An Interaction-Coordination Algorithm with Time- and State-Decompositions for Nonlinear Optimal Control Problems

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Abstract

This paper proposes a computational algorithm for the multilevel control of a composite nonlinear dynamical system with the performance index of a quadratic type.

First, for the solution of a linear two-point boundary-value problem (TPBVP), a computational technique, termed time-decomposition, is proposed. The time-decomposition implies decomposition of the system equations along the subintervals of the independent variable. The boundary conditions of each subinterval are determined algebraically in such a way that the continuity condition of the variables at the boundary points is satisfied. This technique plays an important role in the subsequent discussions.

Second, a nonlinear optimal control problem is considered. The objective is to minimize the performance index of a quadratic type. For this problem, the authors have previously presented 'the interaction-coordination algorithm with modified performance index.' The basic idea of this algorithm is to decompose the overall nonlinear problem into a number of smaller linear subproblems. Here this kind of decomposition is termed state-decomposition.

In the present paper, a computational algorithm by use of both the time- and the state-decomposition is proposed. The algorithm essentially constructs a three-level computational structure, and results in a fast convergence even for problems with strong nonlinearities and/or a long control interval.

1. Introduction

A fundamental idea for the study of composite systems is to decompose the overall problem into a number of smaller and simpler subproblems which can be dealt with by some conventional mathematical or computational means. The subsystems

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have to be coordinated later to reconstruct the overall system, and then the decomposition and coordination algorithm inevitably involves a multilevel structure (Refs. 1-3).

This paper proposes two approaches for the multilevel control of composite nonlinear dynamical systems. That is to say, first, the two-point boundary-value problem (TPBVP) for a linear ordinary differential equation is considered. In general, a linear TPBVP derived from an optimal control problem can be solved by the principle of superposition (Refs. 4, 5). However, for a problem with a 'stiff' structure (Ref. 6) or a problem whose control interval is quite large, it is often difficult for the principle of superposition to satisfy the given terminal conditions because of numerical errors such as round-off errors.

In order to overcome the difficulty, a 'time-decomposition' method is proposed (Ref. 3). This decomposition implies the division of the system equations along the subintervals of the independent variable, i.e., time. A time instant, at which the interval is decomposed, is termed a 'torn time', and the system in each subinterval is called 'subarc' (Ref. 7). The principal idea for the decomposition proposed here is to determine algebraically the values of a set of state variables at a torn time so that the continuity of the remaining state variables at this time is always kept.

Second, a nonlinear optimal control problem is considered. The objective is to minimize the performance index of a quadratic type. For this problem, the interaction-coordination algorithm with use of a modified performance index has been proposed previously by the authors (Ref. 8). In the algorithm, the nonlinearities and interactions are removed from the system variables, and the overall problem is decomposed into a number of linear subproblems. This decomposition is termed here a 'state-decomposition'.

Based upon the above mentioned approaches, the present paper proposes a computational algorithm with the time- and the state-decomposition. The algorithm essentially constructs a three-level structure. The first level calculates the linear subproblems of each subsystem in each subinterval. The second level determines algebraically the boundary values of the states at the torn times. The third level adjusts the interaction variables directly by a gradient algorithm with a constant step size until the interaction balances are attained (Refs. 2, 8).

The algorithm results in a fast convergence even for problems with strong nonlinearities and/or a long control interval. Several features of the algorithm are illustrated by examining an example.

2. Time-Decomposition Method for a Linear TPBVP

According to the variational principle, a nonlinear optimal control problem is

reduced to solve a nonlinear TPBVP. However, since usually the nonlinear TPBVP can not be solved analytically or even numerically, one has to resort to linear approximation techniques with the capabilities of a digital computer in various fashions.

In this section, a computational technique with time-decomposition for solving a linear TPBVP is considered (Ref. 3).

