Uniform Plane Graphs

By

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Abstract

This paper presents two special classes of plane graphs characterized by the sequences $M(v)$ and $W(e)$. $M(v)$ is the circular sequence consisting of the numbers of vertices on the meshes around vertex v; and $W(e)$ is the sequence consisting of the numbers of vertices on the meshes to the right and the left of edge e and also of the degrees of the head and the tail of e. A graph is called uniform with respect to $M(v)$ or $W(e)$ if its vertices all have the same $M(v)$, or if its edges all have the same $W(e)$, respectively. It is shown that if such a uniform plane graph exists for the given $M(v)$ or $W(e)$, the numbers of its vertices, edges and meshes are uniquely determined. Then, the conditions on $M(v)$ or $W(e)$ for the existence of a graph are investigated. Tables of plane graphs which are uniform with respect to $M(v)$ or $W(e)$ are presented. Besides regular polyhedrons, there are thirteen types of graphs which are uniform with respect to $M(v)$, and only four graphs which are uniform with respect to $W(e)$.

1. Introduction

A graph is often characterized by numbers associated with its vertices and edges. For instance, the degree of a vertex is the number of edges incident with the vertex, and a graph is called regular (vertex-regular) if all its vertices have the same degree.[1] The sequence of numbers consisting of the degrees of the vertices is called the degree sequence, and the synthesis of a graph having the given degree sequence is often considered by graph theorists. Now a graph is planar if it can be embedded in the plane, and if it has already been embedded in the plane, it is called a plane graph.[1] For a plane graph, numbers associated with its meshes can also be used to characterize the graph.

In this paper, we consider special classes of plane graphs which are characterized by constant sequences of numbers associated with vertices or edges. Let G be a plane graph. Although G is not directed, we treat each edge of G as two directed

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parallel edges of opposite direction.[2)] If one of the directed edges is e, the other is denoted by e^r. The sequences of numbers $M(v)$ and $W(e)$ defined below are associated with a vertex v and a directed edge e in G, respectively.

Definition: $M(v)$ is the circular sequence consisting of the numbers of vertices on the meshes around v. The order of the numbers in $M(v)$ corresponds to the order of the meshes drawn on the plane.

Definition : $W(e)$ is the sequence of four numbers which sequence consists of the numbers of vertices on the meshes to the right and the left of e, and of the degrees of the head and the tail of e. The numbers in $W(e)$ are ordered as given above.

Since $M(v)$ is a circular sequence, its expression by a linear sequence is not unigue. For example, the expression of $M(v_1)$ for vertex v_1 in graph G_1, shown in Fig. 1, may be $\{3, 3, 5, 4\}$, $\{3, 5, 4, 3\}$, $\{5, 4, 3, 3\}$, or $\{4, 3, 3, 5\}$. These sequences are regarded all equal and any one of them can represent $M(v_1)$. One way to recognize whether two circular sequences represent the same $M(v)$ or not, is to move the numbers in the sequences cyclically so that they are in a certain lexicographic order. It may also happen that a number in $M(v)$ has no one-to-one correspondence with a mesh around v. For example, $M(v_2)$ for vertex v_2 in G_1 is $\{4, 3, 4, 3\}$. The first number 4 in this sequence can be the number of vertices on either of the two meshes, m_1 and m_2.

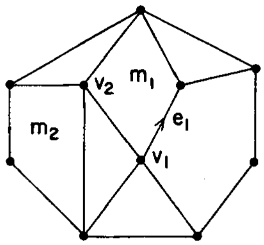


Fig. 1. Graph G_1

As an example of $W(e)$, $W(e_1)=\{5, 4, 3, 4\}$ for edge e_1 in G_1. If $W(e)$ is given, $W(e^r)$ can be determined. For example, $W(e_1{}^r)=\{4, 5, 4, 3\}$.

Graph G is called uniform with respect to $M(v)$, if its vertices all have the same $M(v)$. We treat each edge as two directed edges e and e^r. Graph G is called uniform with respect to $W(e)$, if its edges all have the same $W(e)$ and $W(e^r)$. $W(e)$ and $W(e^r)$ may not be the same.

2. Uniform Graphs with respect to M(v)

We are first interested in uniform plane graphs with respect to $M(v)$. They are regular graphs. We give the definitions of the symbols as follows.

