A Time-Decomposition Algorithm for the Solution of Multiple-Target Problems

By

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Abstract

An algorithm for solving the nonlinear optimal control problem whose system equation contains discontinuities is proposed. The boundary conditions are specified at several corner times as well as at the initial and the final times. Assuming that the corner times are known, by using the variational principle, the problem is reduced to a nonlinear multipoint boundary-value problem (MPBVP), which is further reduced to a linear one by use of an interaction-coordination algorithm. The linear MPBVP is solved by a discontinuous version of a time-decomposition algorithm, which decomposes the problem into a number of subinterval TPBVP's. The exact boundary conditions of these TPBVP's are determined by an algebraic method which utilizes the solution obtained with arbitrarily chosen boundary conditions. After solving the nonlinear MPBVP, the assumed corner times are corrected by a gradient method. Correction is iterated until the optimum is attained. The solution in each iteration satisfies the boundary conditions exactly.

In order to verify the effectiveness of the present algorithm, a linear and a nonlinear problem are solved numerically and the solution to the linear problem is compared with the analytical one.

1. Introduction

A multiple-target problem, or an optimal control problem with discontinuities, has been investigated by several authors[1-5]. The system equation of the problem has discontinuities at the 'corner times' of the control duration. Some of the state variables, as well as the initial state (and possibly the final state), are specified at the corner times. Swiger[1] and Tomizuka & Tsujioka[2] solved this problem by using the orthogonal projection theorem. Also, the problem was solved by use of the gradient methods in Refs. 3-5. However, the former algorithm is restricted to a problem which does not contain nonlinearities in each subinterval between adjacent corner times. The latter, though they can deal with nonlinearities, suffers from an

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unsatisfactory accuracy in satisfying the boundary conditions, because they employ a penalty function method.

Recently, the present authors have proposed algorithms for solving nonlinear optimal control problems[6-8]. The interaction-coordination algorithm (ICA)[6,7] is one which decomposes the overall problem into a number of smaller linear subproblems described by a two-point boundary-value problem (TPBVP), and then coordinates them. The time-decomposition algorithm (TDA)[8] decomposes a linear TPBVP into a number of TPBVP's defined in the subintervals, and then determines algebraically the exact boundary conditions for each subinterval problem.

In this paper, the TDA is extended to the solution of the linear MPBVP with discontinuities. Together with use of the ICA, the nonlinear MPBVP with fixed corner times is solved, and then the corner times are corrected by a gradient method until the optimum is attained.

In Section 2, the problem considered in this paper is formulated and the necessary conditions for optimality are derived. Section 3 sketches the ICA briefly, and Section 4 discusses the details of the discontinuous version of the TDA. The algorithm including optimal correction of the corner times is summarized in Section 5, and the applications to two physical problems are illustrated in Section 6.

2. Problem Statement

In this paper, we discuss a solution of the following multiple-target problem. The system equation is described by

$$\dot{x}=A_i(t)x+B_i(t)u+f_i(t,\ x),\quad t_{i-1}\leqq t<t_i \qquad (i=1,\ 2,\ \ldots,\ N) \qquad\cdots\cdots\cdots\cdots(1)$$

where $x(t)$ is an n-dimensional state vector, $u(t)$ an m-dimensional control vector. A_i and B_i are $n\times n$- and $n\times m$-dimensional matrices, respectively, and f_i is an n-dimensional vector function of the class C^2 with respect to x, and these are continuous in $t\epsilon[t_{i-1},\ t_i)$. The boundary conditions are given by

$$L_i x(t_i)=\pi_i \qquad (i=0,\ 1,\ 2,\ \ldots,\ N) \qquad\cdots\cdots\cdots\cdots(2)$$

where π_i is an r_i-dimensional prescribed vector and L_i is an $r_i\times n$-dimensional matrix containing only one nonzero element in each row, and it is assumed that $t_i(i=1,\ 2,\ \ldots,\ N\text{-}1)$ is not specified.

The objective is to minimize the following performance index of the quadratic type:

$$J=\frac{1}{2}\int_{t_0}^{t_N}(x'Q(t)x+u'R(t)u)\,dt \qquad\cdots\cdots\cdots\cdots(3)$$

with respect to u and $t_i(i=1,\ 2,\ \ldots,\ N\text{-}1)$, where Q is an $n\times n$-dimensional symmetric positive semidefinite matrix and R an $m\times m$-dimensional symmetric positive definite matrix.

Now define the Hamiltonian of (1) and (3) as

$$H^{(i)}=\frac{1}{2}(x'Qx+u'Ru)+p'(A_ix+B_iu+f_i) \qquad \cdots\cdots\cdots\cdots(4)$$

Then, according to the variational principle, the necessary conditions for optimality are obtained as follows:[9]

$$\left.\begin{aligned} &\dot{x}=\left(\frac{\partial H^{(i)}}{\partial p}\right)'=A_i(t)x+B_i(t)u+f_i(t,x) &&\cdots\cdots\cdots\cdots(5)\\ &\dot{p}=-\left(\frac{\partial H^{(i)}}{\partial x}\right)'=-Q(t)x-A_i'(t)p-\left(\frac{\partial f_i}{\partial x}\right)'p \quad (i=1,2,\ldots,N) &&\cdots\cdots(6)\\ &\left(\frac{\partial H^{(i)}}{\partial u}\right)'=R(t)u+B_i'(t)p=0 &&\cdots\cdots\cdots\cdots(7)\end{aligned}\right.$$

$$g_{t_i}\triangleq H^{(i)}(t_i^-)-H^{(i+1)}(t_i^+)=0 \qquad (i=1,2,\ldots,N\text{-}1) \qquad \cdots\cdots\cdots\cdots(8)$$

with the boundary conditions

$$\begin{aligned} &L_ix(t_i)=\pi_i \qquad (i=0,1,\ldots,N)\\ &p_j(t_i^-)=p_j(t_i^+)=\nu_{ji} \quad (\text{if } x_j(t_i) \text{ is not specified}; \ i=1,2,\ldots,N\text{-}1)\end{aligned} \qquad \cdots(9)$$

where $p_j(t_i)$ denotes the j-th element of $p(t_i)$ and ν_{ji} is a Lagrange multiplier. Substitution of $u=-R^{-1}B_i'p$ into (5) yields the following multipoint boundary-value problem (MPBVP):

$$\frac{d}{dt}\begin{bmatrix} x\\ p\end{bmatrix}=\begin{bmatrix} A_i(t) & -E_i(t)\\ -Q(t) & -A_i'(t)\end{bmatrix}\begin{bmatrix} x\\ p\end{bmatrix}+\begin{bmatrix} h_{1i}(t,x)\\ h_{2i}(t,x,p)\end{bmatrix} \quad (i=1,2,\ldots,N) \qquad \cdots\cdots\cdots\cdots(10)$$

constrained by the boundary conditions (9) and the optimality condition for t_i (8), where $E_i=B_iR^{-1}B_i'$, $h_{1i}(t,x)=f_i(t,x)$, and $h_{2i}(t,x,p)=-\left(\frac{\partial f_i}{\partial x}\right)'p$.

