_2\neq 0. \qquad \cdots\cdots\cdots\cdots(3.11)$$

The matrix $[\boldsymbol{M}]$ relative to the second basis is represented by

$$[\boldsymbol{M}]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \kappa_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \end{bmatrix}. \qquad \cdots\cdots\cdots\cdots(3.12)$$

The matrix (3.7) can be regarded as a special case of the matrix (3.12) with $\kappa_2=0$.

3.4 *Characteristic Matrices in Five-Dimensional Space*

For five-dimensional space, we have $r=1$ or $r=2$. By the same process adopted in four-dimensional space we can obtain the representations of the matrix $[\boldsymbol{M}]$. In the case of $r=1$, we have

$$\boldsymbol{M}\boldsymbol{x}_1=\kappa\boldsymbol{x}_2,\quad \boldsymbol{M}\boldsymbol{x}_2=\boldsymbol{M}\boldsymbol{x}_3=\boldsymbol{M}\boldsymbol{x}_4=\boldsymbol{M}\boldsymbol{x}_5=\mathbf{0}, \qquad \cdots\cdots\cdots\cdots(3.13)$$

$$[\boldsymbol{M}]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \cdots\cdots\cdots\cdots(3.14)$$

In the case of $r=2$, we can take the orthogonal basis, $\boldsymbol{x}_1$ and $\boldsymbol{x}_3$ in $\mathscr{G}$ and $\boldsymbol{x}_2$, $\boldsymbol{x}_4$ and $\boldsymbol{x}_5$ in $\mathscr{F}$ such that

$$\boldsymbol{M}\boldsymbol{x}_1=\kappa\boldsymbol{x}_2,\quad \boldsymbol{M}\boldsymbol{x}_3=\kappa'\boldsymbol{x}_2+\kappa''\boldsymbol{x}_4,\quad \boldsymbol{M}\boldsymbol{x}_2=\boldsymbol{M}\boldsymbol{x}_4=\boldsymbol{M}\boldsymbol{x}_5=\mathbf{0}$$
$$\kappa\neq 0,\quad \kappa''\neq 0. \qquad \cdots\cdots\cdots\cdots(3.15)$$

By a rotation (3.9) with (3.10) we can take a new set of the orthogonal basis such that

$$\boldsymbol{M}\boldsymbol{y}_1=\kappa_1\boldsymbol{y}_2,\quad \boldsymbol{M}\boldsymbol{y}_3=\kappa_2\boldsymbol{y}_4,\quad \boldsymbol{M}\boldsymbol{y}_2=\boldsymbol{M}\boldsymbol{y}_4=\boldsymbol{M}\boldsymbol{y}_5=\mathbf{0}\quad \kappa_1\neq 0,\quad \kappa_2\neq 0 \qquad \cdots\cdots\cdots(3.16)$$

and we have

$$[\boldsymbol{M}]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \kappa_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \cdots\cdots\cdots\cdots(3.17)$$

By the similar process we can obtain the matrices $[\boldsymbol{M}]$ in a space of dimension more than five.

The magnitude of the viscometric flows may be represented by the magnitudes of scalars κ, κ_1 and κ_2. They are called the *rates of shear.* The viscometric flows in two- and three-dimensional space have *one* rate of shear, while those in four- and five-dimensional space have *two* rates of shear.

4. Viscometric Functions of Incompressible Fluids

The constant stretch history is characterized by the relative deformation gradient (2.10), and the *response functional* $\mathfrak{G}$ of the simple material in the flow is reduced to a *response function* of tensors $\boldsymbol{M}$ and $\boldsymbol{C}$. The constitutive equation (2.7) of an incompressible fiuid is reduced to

$$\boldsymbol{T} = -p\boldsymbol{1} + \boldsymbol{L}(\boldsymbol{M}), \qquad \cdots\cdots\cdots\cdots(4.1)$$

where we can take the normalization condition

$$\boldsymbol{L}(\boldsymbol{0}) = \boldsymbol{0}. \qquad \cdots\cdots\cdots\cdots(4.2)$$

The identity $(2.8)_2$ for the *principle of frame-indifference* is, then, given by

$$\boldsymbol{L}(\boldsymbol{Q}\boldsymbol{M}\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{L}(\boldsymbol{M})\boldsymbol{Q}^T. \qquad \cdots\cdots\cdots\cdots(4.3)$$

