Electrical Dissipation Effects on Propagation of Magneto-hydrodynamic Waves in Anisotropic Media

By

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Abstract

This paper deals with the investigation of the modification of magneto-hydrodynamic wave propagation by electrical dissipation and the Hall effect. A dispersion relation is obtained from magneto-hydrodynamic equations, Maxwell's equations and the generalized Ohm's law which includes the Hall effect. The effect of electrical dissipation is well represented by a magnetic Reynolds number defined for the Alfvén wave; and the inclusion of the Hall effect is equivalent to taking account of the ion cyclotron motion.

It is found that there exist three waves, in general, as in the case without electrical dissipation or the Hall effect. It is also found that, in a perfectly conducting medium, the slowest wave evanesces in a range of propagation direction determined from the frequency, due to ion cyclotron resonance. However, when the electrical conductivity of the medium is finite, three waves are observable in any direction, and with the decrease of electrical conductivity, one of the waves is reduced to an ordinary acoustic wave, whie the other two waves are reduced to electromagnetic waves.

1. Introduction

As is well known, there are three modes of magneto-hydrodynamic waves in perfectly conducting and compressible media[1]. In a medium of finite electrical conductivity, these waves are modified in their phase velocities and become weakened due to electrical dissipation In fact, it is expected that in an electrically poor conductive medium, these three waves will be reduced to an ordinary sound wave and electromagnetic waves.

The modification of magneto-hydrodynamic wave propagation in a medium due to electrical dissipation has been investigated by many authors[1],[2]. However, most of the authors assumed incompressibility of the medium and smallness of the inverse magnetic Reynolds number. However, complete treatment of the problem which takes

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into account compressibility and finiteness of electrical conductivity of the medium is not found in the literature. The purpose of this work is to investigate the transition of magneto-hydrodynamic waves to acoustic and electromagnetic waves due to the variation of electrical conductivity. Thus, for the sake of clarity the viscosity of the medium will be ignored in this work. Furthermore, the Hall effect will be taken into account in the present analysis, which will permit the correct result up to the ion cyclotron frequency to be obtained.

2. Dispersion Relation

In this chapter, the dispersion relation will be found, which will be used to investigate the modification of magneto-hydrodynamic waves due to the variation of electrical conductivity For simplicity, it will be assumed that the wave amplitude is sufficiently small, and that the wave frequency ω is much smaller than the characteristic relaxation frequency of the medium. Thus, the electrical conductivity σ and the sound speed c_s can be regarded as independent of the wave frequency.[3),4)] It should be noted that the result to be obtained is dependent on the form of Ohm's law that is employed. If we adopt the generalized Ohm's law including the Hall effect[5)], we have*

$$\boldsymbol{j}=\sigma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})-\frac{\beta}{B}\boldsymbol{j}\times\boldsymbol{B}. \qquad \cdots\cdots\cdots\cdots(1)$$

The result will be valid up to the frequency comparable to the ion cyclotron frequency. In this equation (1), $\boldsymbol{j}$ is the electric current density, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, $\boldsymbol{v}$ is the fluid velocity, and β is the Hall parameter, i.e. the ratio of the electron cyclotron frequency to the electron collisional frequency. The dispersion relation for magneto-hydrodynamic waves may be obtained from magneto-hydrodynamic equations, Maxwell's equations and eq. (1) as follows:

$$X^3-(1+\xi+\cos^2\theta-2i\bar{\omega})X^2+[(\cos^2\theta-i\bar{\omega})(1+2\xi-i\bar{\omega})-\bar{\beta}^2\cos^2\theta]X$$
$$-\xi[(\cos^2\theta-i\bar{\omega})^2-\bar{\beta}^2\cos^2\theta)]=0 \qquad \cdots\cdots\cdots\cdots(2)$$

Here, θ is the angle between the magnetic field $\boldsymbol{B}$ and the wave vector $\boldsymbol{k}$ and

$$X=\omega^2/k^2a^2$$
$$\xi=c_s^2/a^2$$
$$\bar{\omega}=\omega/\sigma\mu_0a^2 \qquad \cdots\cdots\cdots\cdots(3)$$
$$\bar{\beta}=\beta\bar{\omega} \qquad \cdots\cdots\cdots\cdots(4)$$

where a is the Alfvén velocity and μ_0 is the magnetic permeability.