Once the t_i's are assumed, the problem is reduced to solving the MPBVP of (9) and (10) with discontinuities at the fixed corner times. This problem will be dealt with in the following sections. The solution, however, does not satisfy the optimality condition (8). Then the assumed values of the t_i's are to be corrected by an appropriate algorithm. The problem is then solved by a three-level algorithm. The first and the second levels solve a nonlinear MPBVP. Using the solution thus obtained, the third level seeks the optimal corner times by a gradient method, that is, the corner times are corrected by

$${}^{l+1}t_i={}^{l}t_i-\eta\cdot\text{sign}\,({}^{l}g_{t_i}) \qquad \begin{pmatrix} i=1,2,\ldots,N\text{-}1\\ l=0,1,\ldots\ldots\end{pmatrix} \qquad \cdots\cdots\cdots\cdots(11)$$

until (8) is satisfied, where l denotes the iteration number and η is a positive step size which is reduced according to the change of the sign of g_{t_1}.

3. A Linearization Technique

Once the problem is reduced to a nonlinear MPBVP with fixed corner times, conventional mathematical or computational means[6-12] are available for the solution. Among them, the interaction-coordination algorithm[6,7] has proved to apply successfully to a problem whose system equation is a perturbed form as given by (1). The algorithm, like the quasilinearization method[10], reduces the problem to the linear MPBVP, and then the time-decomposition algorithm is applicable, as will be seen in the succeeding section.

In the following, the ICA is briefly skecthed. Consider a nonlinear TPBVP whose differential equation is given, in the time interval $t\epsilon[t_0, t_f]$, by (10) without the subscript i. The boundary conditions are given by

$$x(t_0)=\pi_0,\ x(t_f)=\pi_f \qquad \cdots\cdots\cdots\cdots(12)$$

Introducing the interaction variables y and q corresponding to x and p, respectively, we replace (10) by:

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix}=\begin{bmatrix} A & -\beta E \\ -\kappa Q & -A' \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix}+\begin{bmatrix} 0 & (\beta-1)E \\ (\kappa-1)Q & 0 \end{bmatrix}\begin{bmatrix} y \\ q \end{bmatrix}+\begin{bmatrix} h_1(t,\ y) \\ h_2(t,\ y,\ q) \end{bmatrix}\cdots(13)$$

where β and κ are scalar parameters to accelerate the convergence tendency of the iteration. Once y and q are given, (13) is linear in x and p, so that the linear TPBVP of (12) and (13) is solved easily by the superposition principle[10]. If the solutions x and p thus obtained coincide with the assumed y and q, respectively, they are the solution to the original nonlinear TPBVP[6,7]. However, this is generally not the case. Therefore, the interaction variables y and q are corrected by

$$\begin{bmatrix} {}^{k+1}y \\ {}^{k+1}q \end{bmatrix}=\begin{bmatrix} {}^{k}y \\ {}^{k}q \end{bmatrix}+\alpha\begin{bmatrix} {}^{k}x-{}^{k}y \\ {}^{k}p-{}^{k}q \end{bmatrix} \qquad \cdots\cdots\cdots\cdots(14)$$

where k denotes the iteration number and α is a constant step size. The correction is iterated until the coordination error

$${}^{k}G=\left[\int_{t_0}^{t_f}\{({}^{k}x-{}^{k}y)^2+({}^{k}p-{}^{k}q)^2\}dt/2n(t_f-t_0)\right]^{\frac{1}{2}} \qquad \cdots\cdots\cdots\cdots(15)$$

is reduced to zero or a sufficiently small enough unit. For details see References 6 and 7.

4. A Time-Decomposition Algorithm for a Linear MPBVP

As shown in the preceding section, the solution to the nonlinear MPBVP is obtained by solving a sequence of the linear MPBVP's. Therefore, in this section we consider the solution of the linear MPBVP.

Recently the authors have proposed a time-decomposition algorithm for solving a linear TPBVP[8]. Here, we extend the algorithm to the solution of a linear MPBVP with discontinuities.

4.1. A Time-Decomposition Algorithm for Solution of the Linear TPBVP To begin with, we consider a solution of a linear TPBVP:

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = D(t)\begin{bmatrix} x \\ p \end{bmatrix} + h(t) \qquad \cdots\cdots\cdots\cdots(16)$$

with the boundary conditions

$$x(t_0) = \pi_0,\ x(t_N) = \pi_N \qquad \cdots\cdots\cdots\cdots(17)$$

Both x and p are n-dimensional vectors, $D(t)$ is a $2n\times 2n$-dimensional matrix, and $h(t)$ is a $2n$-dimensional vector function. Both π_0 and π_N are prescribed n-dimensional vectors. The inital time t_0 and the final time t_N are assumed to be specified.

Let $\Phi(t, t_0)$ denote the transition matrix of the homogeneous part of (16) with $\Phi(t_0, t_0) = I_{2n}$, $2n\times 2n$-dimensional identity matrix. Then the general solution to (16) is written as

$$x(t) = \Phi_{11}(t, s)x(s) + \Phi_{12}(t, s)p(s) + v_1(t, s) \qquad \cdots\cdots\cdots\cdots(18)$$

$$p(t) = \Phi_{21}(t, s)x(s) + \Phi_{22}(t, s)p(s) + v_2(t, s) \qquad \cdots\cdots\cdots\cdots(19)$$

where

$$\left.\begin{aligned} &\begin{bmatrix} \Phi_{11}(t, s)\,\Phi_{12}(t, s) \\ \Phi_{21}(t, s)\,\Phi_{22}(t, s) \end{bmatrix} = \Phi(t, s) \\ &\begin{bmatrix} v_1(t, s) \\ v_2(t, s) \end{bmatrix} = \int_s^t \Phi(t,\tau)\,h(\tau)\,d\tau \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots(20)$$