4.1 *Viscometric Functions in Two-Dimensional Space*

For two-dimensional space we have the matrix representation (3.3). The stress components with respect to the orthonormal basis representing (3.2) are expressed by two functions $\tau(\kappa)$ and $\sigma(\kappa)$ such that

$$T\langle 12\rangle = \tau(\kappa), \quad T\langle 11\rangle - T\langle 22\rangle = \sigma(\kappa), \qquad \cdots\cdots\cdots\cdots(4.4)$$

where $T\langle ij\rangle$ denotes the physical component of the stress tensor. If we take

$$[\boldsymbol{Q}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots(4.5)$$

the identity (4.3) gives the following relations:

$$\tau(-\kappa) = -\tau(\kappa), \quad \sigma(-\kappa) = \sigma(\kappa). \qquad \cdots\cdots\cdots\cdots(4.6)$$

The normalization condition (4.2) gives

$$\tau(0) = \sigma(0) = 0. \qquad \cdots\cdots\cdots\cdots(4.7)$$

These two functions are called the *viscometric functions* and τ is the *shear stress function* and σ is the *normal stress difference function.*

4.2 *Viscometric Functions in Three-Dimensional Space*

As Noll[3] showed with respect to the orthonormal basis depicted in (3.4) and

(3.5), there are three viscometric functions, one is the shear stress function and two are the normal stress difference functions. They have the following symmetry properties

$$\tau(-\kappa)=-\tau(\kappa),\quad \sigma_1(-\kappa)=\sigma_1(\kappa),\quad \sigma_2(-\kappa)=\sigma_2(\kappa), \qquad \cdots\cdots\cdots\cdots(4.8)$$

$$\tau(0)=\sigma_1(0)=\sigma_2(0)=0, \qquad \cdots\cdots\cdots\cdots(4.9)$$

and the stress components are expressed as

$$T\langle 12\rangle=\tau(\kappa),\quad T\langle 11\rangle-T\langle 33\rangle=\sigma_1(\kappa),\quad T\langle 22\rangle-T\langle 33\rangle=\sigma_2(\kappa). \qquad \cdots\cdots\cdots\cdots(4.10)$$

4.3 *Viscometric Functions in Four-Dimensional Space*

For four-dimensional space we have two matrix representations (3.7) and (3.12). In the case of (3.7), if we choose two orthogonal tensors

$$[Q]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad [Q]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots(4.11)$$

the identity (4.3) yields

$$T\langle 13\rangle=T\langle 14\rangle=T\langle 23\rangle=T\langle 24\rangle=T\langle 34\rangle=0. \qquad \cdots\cdots\cdots\cdots(4.12)$$

If we choose

$$[Q]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots(4.13)$$

we have

$$T\langle 33\rangle=T\langle 44\rangle. \qquad \cdots\cdots\cdots\cdots(4.14)$$

Then there are three viscometric functions

$$T\langle 12\rangle=\tau(\kappa),\quad T\langle 11\rangle-T\langle 44\rangle=\sigma_1(\kappa),\quad T\langle 22\rangle-T\langle 44\rangle=\sigma_2(\kappa), \qquad \cdots\cdots\cdots\cdots(4.15)$$

and we can prove easily that

$$\tau(-\kappa)=-\tau(\kappa),\quad \sigma_1(-\kappa)=\sigma_1(\kappa),\quad \sigma_2(-\kappa)=\sigma_2(\kappa) \qquad \cdots\cdots\cdots\cdots(4.16)$$

if we choose

$$[Q]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \cdots\cdots\cdots\cdots(4.17)$$

The normalization (4. 2) gives

$$\tau(0)=\sigma_1(0)=\sigma_2(0)=0. \qquad \cdots\cdots\cdots\cdots(4.18)$$

In the case of (3. 12), if we choose

$$[Q]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots(4.19)$$

the identity (4. 3) gives

$$T\langle 13\rangle=T\langle 14\rangle=T\langle 23\rangle=T\langle 24\rangle=0. \qquad \cdots\cdots\cdots\cdots(4.20)$$

We have then the following five functions

$$T\langle 12\rangle=\tau(\kappa_1,\kappa_2),\quad T\langle 34\rangle=\tau'(\kappa_1,\kappa_2), \qquad \cdots\cdots\cdots\cdots(4.21\text{ a})$$

$$T\langle 11\rangle-T\langle 22\rangle=\sigma_1(\kappa_1,\kappa_2),\quad T\langle 33\rangle-T\langle 44\rangle=\sigma_1'(\kappa_1,\kappa_2),\ T\langle 22\rangle-T\langle 44\rangle=\sigma_2(\kappa_1,\kappa_2). \qquad \cdots\cdots\cdots\cdots(4.21\text{ b})$$