Solutions of eq. (2) for X determine the propagation properties, i. e. phase

* The pressure term has been omitted, since it does not contribute to the result.

velocity V and attenuation coefficient γ, from the following relations:

$$\frac{V}{a}=\frac{|X|}{\mathrm{Re}\sqrt{X}},\quad \frac{\gamma}{a\sigma\mu_0}=-\frac{\bar{\omega}\ \mathrm{Im}\sqrt{X}}{|X|} \qquad \cdots\cdots\cdots\cdots(5)$$

It should be noted that the electrical dissipative effect is represented by the parameter $\bar{\omega}$, defined by eq. (3). This parameter is equal to the inverse magnetic Reynolds number, in which the Alfvén velocity a and the wave length a/ω (the factor 2π has been discarded) are taken respectively as the characteristic speed and length. Another parameter $\bar{\beta}$ appearing in eq. (2) and defined by eq, (4), is independent of the electrical conductivity and, for completely ionized plasmas, is equal to the ratio of the wave frequency to the ion cyclotron frequency. Therefore, in the limit of low frequency ($\bar{\beta}=0$) and infinitely large conductivity ($\bar{\omega}=0$), eq. (2) recovers the well-known three kinds of magneto-hydrodynamic waves:

$$X(\theta)\equiv V^2(\theta)/a^2=\cos^2\theta,\ \frac{1}{2}[1+\xi\pm\sqrt{(1+\xi)^2-4\xi\ \cos^2\theta}] \qquad \cdots\cdots\cdots\cdots(6)$$

Since eq. (2) is cubic with respect to X, it is expected that there exist three waves, in general, for the given values of parameters ξ, $\bar{\omega}$ and $\bar{\beta}$. However, when $\bar{\omega}=0$ and $\bar{\beta}\neq 0$, the slowest wave does not propagate in the domain

$$\cos^{-1}\bar{\beta}<\theta<\pi-\cos^{-1}\bar{\beta} \qquad \cdots\cdots\cdots\cdots(7)$$

This is due to the ion cyclotron resonance. (See the following chapter.)

3. Results and Discussion

By making use of eq. (5), the phase velocity and the attenuation coefficient can be determined. First of all, a limiting case will be briefly considered in which the fluid is assumed to be incompressible. Therefore, the ordinary sound speed c_s is sufficiently large compared with the Alfvén velocity a. In this case, eq. (2) reduces to

$$X_{\pm}=(\cos\theta\pm\bar{\beta})\cos\theta+i\bar{\omega} \qquad \cdots\cdots\cdots\cdots(8)$$

This is the dispersion relation for the ordinary Alfvén wave propagating in an incompressible and electrically dissipative medium. Substitution of eq. (8) into eq. (5) yields

$$\frac{V_{\pm}(\bar{\omega})}{a}=\frac{2|X_{\pm}|}{\sqrt{|X_{\pm}|+\bar{\omega}}+\varepsilon_{\pm}\sqrt{|X_{\pm}|-\bar{\omega}}} \qquad \cdots\cdots\cdots\cdots(9)$$

$$\frac{\gamma_{\pm}(\bar{\omega})}{a\sigma\mu_0}=\frac{\bar{\omega}^2}{2|X_{\pm}|^2}\ \frac{V_{\pm}(\bar{\omega})}{a} \qquad \cdots\cdots\cdots\cdots(10)$$

where $\varepsilon_{\pm}=\mathrm{sgn}(\mathrm{Re}\ X_{\pm})$. In the above equations, the plus sign and the minus sign represent the right-hand and the left-hand circularly polarized waves, respectively. Making an expansion of eqs. (9) and (10) with respect to $\bar{\omega}$, the following result is obtained for small $\bar{\omega}$:

$$\frac{V_{\pm}(\bar{\omega})}{a}=\sqrt{(\cos\theta\pm\bar{\beta})\cos\theta}\left\{1+\frac{3}{8}\frac{\bar{\omega}^2}{(\cos\theta\pm\bar{\beta})^2\cos^2\theta}+\cdots\cdots\right\}$$

$$\frac{\gamma_{\pm}(\bar{\omega})}{a\sigma\mu_0}=\frac{\bar{\omega}^2}{2(\cos\theta\pm\bar{\beta})\cos\theta}\left\{1-\frac{5}{8}\frac{\bar{\omega}^2}{(\cos\theta\pm\bar{\beta})^2\cos^2\theta}+\cdots\cdots\right\}$$

The electrical dissipation effect on the phase velocity $V_{\pm}(\bar{\omega})$ and the attenuation coefficient $\gamma_{\pm}(\bar{\omega})$ is shown in Fig. 1 ($\bar{\beta}=0$) and Figs. 2 and 3 ($\bar{\beta}=0.5$). For the low frequency limit ($\bar{\beta}\longrightarrow 0$), the two waves become identical to each other, and the propagation properties are the same for θ and $\pi-\theta$. Thus, in Fig. 1 are shown only the curves for $0<\theta<\pi/2$.