2.1. The Principle of Superposition

Consider a linear dynamical system governed by the differential equation

$$\begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix}=\begin{bmatrix}D_{11}(t), & D_{12}(t)\\ D_{21}(t), & D_{22}(t)\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix}+\begin{bmatrix}h_1(t)\\ h_2(t)\end{bmatrix} \tag{1}$$

where both $x(t)$ and $p(t)$ are n-dimensional vectors, $D_{ij}(t)$ $(i,j=1,2)$ is an $n\times n$-dimensional matrix continuous in $t\ \varepsilon[t_0, t_f]$, and $h_i(t)$ $(i=1, 2)$ is an n-dimensional vector function continuous in t. The boundary conditions of (1) are given by

$$x(t_0)=x_0,\quad p(t_f)=p_f \tag{2}$$

Let $\Phi(t, t_0)$ be the $2n\times 2n$-dimensional transition matrix corresponding to the homogeneous part of (1) with the initial condition $\Phi(t_0, t_0)=I_{2n}$, where I_{2n} is the $2n\times 2n$-dimensional identity matrix. Then the solution of (1) subject to a set of initial conditions $[x'(t_0), p'(t_0)]'$ can be written as

$$\begin{bmatrix}x(t)\\ p(t)\end{bmatrix}=\Phi(t, t_0)\begin{bmatrix}x(t_0)\\ p(t_0)\end{bmatrix}+v(t, t_0) \tag{3}$$

where

$$\Phi(t, t_0)\triangleq\begin{bmatrix}\Phi_{11}(t, t_0), & \Phi_{12}(t, t_0)\\ \Phi_{21}(t, t_0), & \Phi_{22}(t, t_0)\end{bmatrix}$$

$$v(t, t_0)=\begin{bmatrix}v_1(t, t_0)\\ v_2(t, t_0)\end{bmatrix}\triangleq\int_{t_0}^{t}\Phi(t, \tau)\,h(\tau)\,d\tau \tag{4}$$

$$h(t)=\begin{bmatrix}h_1(t)\\ h_2(t)\end{bmatrix}$$

Let $\hat{p}_0$ be an initial approximation to the initial condition of $p(t_0)$. Then, from (3) with $p(t_0)=\hat{p}_0$ and $x(t_0)=x_0$, the corresponding terminal value $\hat{p}_f\triangleq p(t_f)$ is given by

$$\hat{p}_f=\Phi_{21}(t_f, t_0)\,x_0+\Phi_{22}(t_f, t_0)\,\hat{p}_0+v_2(t_f, t_0) \tag{5}$$

Similarly, let p_0 be the exact initial condition of p which satisfies the given terminal condition $p(t_f)=p_f$. Then, from (3) with $p(t_0)=p_0$ and $x(t_0)=x_0$, we have

$$p_f=\Phi_{21}(t_f, t_0)\,x_0+\Phi_{22}(t_f, t_0)\,p_0+v_2(t_f, t_0) \tag{6}$$

Subtracting (5) from (6) gives

$$\Phi_{22}(t_f, t_0)\,(p_0-\hat{p}_0)=p_f-\hat{p}_f \tag{7}$$

The results obtained above are summarized in the following lemma.

Lemma 1.

Let $\hat{p}_f$ be the terminal value of p obtained from (3) with a set of initial conditions $[x_0', \hat{p}_0']'$, where $\hat{p}_0$ is an arbitrary n-dimensional vector. If the matrix $\Phi_{22}(t_f, t_0)$ is nonsingular, then, from (7) the exact initial condition of p is given by

$$p_0 = \hat{p}_0 + \Phi_{22}^{-1}(t_f, t_0)\,(p_f - \hat{p}_f) \tag{8}$$

2.2. Time-Decomposition

In this subsection, a computational technique, termed a time-decomposition, is proposed in order to overcome the difficulties caused by numerical errors in applying the superposition principle. The time-decomposition is to decompose the control interval into several subintervals. In the following discussion, for simplicity, only two subintervals $[t_0, t_1]$ and $[t_1, t_f]$ are considered, where t_1 is the torn time such that $t_1 \varepsilon (t_0, t_f)$. Generalization to an arbitrary number of subintervals is straightforward.