From (18) and (19), it is seen that the solution to (16) in the subinterval $[t_i, t_{i+1}]$ with the boundary conditions $x(t_i) = \overline{x(t_i)}$ and $x(t_{i+1}) = \overline{x(t_{i+1})}$ satisfies

$$\overline{x(t_{i+1})} = \phi_{11}(i+1, i)\overline{x(t_i)} + \phi_{12}(i+1, i)\overline{p(t_i^+)} + v_1(t_{i+1}, t_i) \qquad \cdots\cdots\cdots\cdots(21)$$

$$\overline{p(t_{i+1}^-)} = \phi_{21}(i+1, i)\overline{x(t_i)} + \phi_{22}(i+1, i)\overline{p(t_i^+)} + v_2(t_{i+1}, t_i) \qquad \cdots\cdots\cdots\cdots(22)$$

$$(i=0, 1, \ldots, N-1)$$

where $\phi_{ij}(\lambda, \mu)$ denotes $\Phi_{ij}(t_\lambda, t_\mu)$. If $\phi_{12}(i+1, i)$ is nonsingular, we can rewrite (21) as follows:

$$\overline{p(t_i^+)} = \phi_{12}^{-1}(i+1, i)\{\overline{x(t_{i+1})} - \phi_{11}(i+1, i)\overline{x(t_i)} - v_1(t_{i+1}, t_i)\} \qquad \cdots\cdots\cdots\cdots(23)$$

Substitution of (23) into (22) yields

$$\overline{p(t_{i+1}^-)}=\phi_{21}(i+1,\ i)\overline{x(t_i)}+\phi_{22}(i+1,\ i)\phi_{12}^{-1}(i+1,\ i)\{\overline{x(t_{i+1})}-\phi_{11}(i+1,\ i)\overline{x(t_i)}-v_1(t_{i+1},\ t_i)\}+v_2(t_{i+1},\ t_i) \quad \cdots\cdots(24)$$

By replacing i in (23) by $i+1$ and subtracting it from (24), we obtain

$$\overline{p(t_{i+1}^-)}-\overline{p(t_{i+1}^+)}=S_i\overline{x(t_i)}+T_i\overline{x(t_{i+1})}+U_i\overline{x(t_{i+2})}+V_i \quad \cdots\cdots(25)$$

where

$$\left.\begin{aligned}
S_i&=\phi_{21}(i+1,\ i)-\phi_{22}(i+1,\ i)\phi_{12}^{-1}(i+1,\ i)\phi_{11}(i+1,\ i)\\
T_i&=\phi_{22}(i+1,\ i)\phi_{12}^{-1}(i+1,\ i)+\phi_{12}^{-1}(i+2,\ i+1)\phi_{11}(i+2,\ i+1)\\
U_i&=-\phi_{12}^{-1}(i+2,\ i+1)\\
V_i&=-\phi_{22}(i+1,\ i)\phi_{12}^{-1}(i+1,\ i)v_1(t_{i+1},\ t_i)+v_2(t_{i+1},\ t_i)\\
&\quad+\phi_{12}^{-1}(i+2,\ i+1)v_1(t_{i+2},\ t_{i+1})
\end{aligned}\right\} \quad \cdots\cdots(26)$$

$(i=0,\ 1,\ \ldots\ N\text{-}2)$

Taking (25) and (26) for all i ($i=0,\ 1,\ \ldots,\ N\text{-}2$) into account, we can establish the following theorem.

Theorem 1.

Let p be the solution to (16) in the interval $[t_i,\ t_{i+1}]$ with the boundary conditions $x(t_i)=\overline{x(t_i)}$ and $x(t_{i+1})=\overline{x(t_{i+1})}$ $(i=0,\ 1,\ \ldots,\ N\text{-}1;\ \overline{x(t_0)}=\pi_0,\ \overline{x(t_N)}=\pi_N)$. Let $\overline{p(t_i^+)}$ and $\overline{p(t_{i+1}^-)}$ be the value of p at $t=t_i^+$ and $t=t_{i+1}^-$, respectively. Then the following relation holds:

$$\bar{P}=\Gamma\bar{X}+V \quad \cdots\cdots(27)$$

where

$$\Gamma=\begin{bmatrix}
T_0 & U_0 & & & 0\\
S_1 & T_1 & U_1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & S_{N-3} & T_{N-3} & U_{N-3}\\
0 & & & S_{N-2} & T_{N-2}
\end{bmatrix} \quad \cdots\cdots(28)$$

$$\bar{X}=[\overline{x'(t_1)},\ \overline{x'(t_2)},\ \ldots,\ \overline{x'(t_{N-1})}]' \quad \cdots\cdots(29)$$

$$\bar{P}=[(\overline{p(t_1^-)}-\overline{p(t_1^+)})',\ \ldots,\ (\overline{p(t_{N-1}^-)}-\overline{p(t_{N-1}^+)})']' \quad \cdots\cdots(30)$$

$$V=[(S_0\pi_0+V_0)',\ V_1',\ \ldots,\ V_{N-3}',\ (V_{N-2}+U_{N-2}\pi_N)']' \quad \cdots\cdots(31)$$

Note that Γ and V are independent of the choice of the boundary conditions $\overline{x(t_i)}$ $(i=1,\ 2,\ \ldots,\ N\text{-}1)$. Hence we have the following corollary.

Corollary 1.1.

Suppose Γ of (28) be nonsingular. Let $X=[x'(t_1), x'(t_2), \ldots, x'(t_{N-1})]'$, where $x(t_i)$ is the value of the exact solution to (16) at $t=t_i$ $(i=1, 2, \ldots, N-1)$. Then $\bar{X}$ and X of (29) are related by the following algebraic equation:

$$X=\bar{X}-\Gamma^{-1}\bar{P} \qquad \cdots\cdots\cdots\cdots(32)$$

Proof.

For the exact solution $x(t_i)$, obviously p is continuous in $t \in [t_0, t_N]$. Hence,

$$0=\Gamma X+V \qquad \cdots\cdots\cdots\cdots(33)$$

Subtract (27) from (33). Then the nonsingularity of Γ proves the validity of (32)

Q. E. D.

Remark 1.

Corollary 1.1 means that the solution to the given TPBVP can be obtained by solving several numbers of the subinterval TPBVP's. Hence, it is suggested that the TDA is also applicable to the problem, having discontinuities in the system equation.

Now let us consider the nonsingularity of Γ.

Theorem 2.

Suppose that $\phi_{12}(\lambda, 0)$ $(\lambda=1, 2, \ldots, N)$ and $\phi_{12}(\lambda+1, \lambda)$ $(\lambda=1, 2, \ldots, N-1)$ be nonsingular. Then, Γ is nonsingular.

Before proceeding to the proof of Theorem 2, we prove the following two lemmas.

Lemma 1.