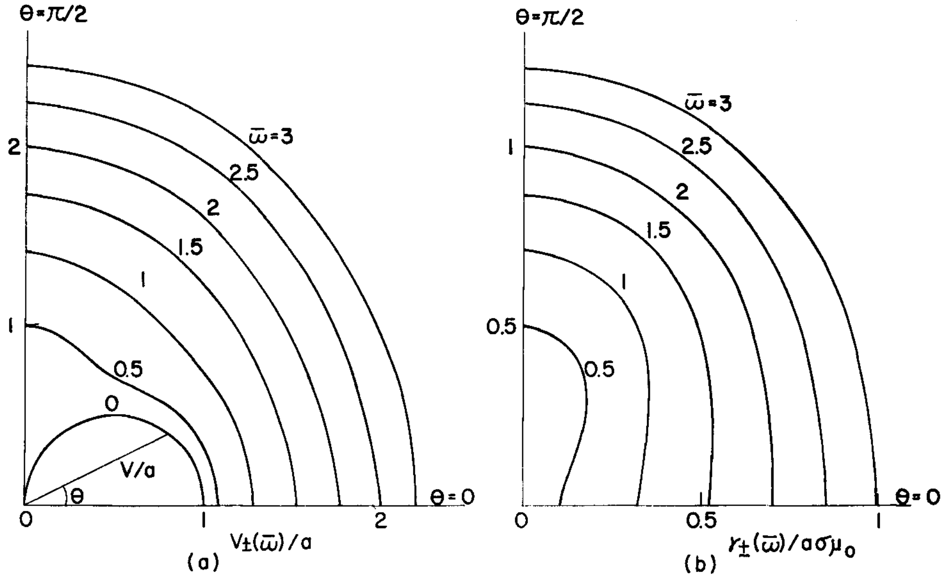

Fig. 1. (a) Phase Velocity and (b) Attenuation Coefficient of torsional Alfvén Wave for $\bar{\beta}=0$.

For $\bar{\beta}\neq 0$, the right-hand and the left-hand circularly polarized waves have different phase velocities and attenuation coefficients. The phase velocities $V_{\pm}(\bar{\omega})$ given by eq. (9), and the attenuation coefficients $\gamma_{\pm}(\bar{\omega})$ given by eq. (10), are plotted in Figs. 2 and 3, respectively, for $\bar{\beta}=0.5$. Here, it may be emphasized that as can be seen from eqs. (9) and (10), the properties of the right-hand and the left-hand circularly polarized waves propagating in the direction θ are identical to those of the left-hand and the right-hand circularly polarized waves propagating in the

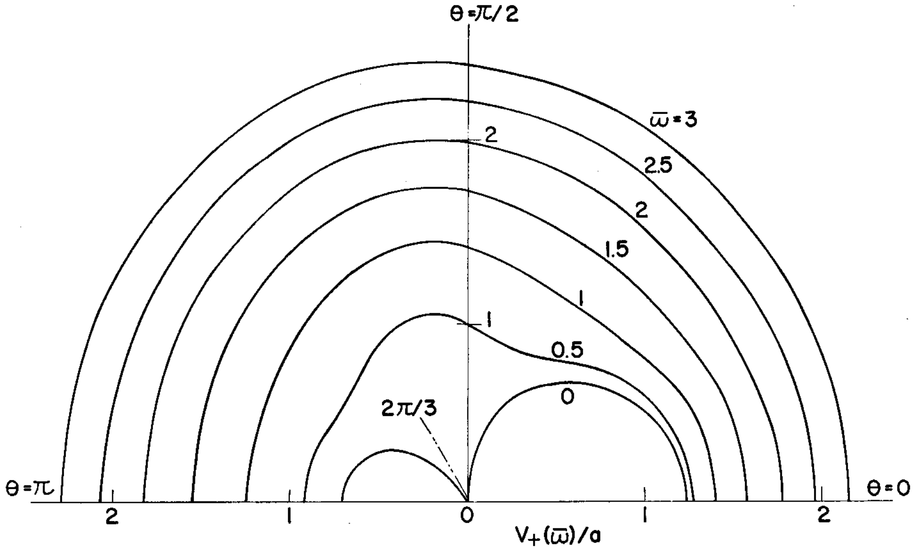

Fig. 2. Phase Velocity of Right-hand Circularly Polarized Wave for $c_s \longrightarrow \infty$ and $\bar{\beta}=0.5$.