G: uniform plane graph with respect to $M(v)$

V: number of vertices in G

E: number of edges in G

F: number of meshes in G

F_i: number of meshes in G which have L_i vertices

R_i: number of meshes around a vertex which have L_i vertices

D: degree of a vertex

Then

$$V-E+F=2 \qquad \cdots\cdots\cdots\cdots(1)$$

$$F=\sum_i F_i \qquad \cdots\cdots\cdots\cdots(2)$$

$$E=DV/2 \qquad \cdots\cdots\cdots\cdots(3)$$

$$D=\sum_i R_i \qquad \cdots\cdots\cdots\cdots(4)$$

$$F_i=R_iV/L_i. \qquad \cdots\cdots\cdots\cdots(5)$$

Eq. (1) is the famous Euler's theorem for polyhedrons. The derivation of eq. (5) is as follows. There are R_i meshes around each vertex which have L_i vertices, and there are V vertices in G. Since each mesh contributes L_i times in R_iV, the total number of meshes which have L_i vertices is R_iV/L_i. The rest of the equations are easily obtained from the definitions.

In order to avoid any unnecessary complication we assume $D\geq 3$ and $L_i\geq 3$, which means that there are no series edges or parallel edges in G. Eliminating E and F from eqs. (1)—(5), we get

$$V=\frac{2}{1-D/2+\sum_i(R_i/L_i)}\,. \qquad \cdots\cdots\cdots\cdots(6)$$

If $M(v)$ is given, D, R_i, and L_i can be determined. Thus eq. (6) shows that V can be determined uniquely for the given $M(v)$. Then E and F can also be determined from eqs. (3) and (1). Since V must be a positive interger, there are certain restrictions on $M(v)$. First

$$1>D/2-\sum_i(R_i/L_i)\geq D/2-\sum_i R_i/3=D/6. \qquad \cdots\cdots\cdots\cdots(7)$$

Therefore D can be 3, 4 or 5 only. Especially if there is no triangular mesh in G, then $L_i\geq 4$ and we obtain $1>D/4$ instead of inequality (7). Thus D must be 3 only. Further conditions on $M(v)$ and the graphs for $M(v)$ satisfying the conditions are investigated below.

Case a. All the meshes around vertex v have the same number of vertices: In this case G represents a regular polyhedron, and we get Table 1.

Case b. At least one of the meshes around vertex v has a number of vertices different from those of the other meshes. In the following, A, B and C represent all different numbers. We have:

Proposition 1. Suppose vertices v_1 and v_2 are adjacent, as shown in Fig. 2. $M(v_1)$ contains subsequence $\{A, B\}$, and $M(v_2)$ contains subsequence $\{B, A\}$. G is uniform and $M(v_1)=M(v_2)$ if the numbers in the sequences are properly moved. Therefore $M(v)$ must contain either $\{A, B, A\}$, $\{B, A, B\}$ or $\{A, B, \cdots, B, A\}$.

Case b-1. $D=3$. From Proposition 1 at least two numbers in $M(v)$ must be the

Table. 1. regular polyhedrons

$M(v)$	$W(e)$	D	V	E	F	G
$\{3,3,3\}$	$\{3,3,3,3\}$	3	4	6	4	*tetrahedron*
$\{4,4,4\}$	$\{4,4,3,3\}$	3	8	12	6	*hexahedron*
$\{5,5,5\}$	$\{5,5,3,3\}$	3	20	30	12	*dodecahedron*
$\{3,3,3,3\}$	$\{3,3,4,4\}$	4	6	12	8	*octahedron*
$\{3,3,3,3,3\}$	$\{3,3,5,5\}$	5	12	30	20	*icosahedron*

same. Let $M(v)=\{A, A, B\}(=\{A, B, A\})$. Then from eq. (6) and $V>0$,

$$V=\frac{2}{2/A+1/B-1/2} \text{ and } B<\frac{2A}{A-4}. \qquad \cdots\cdots\cdots\cdots(8)$$

Since $B\geq 3$, $3\leq A\leq 11$.

(1) $A=3$; $M(v)=\{3, 3, B\}$: Suppose vertex v_1 has three adjacent vertices v_2, v_3 and v_4 as shown in Fig. 3. We see that the degree of v_2, v_3 or v_4 is 3, and there can be no mesh other than those shown in the figure. Therefore B cannot be more than 3.

(2) $A=4$; $M(v)=\{4, 4, B\}$: We obtain graphs as shown in Fig. 4. For each $B\geq 3$, G is uniquely determined.

(3) $A=6$; $M(v)=\{6, 6, B\}$: For each of $B=3$, 4 and 5, G is uniquely determined. See Fig. 5, Fig. 6 and Fig. 7.

(4) $A=8$; $M(v)=\{8, 8, B\}$: G is uniquely determined for $B=3$. See Fig. 8.

(5) We can check that no graph exists for $A=5$, 7 and 11.