For arbitrary λ, μ, and ν, the following relation holds:

$$\phi_{ij}(\lambda, \nu)=\sum_{k=1}^{2}\phi_{ik}(\lambda, \mu)\phi_{kj}(\mu, \nu) \qquad (i, j=1, 2) \qquad \cdots\cdots\cdots\cdots(34)$$

Proof.

From the transition property of Φ,

$$\Phi(t_\lambda, t_\nu)=\Phi(t_\lambda, t_\mu)\Phi(t_\mu, t_\nu) \qquad \cdots\cdots\cdots\cdots(35)$$

Hence, expansion of (35) proves (34).

Q. E. D.

Lemma 2.

Assert the hypothesis of Theorem 2. Then the following sequence of matrices $\bar{T}_i$ is well-defined:

$$\tilde{T}_i=-S_i\tilde{T}_{i-1}^{-1}U_{i-1}+T_i \qquad (i=0,\ 1,\ \ldots,\ N\text{-}2) \qquad \cdots\cdots\cdots\cdots(36)$$

where $S_0 \neq 0$, and $\tilde{T}_{-1}^{-1}$ and U_{-1} are arbitrary matrices. Furthermore $\tilde{T}_i$ is given by

$$\tilde{T}_i=\phi_{12}^{-1}(i+2,\ i+1)\phi_{12}(i+2,\ 0)\phi_{12}^{-1}(i+1,\ 0) \qquad \cdots\cdots\cdots\cdots(37)$$

Proof.

Clearly, it suffices to prove (37). We prove (37) inductively.

First, by the definition of T_i in (26) and Lemma 1, we have

$$\begin{aligned}\tilde{T}_0=T_0&=\phi_{22}(1,\ 0)\phi_{12}^{-1}(1,\ 0)+\phi_{12}^{-1}(2,\ 1)\phi_{11}(2,\ 1)\\&=\phi_{12}^{-1}(2,\ 1)[\phi_{12}(2,\ 1)\phi_{22}(1,\ 0)+\phi_{11}(2,\ 1)\phi_{12}(1,\ 0)]\phi_{12}^{-1}(1,\ 0)\\&=\phi_{12}^{-1}(2,\ 1)\phi_{12}(2,\ 0)\phi_{12}^{-1}(1,\ 0) \qquad \cdots\cdots\cdots\cdots(38)\end{aligned}$$

(38) shows that (37) holds for $i=0$.

Second, we show that the relation (37) holds for $i=k+1$, if it holds for $i=k$. From (36) and (37),

$$\begin{aligned}\tilde{T}_{k+1}-T_{k+1}&=-S_{k+1}\tilde{T}_k^{-1}U_k\\&=[\phi_{21}(k+2,\ k+1)-\phi_{22}(k+2,\ k+1)\phi_{12}^{-1}(k+2,\ k+1)\phi_{11}(k+2,\ k+1)]\\&\quad\ \phi_{12}(k+1\ ,0)\phi_{12}^{-1}(k+2,\ 0) \qquad \cdots\cdots\cdots\cdots(39)\end{aligned}$$

Substituting

$$\phi_{21}(k+2,\ k+1)\phi_{12}(k+1,\ 0)=\phi_{22}(k+2,\ 0)-\phi_{22}(k+2,\ k+1)\phi_{22}(k+1,\ 0) \qquad \cdots\cdots\cdots\cdots(40)$$

into (39), we obtain

$$\begin{aligned}\tilde{T}_{k+1}&=T_{k+1}+\phi_{22}(k+2,\ 0)\phi_{12}^{-1}(k+2,\ 0)\\&\quad-\phi_{22}(k+2,\ k+1)[\phi_{22}(k+1,\ 0)+\phi_{12}^{-1}(k+2,\ k+1)\phi_{11}(k+2,\ k+1)\phi_{12}(k+1,\ 0)]\\&\quad\ \phi_{12}^{-1}(k+2,\ 0)\\&=T_{k+1}+\phi_{22}(k+2,\ 0)\phi_{12}^{-1}(k+2,\ 0)-\phi_{22}(k+2,\ k+1)\phi_{12}^{-1}(k+2,\ k+1)\\&=\phi_{12}^{-1}(k+3,\ k+2)\phi_{11}(k+3,\ k+2)+\phi_{22}(k+2,\ 0)\phi_{12}^{-1}(k+2,\ 0)\\&=\phi_{12}^{-1}(k+3,\ k+2)\phi_{12}(k+3,\ 0)\phi_{12}^{-1}(k+2,\ 0) \qquad \cdots\cdots\cdots\cdots(41)\end{aligned}$$

(41) implies that (37) holds for $i=k+1$. Thus, the proof is completed.

Q. E. D.

Now we can proceed to prove Theorem 2.

Proof of Theorem 2.

Let Λ_i be the nonsingular matrix defined by

$$\Lambda_i=\overbrace{\left[\begin{array}{ccc|c} I_n & & & 0\\ & \ddots & & \\ & & \cdot & -\tilde{T}_{i-1}^{-1}U_{i-1}\\ \hline & & & I_n\\ & & & \ddots\\ 0 & & & I_n\end{array}\right]}^{ni \ \text{of}\ n(N\text{-}1)} \qquad (i=1,\ 2,\ \ldots,\ N\text{-}2) \qquad \cdots\cdots\cdots\cdots(42)$$

Multiplying Λ_i to Γ from the right successively ($i=1, 2, \ldots, N-2$) transforms Γ into

$$\begin{bmatrix} T_0 & & & 0 \\ & \ddots & & \\ & * & \ddots & \\ & & & T_{N-2} \end{bmatrix} \qquad \cdots\cdots\cdots\cdots(43)$$

Due to Lemma 2 the matrix of (43) is nonsingular. Hence, Theorem 2 is proved.

Q. E. D.

4.2. Extension of the TDA to the Linear MPBVP with Discontinuities

As mentioned in Remark 1, the TDA can be extended to solving the linear MPBVP with discontinuities in the system equation. Hereafter, for simplicity, the discontinuity is assumed to occur only once during the overall control duration. Then, let $N=2$ and let the corner time $t=t_1$.

We restrict our discussion to the case where the boundary conditions are given in the form:

$$\psi_i[x(t_i^-), x(t_i^+)] = \begin{bmatrix} x_{1(i)}(t_i^-) - x_{1(i)}(t_i^+) \\ x_{2(i)}(t_i) - \pi_i \\ x_{3(i)}(t_i^-) - x_{3(i)}(t_i^+) \end{bmatrix} = 0 \qquad \cdots\cdots\cdots\cdots(44)$$

where $x_{j(i)}$ is an $r_{j(i)}$-dimensional vector with $r_{1(i)}+r_{2(i)}+r_{3(i)}=n$ ($i=0, 1, 2$). (44) means that some elements of the state variable x are specified at $t=t_1$ and that x is continuous in t.