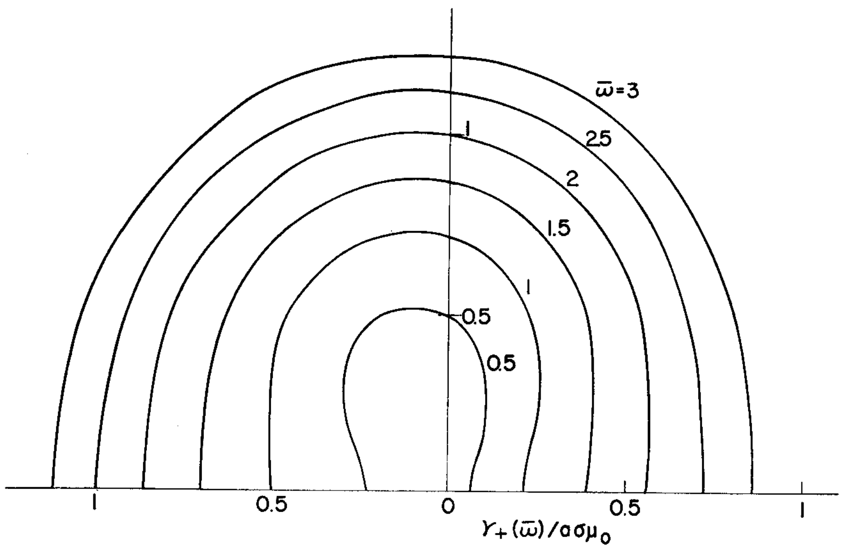

Fig. 3. Attenuation Coefficient of Right-hand Circularly Polarized Wave for $c_s \longrightarrow \infty$ and $\bar{\beta}=0.5$

direction $\pi-\theta$. Therefore, in Figs. 2 and 3, the phase velocity and the attenuation coefficient are shown only for the right-hand circularly polarized wave. The results for the left-hand circularly polarized wave can be obtained simply by transposing the curves in Figs. 2 and 3 symmetrically with respect to the axis $\theta=\pi/2$. It should be emphasized that when the electrical conductivity is infinitely large and the electrical dissipation is absent, a resonance occurs at $\bar{\beta}=\pm\cos\theta$. This is just the ion cyclotron resonance. However, it disappears when the electrical conductivity is finite and the electrical dissipation is present.

As can be seen from eqs. (9) and (10), the two contra-rotating circularly polarized waves have different phase velocities and attenuation coefficients. Therefore, a linearly polarized wave incident on a medium with finite electrical conductivity is converted, in general, into an elliptically polarized wave; and the direction of major axis is rotated. The rotation angle ψ is proportional to the propagation distance x, and is expressed as

$$\psi=\frac{\omega x}{2a}\,\mathrm{Re}\left(\frac{\sqrt{X_-}}{|X_-|}-\frac{\sqrt{X_+}}{|X_+|}\right)$$

This does not agree with Sonnerup's result[6)]

$$\psi\cong\frac{\beta\omega^2}{2\sigma\mu_0 a^3}x \qquad \text{for } \theta=0.$$

Clearly, his result is obtainable by taking the limit $\bar{\omega}\longrightarrow 0$ with β kept constant. However, since the Hall parameter β is proportional to the electrical conductivity, it may not be reasonable in the present research on the effect of electrical dissipation to regard the Hall parameter as constant.

Another limiting case, in which the sound speed c_s is negligibly small compared with the Alfvén speed a, has been discussed amply in the literature of plasma physics[1),2),7)]. Therefore, it will be sufficient here to point out only that the present model gives the correct results up to the ion cyclotron frequency for completely ionized cold plasmas. This implies that inclusion of the Hall effect in eq. (1) is equivalent to taking account of the ion cyclotron motion; and it is possible to obtain correct results from the fluid model up to the ion cyclotron frequency. This holds also for the following more general case.