Case b-2. $D=4$. From Proposition 1 the types of $M(v)$ are restricted to $\{A, A, A, B\}$, $\{A, A, B, B\}$, $\{A, B, A, B\}$ and $\{A, B, A, C\}$.

Case b-2-i. $M(v)=\{A, A, A, B\}$: From eq. (6) and $V>0$,

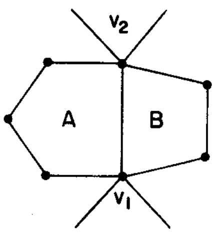


Fig. 2. Graph for Proposition 1.

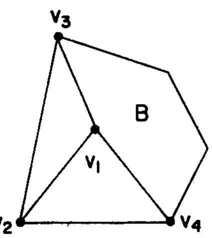


Fig. 3. Graph for $M(v)=\{3,3,B\}$.

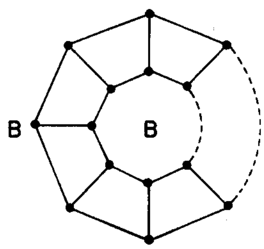


Fig. 4. Graph for $M(v)=\{4,4,B\}$

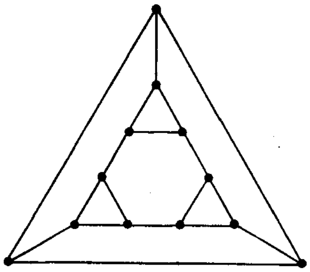


Fig. 5. Graph for $M(v)=\{6,6,3\}$.

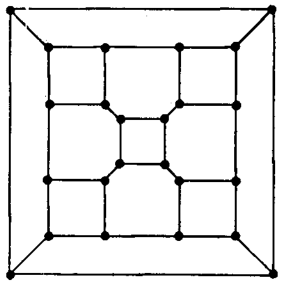


Fig. 6. Graph for $M(v)=\{6,6,4\}$.

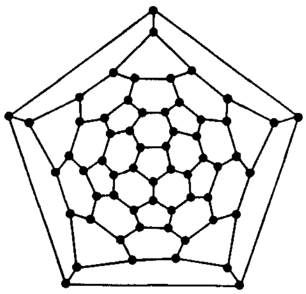


Fig. 7. Graph for $M(v)=\{6,6,5\}$.

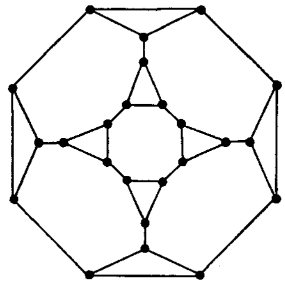


Fig. 8. Graph for $M(v)=\{8,8,3\}$.

$$V=\frac{2}{3/A+1/B-1} \text{ and } B<\frac{A}{A-3}\,. \qquad \cdots\cdots\cdots\cdots(9)$$

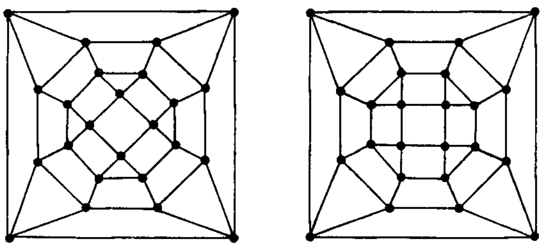
Since $B\geq 3$, $A=3$ or 4.

(1) $A=3$; $M(v)=\{3,\ 3,\ 3,\ B\}$: G can be uniquely determined for each $B\geq 4$.

(2) $A=4$; $M(v)=\{4,\ 4,\ 4,\ B\}$: B can be 3 only. There are two non-isomorphic plane graphs for $M(v)=\{4,\ 4,\ 4,\ 3\}$ as shown in Fig. 9.

Case b-2-ii. $M(v)=\{A,\ A,\ B,\ B\}$ and $\{A,\ B,\ A,\ B\}$: From eq. (6) and $V>0$,

$$V=\frac{2}{2/A+2/B-1} \text{ and } B<\frac{2A}{A-2}\,. \qquad \cdots\cdots\cdots\cdots(10)$$



(a) (b)

Fig. 9. Graph for $M(v)=\{4,4,4,3\}$.

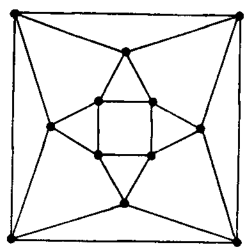


Fig. 10. Graph for $M(v)=\{4,3,4,3\}$, $W(e)=\{3,4,4,4\}$.

Since $B \geq 3$, $A<6$.