Suppose now that $\hat{x}(t_1)$ be an estimated value of x at the torn time t_1. Then, the boundary conditions for Subarc 1 in the subinterval $[t_0, t_1]$ are given by $x(t_0) = x_0$ and $x(t_1) = \hat{x}(t_1)$, and for Subarc 2 in the subinterval $[t_1, t_f]$, $x(t_1) = \hat{x}(t_1)$ and $p(t_f) = p_f$. Hence, from (3), the solution in each subinterval subject to these boundary conditions can be written as follows:

Subarc 1:

$$\hat{x}(t_1) = \Phi_{11}(t_1, t_0)\,x_0 + \Phi_{12}(t_1, t_0)\,\hat{p}(t_0) + v_1(t_1, t_0) \tag{9.1}$$

$$\hat{p}^{(1)}(t_1) = \Phi_{21}(t_1, t_0)\,x_0 + \Phi_{22}(t_1, t_0)\,\hat{p}(t_0) + v_2(t_1, t_0) \tag{9.2}$$

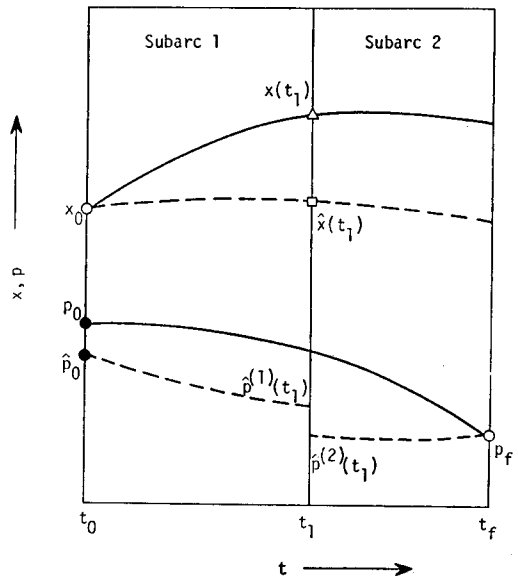

Fig. 1. Solutions by Time-decomposition.

Subarc 2:

$$x(t_f)=\Phi_{11}(t_f,t_1)\hat{x}(t_1)+\Phi_{12}(t_f,t_1)\hat{p}^{(2)}(t_1)+v_1(t_f,t_1) \tag{10.1}$$

$$p_f=\Phi_{21}(t_f,t_1)\hat{x}(t_1)+\Phi_{22}(t_f,t_1)\hat{p}^{(2)}(t_1)+v_2(t_f,t_1) \tag{10.2}$$

where $\hat{p}^{(l)}(t_1)$ denotes the value of the variable p at t_1 in the l-th subarc ($l=1, 2$). For the boundary condition $\hat{x}(t_1)$ prescribed, the continuity condition for p at $t=t_1$, i.e., $\hat{p}^{(1)}(t_1)-\hat{p}^{(2)}(t_1)=0$, is not always satisfied (see Fig. 1).

By calculating the difference $\hat{p}^{(1)}(t_1)-\hat{p}^{(2)}(t_1)$, let us now determine the exact value of x at $t=t_1$, with which both the given boundary conditions and the continuity condition of p at $t=t_1$ are satisfied.

The following theorem plays an important role in the discussion.

Theorem 1.

Suppose that $\Phi_{12}(t_1,t_0)$ and $\Phi_{22}(t_f,t_1)$, the element matrices of the transition matrix Φ, are nonsingular. Then the following relation holds:

$$\hat{p}^{(1)}(t_1)-\hat{p}^{(2)}(t_1)=\Gamma\hat{x}(t_1)+\Delta_0 x_0+\Delta_f p_f+\delta \tag{11}$$

where

$$\left.\begin{aligned}
&\Gamma\triangleq\Phi_{22}(t_1,t_0)\Phi_{12}^{-1}(t_1,t_0)+\Phi_{22}^{-1}(t_f,t_1)\Phi_{21}(t_f,t_1)\\
&\Delta_0\triangleq\Phi_{21}(t_1,t_0)-\Phi_{22}(t_1,t_0)\Phi_{12}^{-1}(t_1,t_0)\Phi_{11}(t_1,t_0)\\
&\Delta_f\triangleq-\Phi_{22}^{-1}(t_f,t_1)\\
&\delta\triangleq-\Phi_{22}(t_1,t_0)\Phi_{12}^{-1}(t_1,t_0)v_1(t_1,t_0)+\Phi_{22}^{-1}(t_f,t_1)v_2(t_f,t_1)+v_2(t_1,t_0)
\end{aligned}\right\} \tag{12}$$

Proof.