By the rotation

$$[Q]=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots(4.22)$$

we have

$$[QMQ^T]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \kappa_2 & 0 & 0 & 0 \\ 0 & 0 & \kappa_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad [QMQ^T]=\begin{bmatrix} T\langle 33\rangle & T\langle 34\rangle & 0 & 0 \\ T\langle 43\rangle & T\langle 44\rangle & 0 & 0 \\ 0 & 0 & T\langle 11\rangle & T\langle 12\rangle \\ 0 & 0 & T\langle 21\rangle & T\langle 22\rangle \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots(4.23)$$

which show that the interchange of κ_1 and κ_2 yield the identities

$$\tau'(\kappa_1,\kappa_2)=\tau(\kappa_2,\kappa_1), \qquad \cdots\cdots\cdots\cdots(4.24\text{ a})$$

$$\sigma_1'(\kappa_1,\kappa_2)=\sigma_1(\kappa_2,\kappa_1),\quad \sigma_2(\kappa_1,\kappa_2)=-\sigma_2(\kappa_2,\kappa_1). \qquad \cdots\cdots\cdots\cdots(4.24\text{ b})$$

Then, the five functions presented in (4. 21) are reduced to three independent viscometric functions and we have

$$T\langle 12\rangle=\tau(\kappa_1,\kappa_2),\quad T\langle 34\rangle=\tau(\kappa_2,\kappa_1), \qquad \cdots\cdots\cdots\cdots(4.25\text{ a})$$

$$\left.\begin{aligned} &T\langle 11\rangle-T\langle 44\rangle=\sigma_1(\kappa_1,\kappa_2)+\sigma_2(\kappa_1,\kappa_2),\\ &T\langle 22\rangle-T\langle 44\rangle=\sigma_2(\kappa_1,\kappa_2),\\ &T\langle 33\rangle-T\langle 44\rangle=\sigma_1(\kappa_2,\kappa_1). \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots(4.25\text{ b})$$

If we choose

$$[Q]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [Q]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots (4.26)$$

the identity (4.3) yields the relations

$$\tau(\kappa_1, \kappa_2) = -\tau(-\kappa_1, \kappa_2) = \tau(\kappa_1, -\kappa_2), \qquad \cdots\cdots\cdots\cdots (4.27\text{a})$$

$$\sigma_\alpha(\kappa_1, \kappa_2) = \sigma_\alpha(-\kappa_1, \kappa_2) = \sigma_\alpha(\kappa_1, -\kappa_2) \qquad (\alpha=1, 2). \qquad \cdots\cdots\cdots\cdots (4.27\text{b})$$

The normalization (4.2) and the relations (4.24 b) and (4.27) give the identities

$$\tau(0, \kappa) \equiv \sigma_2(\kappa, \kappa) \equiv 0 \qquad \cdots\cdots\cdots\cdots (4.28)$$

for any value of κ.

When $\kappa_2=0$, the matrix (3.12) is reduced to the matrix (3.7). Then, the comparison of (4.4) and (4.15) with (4.25) gives

$$\tau(\kappa) \equiv \tau(\kappa, 0), \qquad \cdots\cdots\cdots\cdots (4.29\text{a})$$

$$\sigma_1(\kappa) \equiv \sigma_1(\kappa, 0) + \sigma_2(\kappa, 0), \quad \sigma_2(\kappa) \equiv \sigma_2(\kappa, 0), \qquad \cdots\cdots\cdots\cdots (4.29\text{b})$$

$$\sigma_1(0, \kappa) \equiv 0. \qquad \cdots\cdots\cdots\cdots (4.30)$$

Then, the viscometric flow of an incompressible fluid in four-dimensional space can be characterized by three material functions $\tau(\kappa_1, \kappa_2)$, $\sigma_1(\kappa_1, \kappa_2)$ and $\sigma_2(\kappa_1, \kappa_2)$, and they have the symmetry relations (4.27) and the identity relations (4.28).