In this case, the necessary conditions for optimality (5)～(9) can be written as:[6]

Subarc 1: $t\varepsilon[t_0, t_1]$

$$\left.\begin{aligned} \dot{x} &= A_1x+B_1u+f_1(t, x) \\ \dot{p} &= -Qx-A_1'p-\left(\frac{\partial f_1}{\partial x}\right)'p \\ \left(\frac{\partial H^{(1)}}{\partial u}\right)' &= Ru+B_1'p=0 \end{aligned}\right\} \cdots(45\text{-}1)$$

Subarc 2: $t\varepsilon[t_1, t_2]$

$$\left.\begin{aligned} \dot{x} &= A_2x+B_2u+f_2(t, x) \\ \dot{p} &= -Qx-A_2'p-\left(\frac{\partial f_2}{\partial x}\right)'p \\ \left(\frac{\partial H^{(2)}}{\partial u}\right)' &= Ru+B_2'p=0 \end{aligned}\right\} \cdots(45\text{-}2)$$

$$H^{(i)} = \frac{1}{2}[x'Qx+u'Ru]+p'[A_ix+B_iu+f_i(t, x)] \qquad (i=1, 2)$$

$$\left.\begin{aligned} x(t_0) &= \pi_0 \\ x_2(t_1) &= \pi_1 \\ p_1(t_1) &= \nu_{11} \\ p_3(t_1) &= \nu_{31} \end{aligned}\right\} \cdots(46\text{-}1) \qquad \left.\begin{aligned} x_2(t_1) &= \pi_1 \\ x(t_2) &= \pi_2 \\ p_1(t_1) &= \nu_{11} \\ p_3(t_1) &= \nu_{31} \end{aligned}\right\} \cdots\cdots\cdots\cdots(46\text{-}2)$$

$$g_{t_1} \triangleq H^{(1)}(t_1) - H^{(2)}(t_1) = 0 \qquad \cdots\cdots\cdots\cdots(47)$$

Therefore, the problem is reduced to finding the boundary conditions $x_1(t_1)$ and $x_3(t_1)$

which guarantee the continuity of $p_1(t)$ and $p_3(t)$ at $t=t_1$, and also to finding the optimal corner time t_1 which satisfies (47).

Once t_1 is assumed, the MPBVP of (45) and (46) can be solved with use of both the ICA and the TDA. Suppose that (45) is linearized as (10). Let $\breve{p}_1(t_1^{\pm})$ and $\breve{p}_3(t_1^{\pm})$ be the values of the solutions p_1 and p_3 at $t=t_1^{\pm}$, respectively, to the TPBVP's with the boundary conditions $x(t_0)=\pi_0$, $x_1(t_1)=\breve{x}_1(t_1)$, $x_2(t_1)=\pi_1$, $x_3(t_1)=\breve{x}_3(t_1)$, and $x(t_2)=\pi_2$. Then from Theorem 1,

$$\begin{bmatrix} \breve{p}_1(t_1^-)-\breve{p}_1(t_1^+) \\ \breve{p}_3(t_1^-)-\breve{p}_3(t_1^+) \end{bmatrix} = \tilde{\Gamma} \begin{bmatrix} \breve{x}_1(t_1) \\ \breve{x}_3(t_1) \end{bmatrix} + \begin{bmatrix} \Gamma_{12}\pi_1 \\ \Gamma_{32}\pi_1 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_3 \end{bmatrix} \qquad \cdots\cdots\cdots\cdots(48)$$

where

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix} = \Gamma, \qquad \tilde{\Gamma} = \begin{bmatrix} \Gamma_{11} & \Gamma_{13} \\ \Gamma_{31} & \Gamma_{33} \end{bmatrix}, \qquad [W_1', W_2', W_3']' = V \qquad \cdots\cdots\cdots\cdots(49)$$

Hence, similarly to Corollary 1.1, the exact solution $X=[x_1'(t_1), x_3'(t_1)]'$ is given by

$$X=\tilde{X}-\tilde{\Gamma}^{-1}\tilde{P} \qquad \cdots\cdots\cdots\cdots(50)$$

where $\tilde{X}=[\breve{x}_1'(t_1), \breve{x}_3'(t_1)]'$, $\tilde{P}=[(\breve{p}_1(t_1^-)-\breve{p}_1(t_1^+))', (\breve{p}_3(t_1^-)-\breve{p}_3(t_1^+))']'$. Then, the linear TPBVP's are solved again with the exact boundary conditions. The ICA is iterated until the solution reduces (15) to zero or a sufficiently small enough unit.

After the first- and the second-level calculations, the corner time t_1 is corrected by (11). The third-level calculation is carried out until g_{t_1} is reduced to zero or a sufficiently small enough unit.

5. Summary of the Algorithm

In this section, we summarize the results obtained above into the form of an algorithm.

Step 1: Set $l=0$, $k=0$. Assume ${}^0t_1$. Let ${}^0\breve{x}_1({}^0t_1)={}^0\breve{x}_3({}^0t_1)=0$, ${}^0y(t)={}^0q(t)=0$, $t\varepsilon[t_0, t_2]$. Choose appropriate values of β, κ, α, and η.

Step 2: Solve the homogeneous part of (45) with the boundary conditions $\tilde{X}=e_\nu$, the ν-th unit vector, $x(t_0)=0$, $x_2(t_1)=0$, and $x(t_2)=0$. Then the difference $\tilde{P}$ represents the ν-th column of $\tilde{\Gamma}$. Calculate $\tilde{\Gamma}^{-1}$.

Step 3: Solve each linear TPBVP of the subarcs with the boundary conditions

Subarc 1:

$$x(t_0)=\pi_0,\ x_1(t_1)={}^k\breve{x}_1(t_1),\ x_2(t_1)=\pi_1,\ x_3(t_1)={}^k\breve{x}_3(t_1)$$

Subarc 2:

$$x_1(t_1)={}^k\breve{x}_1(t_1),\ x_2(t_1)=\pi_1,\ x_3(t_1)={}^k\breve{x}_3(t_1),\ x(t_2)=\pi_2 \qquad \cdots\cdots\cdots\cdots(51)$$

Let us denote the solutions p_1 and p_3 at $t=t_1^{\mp}$ as $\widetilde{p_1(t_1^{\mp})}$ and $\widetilde{p_3(t_1^{\mp})}$, respectively.
Step 4: By (50), determine the exact solution $x(t_1)$. Solve the subarc TPBVP's again with use of $x(t_1)$ and π_0 and π_2. Let us denote the solution as ${}^kx$ and ${}^kp$.
Step 5: If ${}^kG$ of (15) is small enough, proceed to *Step* 6. Otherwise, correct ${}^ky$ and ${}^kq$ by (14), replace ${}^k\widetilde{x_1(t_1)}$ and ${}^k\widetilde{x_3(t_1)}$ by ${}^kx_1(t_1)$ and ${}^kx_3(t_1)$, respectively, and replace k by $k+1$. Then return to *Step* 3.
Step 6: Compute ${}^lg_{t_1}$ by (47). If ${}^lg_{t_1}$ is small enough, the optimum is attained and the calculation is terminated. Otherwise, correct ${}^lt_1$ by (11), replace l by $l+1$, and return to *Step* 2.