Since $\Phi_{12}(t_1,t_0)$ is nonsingular by assumption, it follows from (9.1) that

$$\hat{p}(t_0)=\Phi_{12}^{-1}(t_1,t_0)[\hat{x}(t_1)-\Phi_{11}(t_1,t_0)x_0-v_1(t_1,t_0)] \tag{13}$$

Substituting (13) into (9.2) leads to

$$\begin{aligned}\hat{p}^{(1)}(t_1)=&\Phi_{21}(t_1,t_0)x_0+\Phi_{22}(t_1,t_0)\Phi_{12}^{-1}(t_1,t_0)\\&[\hat{x}(t_1)-\Phi_{11}(t_1,t_0)x_0-v_1(t_1,t_0)]+v_2(t_1,t_0)\end{aligned} \tag{14}$$

On one hand, since $\Phi_{22}(t_f,t_1)$ is also nonsingular by assumption, we have from (10.2)

$$\hat{p}^{(2)}(t_1)=\Phi_{22}^{-1}(t_f,t_1)[p_f-\Phi_{21}(t_f,t_1)\hat{x}(t_1)-v_2(t_f,t_1)] \tag{15}$$

On taking (12) into account, subtracting (15) from (14) gives (11) Q. E. D.

Let us now discuss the nonsingularity of Γ in (12). To begin with, we state the following lemma.

Lemma 2.

For an arbitrary $t_1\varepsilon(t_0,t_f)$, the matrix $\Phi_{ij}(t_f,t_0)$ $(i,j=1,2)$ satisfies the following relation:

$$\Phi_{ij}(t_f,t_0)=\sum_{k=1}^{2}\Phi_{ik}(t_f,t_1)\Phi_{kj}(t_1,t_0) \tag{16}$$

Proof.

From the transition property of Φ, we have

$$\Phi(t_f, t_0) \not\Delta [\Phi_{ij}(t_f, t_0)] = \Phi(t_f, t_1)\Phi(t_1, t_0) = [\Phi_{ik}(t_f, t_1)][\Phi_{kj}(t_1, t_0)] \tag{17}$$

Hence, the expansion of (17) yields (16). Q. E. D.

Lemma 2 is sufficient to establish the following theorem.

Theorem 2.

If the three matrices $\Phi_{12}(t_1, t_0)$, $\Phi_{22}(t_f, t_1)$, and $\Phi_{22}(t_f, t_0)$ are nonsingular, then Γ in (12) is also nonsingular and the following relation holds:

$$\Gamma^{-1} = \Phi_{12}(t_1, t_0)\Phi_{22}^{-1}(t_f, t_0)\Phi_{22}(t_f, t_1) \tag{18}$$

Proof.

By virtue of Lemma 2, it follows that

$$\Phi_{22}(t_f, t_0) = \Phi_{21}(t_f, t_1)\Phi_{12}(t_1, t_0) + \Phi_{22}(t_f, t_1)\Phi_{22}(t_1, t_0) \tag{19}$$

Multiply (19) on the left by $\Phi_{22}^{-1}(t_f, t_1)$ and on the right by $\Phi_{12}^{-1}(t_1, t_0)$. Then we have

$$\Phi_{22}^{-1}(t_f, t_1)\Phi_{22}(t_f, t_0)\Phi_{12}^{-1}(t_1, t_0) = \Phi_{22}^{-1}(t_f, t_1)\Phi_{21}(t_f, t_1) + \Phi_{22}(t_1, t_0)\Phi_{12}^{-1}(t_1, t_0) = \Gamma \tag{20}$$

Since the left hand side of (20) is nonsingular by assumption, Γ is also nonsingular, and then the inverse of (20) gives (18). Q. E. D.

Thus the nonsingularity of $\Phi_{22}(t_f, t_0)$ is necessary and sufficient for the existence of a unique solution to the linear TPBVP of (1) and (2). Similarly, the nonsingularity of $\Phi_{12}(t_1, t_0)$ and $\Phi_{22}(t_f, t_1)$ is necessary and sufficient for the unique existence of the two subinterval solutions.

Theorem 1 shows that the values of Γ and $\Delta_0 x_0 + \Delta_f p_f + \delta$ are independent of the choice of the value $\hat{x}(t_1)$. Hence, from Theorems 1 and 2 we have the following.

Theorem 3.