4.4 *Viscometric Functions in Five-Dimensional Space*

For five dimensional space we have two matrix representations (3.14) and (3.17). In the case of (3.14), by the similar process adopted in the subsection 4.3 we have

$$T\langle 13\rangle = T\langle 14\rangle = T\langle 15\rangle = T\langle 23\rangle = T\langle 24\rangle = T\langle 25\rangle = T\langle 34\rangle = T\langle 35\rangle = T\langle 45\rangle = 0, \qquad \cdots\cdots\cdots\cdots (4.31)$$

and

$$T\langle 33\rangle = T\langle 44\rangle = T\langle 55\rangle. \qquad \cdots\cdots\cdots\cdots (4.32)$$

There are three viscometric functions

$$T\langle 12\rangle = \tau(\kappa), \quad T\langle 11\rangle - T\langle 55\rangle = \sigma_1(\kappa), \quad T\langle 22\rangle - T\langle 55\rangle = \sigma_2(\kappa), \qquad \cdots\cdots\cdots\cdots (4.33)$$

and the relations (4.16) and (4.18) hold.

In the case of (3.17), we have also

$$T\langle 13\rangle = T\langle 14\rangle = T\langle 15\rangle = T\langle 23\rangle = T\langle 24\rangle = T\langle 25\rangle = T\langle 35\rangle = T\langle 45\rangle = 0. \qquad \cdots\cdots (4.34)$$

Then, there are the following six functions

$$T\langle 12\rangle=\tau(\kappa_1,\kappa_2),\quad T\langle 34\rangle=\tau'(\kappa_1,\kappa_2), \qquad \cdots\cdots\cdots\cdots (4.35\text{ a})$$

$$\left.\begin{aligned} &T\langle 11\rangle-T\langle 12\rangle=\sigma_1(\kappa_1,\kappa_2),\quad T\langle 33\rangle-T\langle 44\rangle=\sigma_1'(\kappa_1,\kappa_2),\\ &T\langle 22\rangle-T\langle 44\rangle=\sigma_2'(\kappa_1,\kappa_2),\quad T\langle 22\rangle-T\langle 55\rangle=\sigma_2(\kappa_1,\kappa_2).\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (4.35\text{ b})$$

By the rotation

$$[Q]=\begin{bmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}, \qquad \cdots\cdots\cdots\cdots (4.36)$$

where we have

$$\left.\begin{aligned} &[QMQ^T]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ \kappa_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \kappa_1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},\\ &[QMQ^T]=\begin{bmatrix} T\langle 33\rangle & T\langle 34\rangle & 0 & 0 & 0\\ T\langle 34\rangle & T\langle 44\rangle & 0 & 0 & 0\\ 0 & 0 & T\langle 11\rangle & T\langle 12\rangle & 0\\ 0 & 0 & T\langle 12\rangle & T\langle 22\rangle & 0\\ 0 & 0 & 0 & 0 & T\langle 55\rangle\end{bmatrix},\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (4.37)$$

which show that the interchange of κ_1 and κ_2 yield the identities

$$\tau'(\kappa_1,\kappa_2)=\tau(\kappa_2,\kappa_1), \qquad \cdots\cdots\cdots\cdots (4.38\text{ a})$$

$$\left.\begin{aligned} &\sigma_1'(\kappa_1,\kappa_2)=\sigma_1(\kappa_2,\kappa_1),\quad \sigma_2'(\kappa_1,\kappa_2)=-\sigma_2'(\kappa_2,\kappa_1),\\ &\sigma_2(\kappa_1,\kappa_2)=-\sigma_2'(\kappa_2,\kappa_1)+\sigma_2(\kappa_2,\kappa_1).\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (4.38\text{ b})$$

Then, the six functions presented in (4.35) are reduced to three independent viscometric functions and we have

$$T\langle 12\rangle=\tau(\kappa_1,\kappa_2),\quad T\langle 34\rangle=\tau(\kappa_2,\kappa_1), \qquad \cdots\cdots\cdots\cdots (4.39\text{ a})$$

$$\left.\begin{aligned} &T\langle 11\rangle-T\langle 55\rangle=\sigma_1(\kappa_1,\kappa_2)+\sigma_2(\kappa_1,\kappa_2),\\ &T\langle 22\rangle-T\langle 55\rangle=\sigma_2(\kappa_1,\kappa_2),\\ &T\langle 33\rangle-T\langle 55\rangle=\sigma_1(\kappa_2,\kappa_1)+\sigma_2(\kappa_2,\kappa_1),\\ &T\langle 44\rangle-T\langle 55\rangle=\sigma_2(\kappa_2,\kappa_1).\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (4.39\text{ b})$$

If we choose

$$[Q]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad [Q]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \cdots(4.40)$$

the identity (4.3) yields the relations

$$\tau(\kappa_1, \kappa_2) = -\tau(-\kappa_1, \kappa_2) = \tau(\kappa_1, -\kappa_2), \qquad \cdots\cdots\cdots\cdots(4.41\,\text{a})$$

$$\sigma_\alpha(\kappa_1, \kappa_2) = \sigma_\alpha(-\kappa_1, \kappa_2) = \sigma_\alpha(\kappa_1, -\kappa_2) \qquad (\alpha=1, 2). \qquad \cdots\cdots\cdots\cdots(4.41\,\text{b})$$

The normalization (4.2) and the relations (4.38b) give the identities

$$\tau(0, \kappa) \equiv 0 \qquad \cdots\cdots\cdots\cdots(4.42)$$

for any value of κ.