6. Examples

Two physical problems are examined to illustrate the applications of the present algorithm. For the numerical integration of the differential equations, the fourth-order Runge-Kutta-Gill scheme is employed, where use is made of one hundred grid-points in the overall interval.

Example 1[2)]

Let us consider the problem of minimizing the functional:

$$J[u]=\int_0^2 u^2 dt \qquad \cdots\cdots\cdots\cdots(52)$$

with respect to the control $u(\cdot)$ and the corner time t_1. The state equations governing the system are

$$\left.\begin{array}{l}\dot{x}_1=x_2\\ \dot{x}_2=u\end{array}\right\} t\,\varepsilon\,[0,\ t_1) \quad\cdots\cdots(53\text{-}1) \qquad \left.\begin{array}{l}\dot{x}_1=x_2\\ \dot{x}_2=2u\end{array}\right\} t\,\varepsilon\,[t_1,\ 2] \quad\cdots\cdots(53\text{-}2)$$

and the boundary conditions are

$$\left.\begin{array}{l}x_1(0)=1\\ x_2(0)=1\end{array}\right\} \quad\cdots\cdots(54\text{-}1) \qquad \left.\begin{array}{l}x_1(2)=0\\ x_2(2)=0\end{array}\right\} \quad\cdots\cdots(54\text{-}2)$$

$$x_1(t_1)=0.5 \qquad \cdots\cdots\cdots\cdots(55)$$

(53) implies that the mass of the article is reduced by half at the corner time t_1.

For this problem, the necessary conditions for optimality are written as follows:

$$\left.\begin{array}{l}\dot{x}_1=x_2\\ \dot{x}_2=-0.5p_2\\ \dot{p}_1=0\\ \dot{p}_2=-p_1\end{array}\right\} t\,\varepsilon\,[0,\ t_1) \quad\cdots\cdots(56\text{-}1) \qquad \left.\begin{array}{l}\dot{x}_1=x_2\\ \dot{x}_2=-2p_2\\ \dot{p}_1=0\\ \dot{p}_2=-p_1\end{array}\right\} t\,\varepsilon\,[t_1,\ 2] \quad\cdots\cdots(56\text{-}2)$$

$$x_1(0)=x_2(0)=1 \quad\cdots\cdots(57\text{-}1) \qquad x_1(2)=x_2(2)=0 \quad\cdots\cdots(57\text{-}2)$$

$$x_1(t_1)=0.5 \qquad \cdots\cdots\cdots\cdots(58)$$

$$p_2(t_1^-)=p_2(t_1^+) \qquad \cdots\cdots\cdots\cdots(59)$$

$$u=-0.5p_2 \quad \cdots\cdots(60\text{-}1) \qquad\qquad u=-p_2 \quad \cdots\cdots(60\text{-}2)$$

$$g_{t_1}=u(t_1^-)^2+p_1(t_1^-)x_2(t_1)+p_2(t_1^-)u(t_1^-)-u(t_1^+)^2-p_1(t_1^+)x_2(t_1)-2p_2(t_1^+)u(t_1^+)=0 \qquad \cdots\cdots\cdots\cdots(61)$$

It is easily shown that the general solutions to (56) are given by

$$\left.\begin{aligned} x_1(t)&=[c_1t^3-3c_2t^2+12c_3t+12c_4]/12\\ x_2(t)&=[c_1t^2-2c_2t+4c_3]/4\\ p_1(t)&=c_1\\ p_2(t)&=-c_1t+c_2 \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots(62\text{-}1)$$

$$\left.\begin{aligned} x_1(t)&=[d_1t^3-3d_2t^2+3d_3t+3d_4]/3\\ x_2(t)&=d_1t^2-2d_2t+d_3\\ p_1(t)&=d_1\\ p_2(t)&=-d_1t+d_2 \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots(62\text{-}2)$$

where $c_1\sim c_4$ and $d_1\sim d_4$ are constants.

Now let us follow the algorithm of Section 5.

Step 1: Let ${}^0t_1=t_1$. Since the problem is linear, it is unnecessary to utilize the ICA.

Step 2: $\tilde{\Gamma}$ is obtained as follows: solve (56) with the boundary conditions $x_1(0)=x_2(0)=x_1(t_1)=x_1(2)=x_2(2)=0$, and $x_2(t_1)=1$. The coefficients c_i and d_i are obtained as follows:

$$\begin{aligned} &c_1=12/t_1^2,\ c_2=4/t_1,\ c_3=c_4=0\\ &d_1=3/(t_1-2)^2,\ d_2=(t_1+4)/(t_1-2)^2,\ d_3=4(t_1+1)/(t_1-2)^2 \qquad \cdots\cdots\cdots\cdots(63)\\ &d_4=-4t_1/(t_1-2)^2 \end{aligned}$$

Hence,

$$\tilde{\Gamma}=\widehat{p_2(t_1^-)}-\widehat{p_2(t_1^+)}=-c_1t_1+c_2+d_1t_1-d_2=-2(3t_1-8)/t_1(t_1-2) \qquad \cdots\cdots\cdots\cdots(64)$$

Step 3: Let $\widehat{x_2(t_1)}=0$. The solutions to (56) with boundary conditions $x_1(0)=x_2(0)=1$, $x_1(t_1)=0.5$, $x_2(t_1)=\widehat{x_2(t_1)}$, and $x_1(2)=x_2(2)=0$ yield

$$\begin{aligned} &c_1=12(t_1+1)/t_1^3,\ c_2=2(4t_1+3)/t_1^2,\ c_3=c_4=1\\ &d_1=-3/(t_1-2)^3,\ d_2=-3(t_1+2)/2(t_1-2)^3,\ d_3=-6t_1/(t_1-2)^3 \qquad \cdots\cdots\cdots\cdots(65)\\ &d_4=(6t_1-4)/(t_1-2)^3 \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{p_2(t_1^-)}-\widehat{p_2(t_1^+)}&=-c_1t_1+c_2+d_1t_1-d_2\\ &=(-8t_1^3+17t_1^2+16t_1-48)/2t_1^2(t_1-2)^2 \qquad \cdots\cdots\cdots\cdots(66) \end{aligned}$$