(1) $A=4$, $B=3$; $M(v)=\{4, 3, 4, 3\}$: G is uniquely determined. See Fig. 10.

(2) $A=5$, $B=3$; $M(v)=\{5, 3, 5, 3\}$: G is uniquely determined.

(3) No graph other than those given above can exist in this case.

Case b-2-iii. $M(v)=\{A, B, A, C\}$: From eq. (6) and $V>0$,

$$V=\frac{2}{2/A+1/B+1/C-1} \text{ and } 2/A+1/B+1/C>1. \qquad \cdots\cdots\cdots\cdots(11)$$

G can exist for $M(v)=\{4, 5, 4, 3\}$ only. For this $M(v)$, G is unique.

Case b-3. $D=5$: From Proposition 1, the possible types of $M(v)$ are $\{A, A, A, A, B\}$, $\{A, A, A, B, B\}$, $\{A, A, B, A, B\}$, $\{A, A, B, A, C\}$ and $\{A, B, B, A, C\}$.

Case b-3-i. $M(v)=\{A, A, A, A, B\}$: From eq. (6) and $V>0$,

$$V=\frac{2}{4/A+1/B-3/2} \text{ and } B<\frac{2A}{3A-8}. \qquad \cdots\cdots\cdots\cdots(12)$$

Since $B \geq 3$, $A=3$ only. B can be 4 or 5. It can be checked that a graph can really exist for each of $M(v)=\{3, 3, 3, 3, 4\}$ and $\{3, 3, 3, 3, 5\}$ and is uniquely determined.

Case b-3-ii. From eq. (6) we see that V can not be positive for the rest of the types of $M(v)$ under the condition that A, B, $C \geq 3$ and $A \neq B \neq C \neq A$.

The results obtained for Case b are summarized in Table 2.

Table 2. Uniform plane graphs with respect to $M(v)$.

$M(v)$	D	V	E	F	G
$\{4,4,B\}$ $B=3,5,6,\cdots$	3	$2B$	$3B$	$B+2$	Fig. 4
$\{6,6,3\}$	3	12	18	8	Fig. 5
$\{6,6,4\}$	3	24	36	14	Fig. 6
$\{6,6,5\}$	3	60	90	32	Fig. 7
$\{8,8,3\}$	3	24	36	14	Fig. 8
$\{10,10,3\}$	3	60	90	32	
$\{3,3,3,B\}$ $B=4,5,\cdots$	4	$2B$	$4B$	$2B+2$	
$\{4,4,4,3\}$	4	24	48	26	Fig. 9
$\{4,3,4,3\}$	4	12	24	14	Fig. 10
$\{5,3,5,3\}$	4	30	60	32	
$\{5,4,3,4\}$	4	60	120	62	
$\{3,3,3,3,4\}$	5	24	60	48	
$\{3,3,3,3,5\}$	5	60	150	92	

3. Uniform Graphs with respect to W(e)

In this chapter, we consider uniform plane graphs with respect to $W(e)$. Let G be such a graph. Eqs. (1) and (2) in the previous chapter also hold for G. In the following, A or B is the number of vertices in a mesh and $A \neq B$, and I or J is the degree of a vertex and $I \neq J$. D is the degree of a vertex when G is regular.

Case a. $W(e)=W(e^r)=\{A, A, D, D\}$: We obtain regular polyhedrons in this case. See Case a and Table 1 in the previous chapter.

Case b. $W(e)=\{A, B, D, D\}$ and $W(e^r)=\{B, A, D, D\}$: G is a regular graph, and eqs. (3)—(5) in the previous chapter also hold. We can assume $A<B$ without loss of generality. The meshes with A vertices and those with B vertices appear alternately around a vertex, and therefore D must be even. Let $D=2K$. Since $D \geq 3$, $K \geq 2$. The numbers of meshes with A vertices and B vertices are both equal to K. From eqs. (3) and (5) we have

$$E=KV, \quad F_A=KV/A \text{ and } F_B=KV/B, \qquad \cdots\cdots\cdots\cdots (13)$$

where F_A and F_B are numbers of meshes with A vertices and B vertices in G respectively. Then from eqs. (1), (2) and (13),

$$V=\frac{2}{1-K+K/A+K/B}. \qquad \cdots\cdots\cdots\cdots (14)$$

From eq. (14) we see that A must be 3 in order to get a positive V, and that B is either 4 or 5.

(1) $A=3$, $B=4$; $W(e)=\{3, 4, 4, 4\}$: G is uniquely determined and is shown in Fig. 10. This graph has a uniform $M(v)$ of $\{4, 3, 4, 3\}$.