Now consider a more general case where c_s is finite. When the medium is perfectly conducting, the coefficients of eq. (2) are real and it has three real roots. Therefore, the magneto-hydronamic waves do not damp and their velocities are given by

$$\frac{V^2(\bar{\beta})}{a^2}=\frac{1}{3}(1+\xi+\cos^2\theta)+2\sqrt{\frac{p}{3}}\cos\left(\phi-\frac{2(n-1)}{3}\pi\right)$$

$$(n=1,2,3)\cdots\cdots\cdots\cdots(11)$$

where

$$\phi=\frac{1}{3}\cos^{-1}\left(\frac{q}{2p}\sqrt{\frac{3}{p}}\right)$$

$$p=\frac{1}{3}[(1+\xi)^2-(1+4\xi)\cos^2\theta+\cos^4\theta]+\bar{\beta}^2\cos^2\theta$$

$$q=\frac{1}{9}[2(1+\xi)^3-3(1+\xi)(1+4\xi)\cos^2\theta-3(1-5\xi)\cos^4\theta+2\cos^6\theta]$$

$$+\bar{\beta}^2(1-2\xi+\cos^2\theta)\cos^2\theta.$$

When $\bar{\beta}=0$, these results reduce to eq. (6). It may be readily seen that the wave propagating perpendicular to the imposed magnetic field is not modified by the Hall effect, and eq. (11) recovers the magneto-acoustic wave, whose velocity is equal to $(a^2+c_s^2)^{\frac{1}{2}}$. On the other hand, the phase velocities of waves propagating in the direction of the imposed magnetic field are

$$V=c_s \qquad \cdots\cdots\cdots\cdots(12)$$

$$V_{\pm}=a\sqrt{1\pm\bar{\beta}} \qquad \cdots\cdots\cdots\cdots(13)$$

The wave described by eq. (12) is a purely longitudinal acoustic wave; and the waves described by eq. (13) are transverse right-hand (+sign) and left-hand (− sign) circularly polarized waves.

For the non-vanishing parameter $\bar{\beta}$, the phase velocities of these waves propagating in an arbitrary direction are separated by the fast and the slow waves for $\bar{\beta}=0$, represented by eq. (6). In particular, when the sound speed c_s is greater than the Alfvén velocity a, the fast wave is almost independent of the parameter $\bar{\beta}$. However, the slowest wave vanishes in the domain determined by the relation (7), since, in this range, the smallest solution X of eq. (2) is negative.

When the electrical conductivity of the medium is finite ($\bar{\omega}\neq 0$), the solutions $X(\theta)$ of eq. (2) are no longer real but complex numbers, which implies wave attenuation. For $\theta=0$, eq. (2) has the following solutions:

$$X_1(0)=\xi$$

$$X_2(0)=1+\bar{\beta}-i\bar{\omega}$$

$$X_3(0)=1-\bar{\beta}-i\bar{\omega}$$

With an increase of the angle θ, the solutions of eq. (2) move from these points in the complex plane, and reach the following points at $\theta=\pi/2$, respectively:

$$X_1(\pi/2)=\{1+\xi-i\bar{\omega}-[(1-\xi-i\bar{\omega})^2+4\xi]^{\frac{1}{2}}\}/2$$

$$X_2(\pi/2)=\{1+\xi-i\bar{\omega}+[(1-\xi-i\bar{\omega})^2+4\xi]^{\frac{1}{2}}\}/2$$

$$X_3(\pi/2)=-i\bar{\omega}$$

In the above expressions, $[(1-\xi-i\bar{\omega})^2+4\xi]^{\frac{1}{2}}$ represents a branch in the lower half-plane, i. e. it becomes asymptotically equal to $-i\bar{\omega}$ for large $\bar{\omega}$.

The first wave described by $X_1(\theta)$ shows characteristics analogous to those of an acoustic wave. As the parameter $\bar{\omega}$ increases, the phase velocity V approaches the ordinary sound speed c_s, independently of the propagation direction θ. Also the attenuation coefficient γ vanishes, since $\gamma/\sigma\mu_0 a$ remains finite. Note that $\sigma\longrightarrow 0$ when $\bar{\omega}\longrightarrow\infty$. In fact,

$$X_1(\theta)\sim\xi(1-\frac{i}{\bar{\omega}}\sin^2\theta)$$

for sufficiently large $\bar{\omega}$, and thus eq. (5) gives

$$V\cong c_s,\quad \gamma/\sigma\mu_0 a\cong\sin^2\theta/2\sqrt{\xi}$$

These results are valid for any value of $\bar{\beta}$.