When $\kappa_2=0$, the matrix (3.17) is reduced to the matrix (3.14). Then the comparison of (4.32) and (4.33) with (4.39) gives

$$\tau(\kappa) \equiv \tau(\kappa, 0), \qquad \cdots\cdots\cdots\cdots(4.43\,\text{a})$$

$$\sigma_1(\kappa) \equiv \sigma_1(\kappa, 0) + \sigma_2(\kappa, 0), \qquad \sigma_2(\kappa) \equiv \sigma_2(\kappa, 0), \qquad \cdots\cdots\cdots\cdots(4.43\,\text{b})$$

$$\sigma_1(0, \kappa) \equiv \sigma_2(0, \kappa) \equiv 0. \qquad \cdots\cdots\cdots\cdots(4.44)$$

The above relations (4.43) are identical with (4.29).

Summarily we can say that the materials have one shear stress function in spaces of two-, three-, four- and five-dimension; and they have one normal stress difference function in a space of two-dimension and two in spaces of three-, four- and five-dimension.

5. Approximation Forms of Viscometric Functions in Small Magnitude of Rates of Shear

For two- and three-dimensional spaces, the shear stress function is an odd function and the normal stress difference functions are even functions of a rate of shear. If we assume that they have four continuous derivatives at zero rate of shear, we have from (4.6), (4.7), (4.8) and (4.9)

$$\tau(\kappa) = \mu_0\kappa + \mu_1\kappa^3 + O(\kappa^5), \qquad \cdots\cdots\cdots\cdots(5.1)$$

$$\sigma(\kappa) = \eta_0\kappa^2 + \eta_1\kappa^4 + O(\kappa^6), \qquad \cdots\cdots\cdots\cdots(5.2)$$

$$\sigma_\alpha(\kappa) = \eta_{\alpha 0}\kappa^2 + \eta_{\alpha 1}\kappa^4 + O(\kappa^6) \quad (\alpha=1, 2), \qquad \cdots\cdots\cdots\cdots(5.3)$$

where $\mu_0, \mu_1, \eta_0, \eta_1, \eta_{\alpha 0}, \eta_{\alpha 1}$ $(\alpha=1, 2)$ are material constants.

The viscometric functions in four- and five-dimensional spaces have two arguments, and they are subjected to the symmetry relations (4.27) and (4.41) and the identities (4.28), (4.30), (4.42) and (4.44). If we assume that they have four continuous derivatives at zero rates of shear, we can express them for five-dimensional space such as

$$\tau(\kappa_1, \kappa_2) = \mu_0\kappa_1 + \mu_1\kappa_1\kappa_2^2 + \mu_2\kappa_1^3 + O(\kappa^5), \qquad \cdots\cdots\cdots\cdots (5.4)$$

$$\sigma_\alpha(\kappa_1, \kappa_2) = \eta_{\alpha 0}\kappa_1^2 + \eta_{\alpha 1}\kappa_1^2\kappa_2^2 + \eta_{\alpha 2}\kappa_1^4 + O(\kappa^6) \quad (\alpha=1, 2), \qquad \cdots\cdots\cdots\cdots (5.5)$$

and the four-dimensional space above $\sigma_2(\kappa_1, \kappa_2)$ must be replaced by

$$\sigma_2(\kappa_1, \kappa_2) = \zeta_0(\kappa_1^2 - \kappa_2^2) + \zeta_1(\kappa_1^2 - \kappa_2^2)^2 + O(\kappa^6), \qquad \cdots\cdots\cdots\cdots (5.6)$$

where μ_0, μ_1, μ_2, $\eta_{\alpha 0}$, $\eta_{\alpha 1}$, $\eta_{\alpha 2}$ $(\alpha=1, 2)$, ζ_0, ζ_1 are material constants and κ denotes the magnitude $\sqrt{\kappa_1^2+\kappa_2^2}$.