(2) $A=3$, $B=5$; $W(e)=\{3, 5, 4, 4\}$: G is unique and has a uniform $M(v)$ of $\{5, 3, 5, 3\}$.

Case c. $W(e)=\{A, A, I, J\}$ and $W(e^r)=\{A, A, J, I\}$: Let G^d be the dual graph of G. Then of course G is the dual of G^d. Let e^d be the edge in G^d corresponding to the edge e in G. If $W(e)=\{A, A, I, J\}$, then $W(e^d)$ in G^d is $\{I, J, A, A\}$. From this fact we see that the graphs which satisy the condition of Case c are the duals of those obtained in Case b. Thus we get graphs for $W(e)=\{4, 4, 3, 4,\}$ and $W(e)=\{4, 4, 3, 5\}$, which are the duals of those for $W(e)=\{3, 4, 4, 4\}$ and $W(e)=\{3, 5, 4, 4\}$ respectively. The graph for $W(e)=\{4, 4, 3, 4\}$ is shown in Fig. 11.

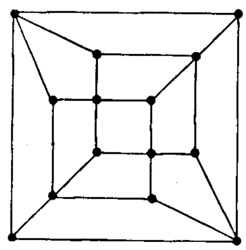


Fig. 11. Graph for $W(e)=\{4, 4, 3, 4\}$.

Case d. $W(e)=\{A, B, I, J\}$ and $W(e^r)=\{B, A, J, I\}$: Since there can be no edge which is incident with two vertices of the same degree, the graph satisfying the condition of this case, if any, must be a bipartite graph. The vertices of G are partitioned into two sets, namely V_I and V_J, consisting of the vertices with degree I and J respectively. Each vertex in V_I is adjacent to I vertices in V_J, and each vertex in V_I to J vertices in V_J. The numbers of vertices in V_I and V_J must be $JV/(I+J)$ and $IV/(I+J)$ respectively. The total number of edges of G is, then, $E=IJV/(I+J)$. The numbers of meshes with A vertices around a vertex in V_I and V_J are $I/2$ and $J/2$ respectively, and then the total number of meshes with A vertices is $IJV/2(I+J)A+JIV/2(I+J)A=E/A$. Similarly, the total number of meshes with B vertices is E/B. Therefore, $F=E/A+E/B$ and it follows from eq. (1) that

$$V=\frac{2}{1+\frac{IJ}{I+J}\left(\frac{1}{A}+\frac{1}{B}-1\right)}\,. \qquad \cdots\cdots\cdots\cdots(15)$$

Since $A\geq 3$, $B\geq 3$ and $A\neq B$, it is required

$$1+\frac{IJ}{I+J}\left(\frac{1}{A}+\frac{1}{B}-1\right)\geq 1-\frac{IJ}{I+J}\cdot\frac{5}{12}>0 \qquad \cdots\cdots\cdots\cdots(16)$$

in order for $V>0$. On the other hand, since meshes with A vertices and those with B vertices appear alternately around a vertex, I and J must be even numbers, and $I\geq 4$ and $J\geq 6$, or $I\geq 6$ and $J\geq 4$. Thus $IJ/(I+J)\geq 12/5$. We see that inequality (16) cannot be satisfied and there is no graph for Case d.

The results obtained for Case b and Case c are summarized in Table 3.

Table 3. Uniform plane graphs with respect to $W(e)$.

$W(e)$	V	E	F	G
{3, 4, 4, 4}	12	24	14	Fig. 10
{3, 5, 4, 4}	30	60	32	
{4, 4, 3, 4}	14	24	12	Fig. 11
{4, 4, 3, 5}	32	60	30	

4. Concluding Remarks

The graphs uniform with respect to $M(v)$ are vertex-regular of special types. Some of them are vertex-symmetric[1] but the others are not. A graph can be uniquely determined for a given sequence $M(v)$, if one exists, except for $M(v)=\{4, 4, 4, 3\}$. Two non-isomorphic graphs can be obtained for $M(v)=\{4, 4, 4, 3\}$. It is interesting that one of them is vertex-symmetric, but the other is not. Besides regular polyhedrons, there are only four plane graphs, which are uniform with respect to $W(e)$. They are edge-regular,[1] and are uniquely determined for $W(e)$.

$W(e)$ was used to partition the set of edges in the algorithm to test the isomorphism or automorphism of planar graphs.[2] $M(v)$ can also be used for the test. The above result on $M(v)$, however, shows that only $M(v)$ is not enough for partitioning the vertices to test isomorphism, since the vertices in the non-isomorphic graphs of Fig. 9 have the same $M(v)$ and can not be partitioned by use of $M(v)$ only.

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