The second wave represented by $X_2(\theta)$ shows the same tendency as the right-hand circularly polarized wave, as was previously discussed. The third wave represented by $X_3(\theta)$ corresponds to the left-hand circularly polarized wave. Note that for the small $\bar{\omega}$, these waves are fairly modified by the compressibility of the medium, which implies a strong coupling with the acoustic wave. However, as $\bar{\omega}$ increases, the above-mentioned tendency is seen and verified by comparing Figs. 2 and 3 with Figs. 4-9, in which the first, the second and the third waves are plotted, respectively, by solid, dashed and dash-and-dotted curves with the values of parameter $\bar{\omega}$. For the sufficiently large $\bar{\omega}$,

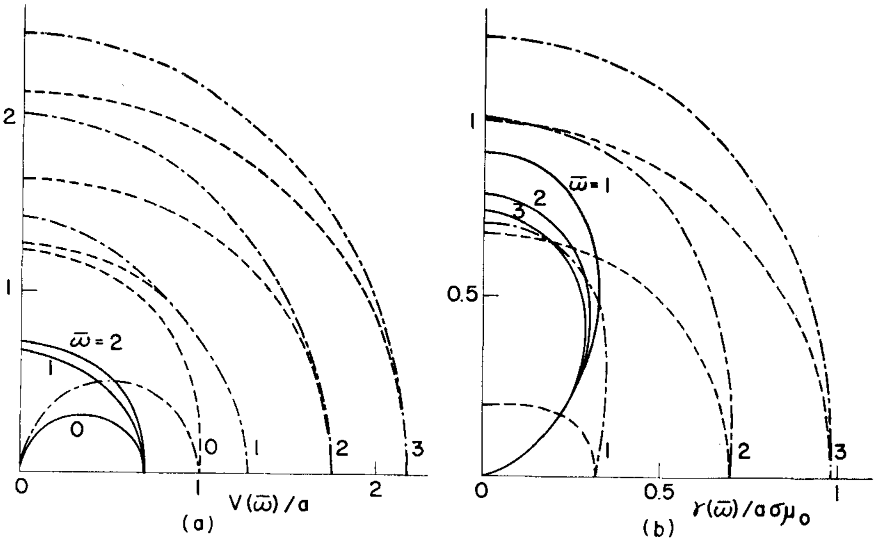

Fig. 4. (a) Phase Velocity and (b) Attenuation Coefficient of Magneto-hydrodynamic Waves for $\xi=0.5$ and $\bar{\beta}=0$.

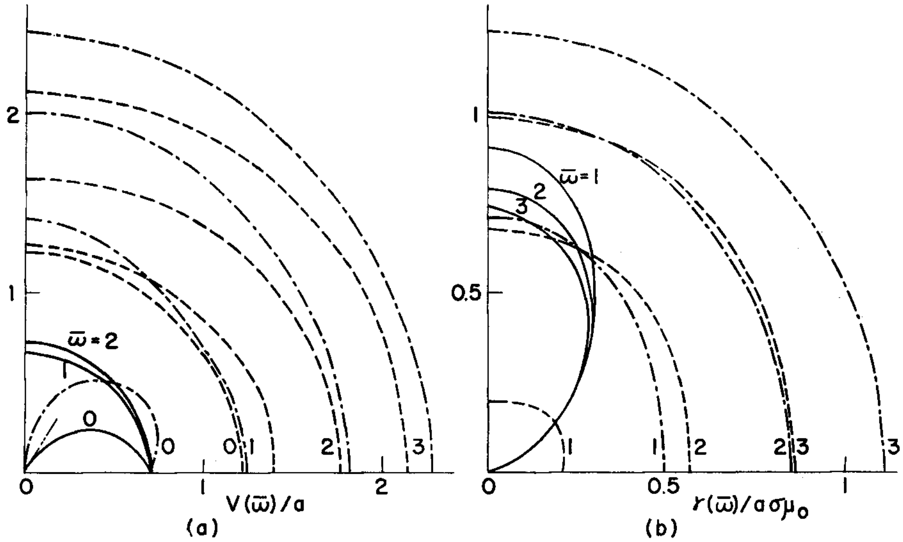

Fig. 5. (a) Phase Velocity and (b) Attenuation Coefficient of Magneto-hydrodynamic Waves for $\xi=0.5$ and $\bar{\beta}=0.5$.