Step 4: Substitution of (64) and (66) into (50) yields

$$x_2(t_1) = \widetilde{x_2(t_1)} - \tilde{\Gamma}^{-1}[\widetilde{p_2(t_1^-)} - \widetilde{p_2(t_1^+)}]$$
$$= (-8t_1^3+17t_1^2+16t_1-48)/4t_1(t_1-2)(3t_1-8) \qquad \cdots\cdots\cdots\cdots (67)$$

Again solve the TPBVP's (56) with the boundary conditions thus obtained. Then the constants are obtained as follows:

$$c_1 = (12t_1^3-81t_1^2+72t_1+48)/t_1^2(t_1-2)(3t_1-8)$$
$$c_2 = (16t_1^3-77t_1^2+60t_1+48)/t_1^2(t_1-2)(3t_1-8)$$
$$c_3 = c_4 = 1 \qquad \cdots\cdots\cdots\cdots (68)$$
$$d_1 = 3(-8t_1^3+5t_1^2+48t_1-48)/4t_1(t_1-2)^3(3t_1-8)$$
$$d_2 = (-8t_1^4-33t_1^3+96t_1^2+112t_1-192)/4t_1(t_1-2)^3(3t_1-8)$$
$$d_3 = (-8t_1^4-9t_1^3+81t_1^2-32t_1-48)/t_1(t_1-2)^3(3t_1-8)$$
$$d_4 = (8t_1^3+t_1^2-76t_1+8)/(t_1-2)^3(3t_1-8)$$

Step 5: This step is skipped since the ICA is not utilized.

Step 6: The gradient g_{t_1} is given by

$$g_{t_1} = H^{(1)}(t_1) - H^{(2)}(t_1) = \frac{3}{4}[p_2(t_1)]^2 + [p_1(t_1^-) - p_1(t_1^+)]x_2(t_1)$$
$$= \frac{3}{4}(-c_1t_1+c_2)^2 + (c_1-d_1)x_2(t_1) \qquad \cdots\cdots\cdots\cdots (69)$$

with (67) and (68). (69) is a rational function of t_1. It is not difficult to solve $g_{t_1}=0$ with the aid of a digital computer. Figure 1 shows the dependence of g_{t_1} and J on t_1. It can be read off the figure that the optimal t_1 is 1.377....

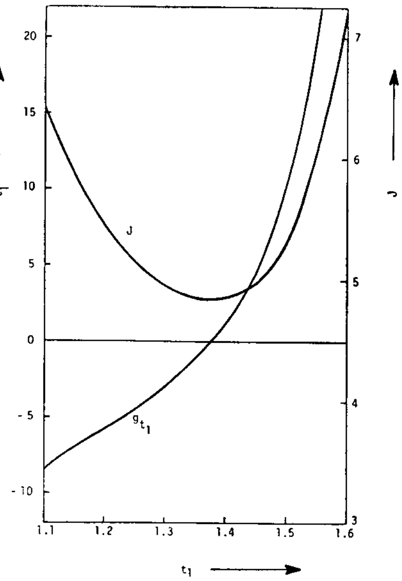

Fig. 1. Dependence of g_{t_1} and J on the corner time t_1.

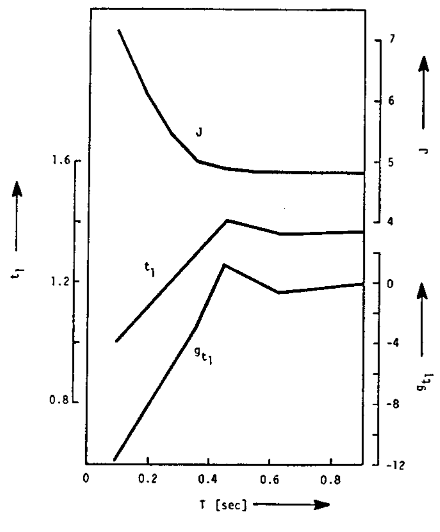

Fig. 2. Variations of t_1, g_{t_1}, and J with the computing time T.

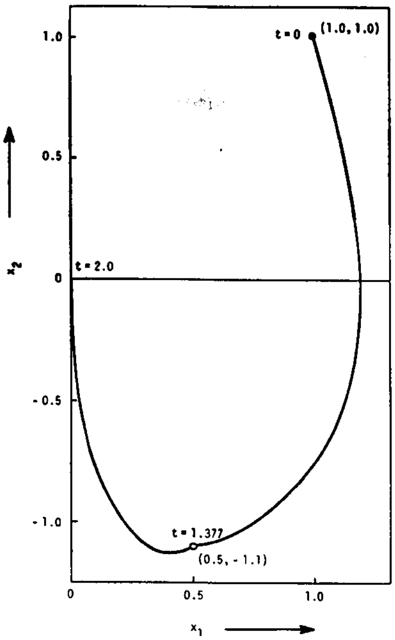

Fig. 3. The optimal trajectory on the x_1-x_2 plane.

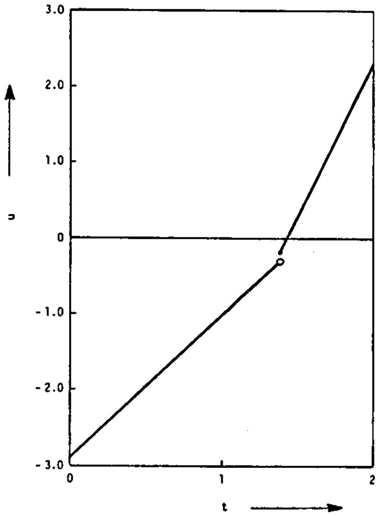

Fig. 4. Time history of the optimal control u.

The algorithm is also carried out on a digital computer. The initial estimate of t_1 is 1.0 and the step size η in (11) is set to 0.1. When the sign of g_{t_1} is changed, η is reduced by a fifth. Figure 2 shows the variations of t_1, g_{t_1}, J with the computing time T. After 16 iterations they converged. As seen from Fig. 2, a too severe criterion for the optimality with respect to t_1 contributes only to the computing time.

Figures 3 and 4 show the optimal trajectory on the x_1-x_2 plane and the time history of the optimal control u, respectively.