The approximation forms mentioned above show that the departures from the classical behavior of proportionality of shear stress to rate of shear is an effect of the third order in κ, and the normal stress effects are of the second order in κ.

6. Generalized Steady Curvilinear Flows

In the third paper[6] of this series, we investigated the *steady curvilinear flow* of the *Stokes fluid* in n-dimensional space. Here we consider a generalized steady curvilinear flow of the *general fluid* in n-dimensional space.

Let us consider an orthogonal curvilinear coordinate system in an n-dimensional space. The line element is assumed to be

$$ds^2 = \sum_{k=1}^{n} (g_k)^2 (dx^k)^2, \qquad \cdots\cdots\cdots\cdots (6.1)$$

where the magnitudes of the base vectors g_k are functions of x^α, i. e.,

$$g_k = g_k(x^\alpha) \qquad \begin{pmatrix} k=1, 2, \cdots, n \\ \alpha=1, 2, \cdots, r \end{pmatrix}. \qquad \cdots\cdots\cdots\cdots (6.2)$$

Now the material point has the velocity components

$$\dot{x}^\alpha = 0, \quad \dot{x}^\beta = u^\beta(x^\alpha) \qquad \begin{pmatrix} \alpha=1, 2, \cdots, r \\ \beta=r+1, r+2, \cdots, n \end{pmatrix}. \qquad \cdots\cdots\cdots\cdots (6.3)$$

Hence, k, α and β run the above mentioned respective ranges. This motion is called the *generalized steady curvilinear flow*. At time τ $(\tau < t)$ the material point has the coordinates ξ^k and its velocity is given by

$$\dot{\xi}^\alpha = 0, \quad \dot{\xi}^\beta = u^\beta(\xi^\alpha). \qquad \cdots\cdots\cdots\cdots (6.4)$$

Integrating the above with the condition $\xi^k = x^k$, we have

$$\xi^\alpha = x^\alpha, \quad \xi^\beta = x^\beta + (\tau - t) u^\beta(x^\alpha). \qquad \cdots\cdots\cdots\cdots (6.5)$$

The history of the relative deformation gradient $\boldsymbol{F}^t_t(s)=\boldsymbol{F}_t(t-s)=\boldsymbol{F}_t(\tau)=\partial\boldsymbol{\xi}/\partial\boldsymbol{x}$ is given by

$$[F^t_t(s)]=\begin{bmatrix} [1] & [0] \\ -s\left[\dfrac{\partial u^\beta}{\partial x^\alpha}\right] & [1] \end{bmatrix}, \qquad \cdots\cdots\cdots\cdots(6.6)$$

where we use the natural basis and their duals. The *physical components* of $\boldsymbol{F}^t_t(s)$ are given by

$$\boldsymbol{F}^t_t(s)<\beta\alpha>=\frac{g_\beta}{g_\alpha}\frac{\partial u^\beta}{\partial x^\alpha}\equiv\kappa_{\beta\alpha}, \qquad \cdots\cdots\cdots\cdots(6.7)$$

and its matrix is written by

$$[\boldsymbol{F}^t_t(s)<kl>]=[1]-s[\boldsymbol{M}], \qquad \cdots\cdots\cdots\cdots(6.8)$$

$$[\boldsymbol{M}]\equiv\begin{bmatrix} [0] & [0] \\ [\kappa_{\beta\alpha}] & [0] \end{bmatrix} \qquad \cdots\cdots\cdots\cdots(6.9)$$

where $[\kappa_{\beta\alpha}]$ is a $(n-r)\times r$ matrix.

It is easily verified that by the routine multiplication rule of matrices we have

$$[\boldsymbol{M}^2]=[0]. \qquad \cdots\cdots\cdots\cdots(6.10)$$

Therefore, the deformation gradient (6.7) can be written as

$$[\boldsymbol{F}^t_t(s)<kl>]=\exp(-s[\boldsymbol{M}]), \qquad \cdots\cdots\cdots\cdots(6.11)$$

where we use (6.9) and the formula (2.11). Then, by (2.2) we have

$$[\boldsymbol{C}^t_t(s)<kl>]=\exp(-s[\boldsymbol{M}]^T)\exp(-s[\boldsymbol{M}])=[\boldsymbol{C}^0_0(s)<kl>]. \qquad \cdots\cdots(6.12)$$

From the definition (2.9) and (2.12), we can say that the generalized steady curvilinear flow is a motion with constant stretch history and also a viscometric flow.

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