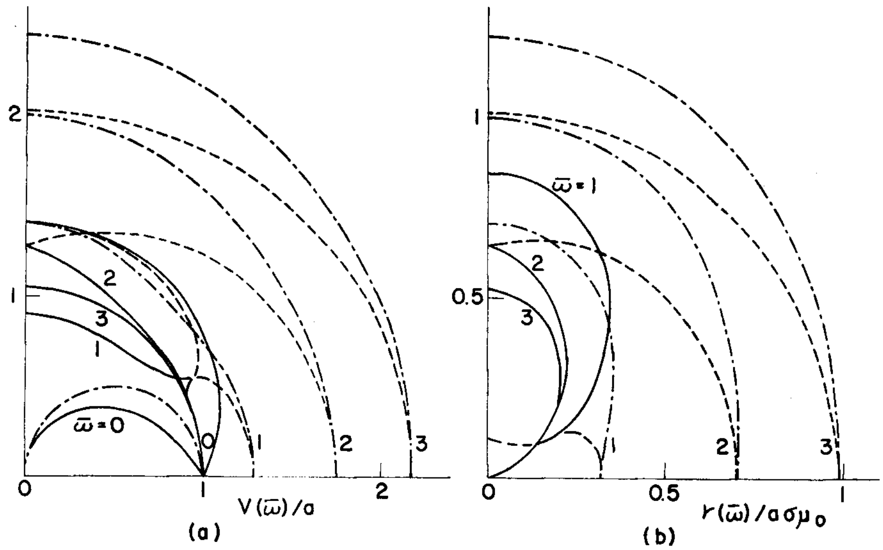

Fig. 6. (a) Phase Velocity and (b) Attenuation Coefficient of Magneto-hydrodynamic Waves for $\xi=1$ and $\bar{\beta}=0$.

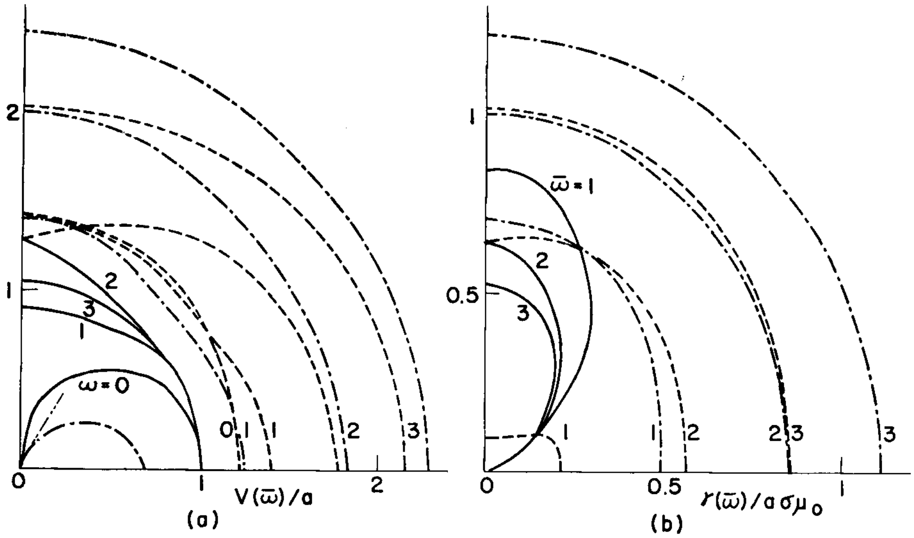

Fig. 7. (a) Phase Velocity and (b) Attenuation Coefficient of Magneto-hydrodynamic Waves for $\xi=1$ and $\bar{\beta}=0.5$.