Example 2[13)]

Next, we consider the problem whose system equation contains nonlinearities. The problem is a discontinuous version of the three-axis attitude-control problem. The system equations are described by

$$\begin{aligned} &\dot{x}_1=x_2, \quad \dot{x}_2=\varepsilon(x_4+x_4x_6+x_3u_3)+u_1 \\ &\dot{x}_3=x_4, \quad \dot{x}_4=-\varepsilon(x_2+x_2x_6+x_1u_3)+u_2 \\ &\dot{x}_5=x_6, \quad \dot{x}_6=\varepsilon(x_2x_4+x_1u_2)+u_3 \end{aligned} \qquad \cdots\cdots\cdots\cdots(70)$$

Suppose that the parameter ε changes discontinuously from 1.0 to 2.0 at $t=t_1$ at which $x_1(t_1)=0.0$, $x_3(t_1)=0.15$, and $x_5(t_1)=0.055$ are to be satisfied. The objective is to find the control u and the corner time t_1 which minimize

$$J=\frac{1}{2}\int_0^5\left(\sum_{i=1}^{6}x_i^2+\sum_{i=1}^{3}u_i^2\right)dt \qquad \cdots\cdots\cdots\cdots(71)$$

starting from $x(0)=[1, 0, 1, 0, 1, 0]'$. The problem is decomposed into the three subsystems of $[x_1, x_2]$, $[x_3, x_4]$, and $[x_5, x_6]$. For solving one of the subsystems, the variables of the other subsystems and those in the nonlinear terms are replaced by interaction vectors. The necessary condition for Subsystem 1 is described by[7]

$$\begin{aligned}
\dot{x}_1&=x_2\\
\dot{x}_2&=-\beta p_2+(\beta-1)q_2+\varepsilon[y_4(1+y_6)-y_3(\varepsilon y_3q_2-\varepsilon y_1q_4+q_6)]\\
\dot{p}_1&=-\kappa x_1+(\kappa-1)y_1+\varepsilon^2[-y_6q_2q_4+y_1(q_4^2+q_6^2)]\\
\dot{p}_2&=-\kappa x_2-p_1+(\kappa-1)y_2+\varepsilon[q_4(1+y_6)-y_4q_6]
\end{aligned} \qquad \cdots\cdots\cdots\cdots(72)$$

Similar problems obtained for subsystems 2 and 3 are omitted here.

Figure 5 shows the variations of t_1, g_{t_1}, and J with the iteration number l for correcting the corner time t_1. It takes 4.8 seconds to attain convergence. The parameters chosen are $\beta=\kappa=1$, $\alpha=1.0$, $\eta=0.5$, ${}^0t_1=2.5$. η is reduced by half when the sign of g_{t_1} is changed.

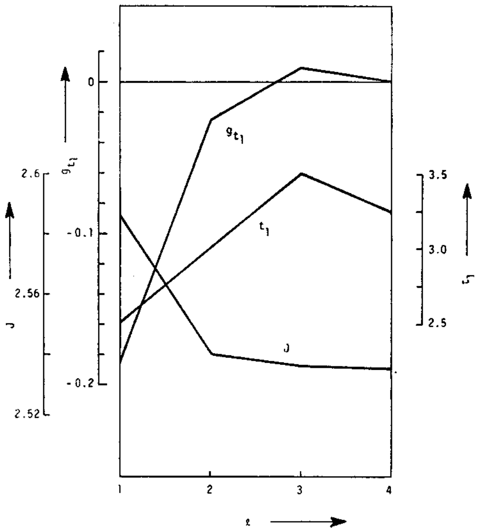

Fig. 5. Variations of t_1, g_{t_1}, and J with the iteration number l.

7. Conclusion

A multipoint boundary-value problem with discontinuities in system equations is solved by a combined use of the interaction-coordination algorithm (ICA) and the time-decomposition algorithm (TDA). The TDA is an extension of the original one for a linear TPBVP.

The algorithm can effectively deal with nonlinear problems and the convergence is quite rapid. The solution satisfies the specified boundary conditions exactly because the idea of a penalty function is not employed.

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References

1) Swiger, J. M.: Application of the Theory of Minimum-Normed Operator to Optimum Control System Problems, in Advances in Control Systems, **3**, 151/218, Academic Press, New York, 1966.
2) Tomizuka, M. and Tsujioka, Y.: Formulation of a Multiple Targets Problem and Its Analysis, Trans. Soc. Instrument and Control Engineers, **6**, No. 3, 228/233 (1970, in Japanese).
3) Dyer, P. and McReynolds, S. P.: "The Computation and Theory of Optimal Control," Academic Press, New York, 1970.
4) Tamura, K.: A Numerical Approach to the Optimal Control Problem with Discontinuity, Trans. Soc. Instrument and Control Engineers, **8**, No. 5, 600/605 (1972, in Japanese).
5) Bryson, A. E. and Denham, W. F.: A Steepest Ascent Method for Solving Optimum Programming Problems, J. Appl. Mech., Ser. E, **29**, 247/257 (1962).
6) Nishikawa, Y., Ojika, T., Shimazutsu, H., and Okudaira, M.: An Interaction-Coordination Algorithm with Modified Performance Index for Nonlinear Optimal Control Problems, JOTA, **25**, No. 2, 229/245 (1978).
7) Ojika, T., Nishikawa, Y., Shimazutsu, H., and Okudaira, M.: An Interaction-Coordination Algorithm with Modified Performance Index for Optimal Control of Coupled Nonlinear Systems, Trans. Soc. Instrument and Control Engineers, **13**, No. 5, 463/469 (1977, in Japanese).
8) Ojika, T., Nishikawa, Y., Okudaira, M., and Sera, T.: Decomposition and Coordination Algorithms for the Optimal Control of Nonlinear Systems II, Preprints of 14th Convention of Soc. Instrument and Control Engineers, 45/46 (1975, in Japanese).
9) Bryson, A. E. and Ho, Y. C.: "Applied Optimal Control," Blaisdell Publishing Co., 1969.
10) McGill, R. and Kenneth, P.: Solution of Variational Problems by Means of a Generalized Newton-Raphson Operator, AIAA J., **2**, No. 10, 1761/1766 (1964).
11) Baird, C. A., Jr.: Modified Quasilinearization Technique for the Solution of Boundary-Value Problems for Ordinary Differential Equations, JOTA, **3**, No. 4, 227/242 (1969).
12) Mataušek, M. R.: Direct Shooting Method for the Solution of Boundary-Value Problems, Ibid, **12**, No. 2, 152/172 (1973).
13) Nishikawa, Y., Hayashi, C., and Sannomiya, N.: Fuel and Energy Minimization in Three Dimensional Attitude Control of an Orbiting Satellite, Proc. IFAC Symp. on Peaceful Uses of Automation in Outer Space, Stravanger, Norway, 287/298 (1965).