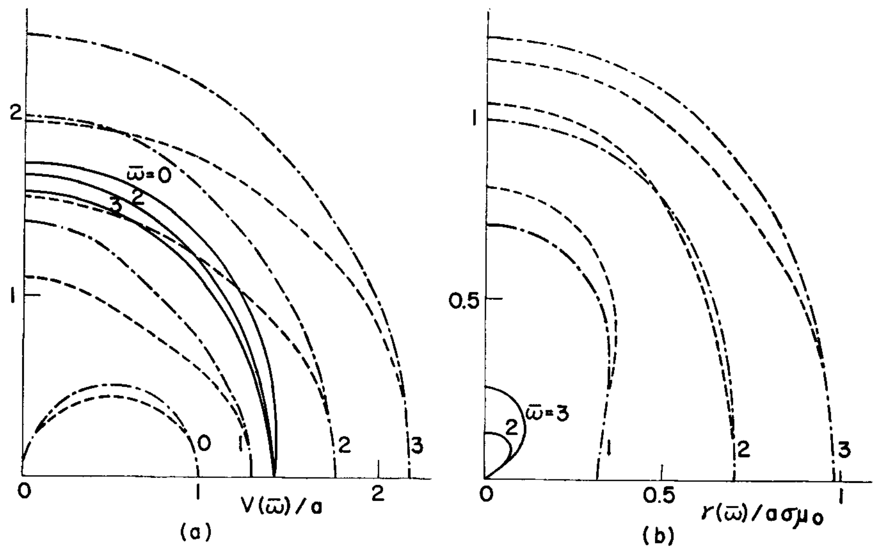

Fig. 8. (a) Phase Velocity and (b) Attenuation Coefficient of Magneto-hydrodynamic Waves for $\xi=2$ and $\bar{\beta}=0$.

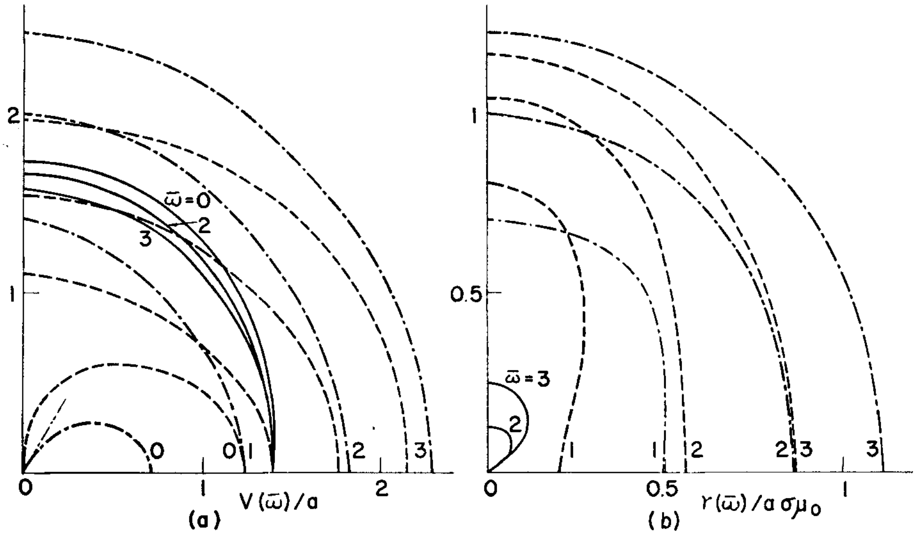

Fig. 9. (a) Phase Velocity and (b) Attenuation Coefficient of Magneto-hydrodynamic Waves for $\xi=2$ and $\bar{\beta}=0.5$.

$$X_{2,3}(\theta)\sim -i\bar{\omega}+\frac{1+\cos^2\theta\pm[\sin^4\theta+4\bar{\beta}^2\cos^2\theta]^{\frac{1}{2}}}{2}$$

Substitution of this relation into eq. (5) yields the phase velocity and the attenua tion coefficient. Since the displacement current was neglected, the phase velocity obtained is not correct, but the attenuation coefficient is correctly obtained as $\gamma=(\sigma\mu_0\omega/2)^{\frac{1}{2}}$. This is identical to the damping coefficient well-known in electromagnetism[8]. Therefore, it can be considered that the second and the third waves are reduced to electromagnetic waves, when the electrical conductivity of the medium becomes sufficiently small.

4. Conclusions

In the preceding chapters, the transition of magneto-hydrodynamic waves to acoustic and electromagnetic waves due to a variation of electrical conductivity of the medium was investigated by analyzing the dispersion relation obtained from magneto-hydrodynamic equations, Maxwell's equations and generalized Ohm's law. The effect of electrical dissipation is well described by a magnetic Reynolds number defined for the Alfvén wave, and inclusion of the Hall effect is equivalent to taking account of the ion cyclotron motion and considering the wave frequency up to the ion cyclotron frequency.

It was found that there exist three waves, in general, as in the case without any electrical dissipation or Hall effect, and that the slowest wave does not propagate in perfectly conducting media in a range of propagation direction determined by the relation (7). This is due to the ion cyclotron resonance.

When the electrical conductivity of the medium is finite, there are three waves in any direction, and with a decrease of electrical conductivity, they are reduced simply to the acoustic and the electromagnetic waves. The propagation properties depend on the wave frequency and the propagation direction, but as the electrical conductivity decreases, the dependence becomes less important.

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