Suppose that the matrices $\Phi_{12}(t_1, t_0)$, $\Phi_{22}(t_f, t_1)$, and $\Phi_{22}(t_f, t_0)$ are all nonsingular. Then, by making use of $\hat{x}(t_1)$, $\hat{p}^{(1)}(t_1)$, $\hat{p}^{(2)}(t_1)$, and Γ in Theorem 1, the exact solution $x(t_1)$ at the torn time is expressed as

$$x(t_1) = \hat{x}(t_1) - \Gamma^{-1}[\hat{p}^{(1)}(t_1) - \hat{p}^{(2)}(t_1)] \tag{21}$$

Proof.

Since p at t_1 is continuous with the exact value of $x(t_1)$, we have from Theorem 1 that

$$\Gamma x(t_1) + \Delta_0 x_0 + \Delta_f p_f + \delta = 0 \tag{22}$$

As Γ is nonsingular from Theorem 2, by subtracting (11) from (22) and then by operating Γ^{-1} to both sides of the resulting equation, it will yield (21). Q. E. D.

Once the value of the transition matrix is obtained, Γ^{-1} could be calculated directly from (18). In practice, however, the following procedure of calculation is more efficient to reduce numerical errors. That is to say, (i) for the linear TPBVP of (1) and (2), set $h(t) = 0$, $x_0 = p_f = 0$, and $\hat{x}(t_1)$ equal to the ν-th unit vector ($\nu = 1, 2, \ldots, n$), (ii) obtain $\hat{p}^{(1)}(t_1)$ and $\hat{p}^{(2)}(t_1)$ by solving the linear TPBVP with the corresponding boundary conditions for each subinterval, (iii) let the difference

$\hat{p}^{(1)}(t_1)-\hat{p}^{(2)}(t_1)$ be the ν-th column of Γ ($\nu=1, 2, \ldots, n$), and then calculate Γ^{-1}.

3. Nonlinear Optimal Control Problems

In this section, an iterative algorithm for solving large-scale optimal control problems with strong nonlinearities and/or a long control interval is considered. In the algorithm, both the time-decomposition method discussed in the previous section and the modified interaction-coordination algorithm proposed previously by the authors (Ref. 8) are employed.

3. 1. Problem Statement

Consider the nonlinear control system defined by

$$\dot{x}=A(t;\varepsilon)x+B(t)u+\varepsilon f(t,x),\quad x(t_0)=x_0,\quad x(t_f):\text{free} \tag{23}$$

where x is the n-dimensional state variable, and u the m-dimensional control variable unbounded. A and B are $n\times n$- and $n\times m$-matrices, respectively, continuous in time t, and f is a nonlinear vector function of the class C^2. The initial time t_0 and the final time t_f are assumed to be fixed.

Let the performance index for the system (23) be a quadratic form:

$$J=\frac{1}{2}\int_{t_0}^{t_f}[x'Q(t;\varepsilon)x+u'R(t)u]dt \tag{24}$$

where the matrices Q and R, both continuous in t, are positive semidefinite and positive definite, respectively.

The parameter ε represents both interconnections among the subsystems and system nonlinearities. That is to say, when $\varepsilon=0$ the problem of (23) and (24) is decomposed into several linear subproblems independent of each other. The decomposition is termed here state-decomposition to differentiate it from the time-decomposition in the previous section.

For simplicity, a system composed of only two subsystems is considered here. Its partitioned form is

$$\begin{aligned}
&x=(x_1', x_2')',\quad u=(u_1', u_2')'\\
&f(t,x)=[f_1'(t,x_1,x_2), f_2(t,x_1,x_2)]'\\
&A(t;\varepsilon)=\begin{bmatrix}A_{11}, \varepsilon A_{12}\\ \varepsilon A_{21}, A_{22}\end{bmatrix},\quad Q(t;\varepsilon)=\begin{bmatrix}Q_{11}, \varepsilon Q_{12}\\ \varepsilon Q_{12}, Q_{22}\end{bmatrix}\\
&B(t)=\operatorname{diag}(B_{11}, B_{22}),\quad R(t)=\operatorname{diag}(R_{11}, R_{22})
\end{aligned} \tag{25}$$

where x_1 and x_2 are n_1- and n_2-dimensional substates, $n_1+n_2=n$, and u_1 and u_2 are m_1- and m_2-dimensional subcontrols, $m_1+m_2=m$. Generalization to an arbitrary number of subsystems is straightforward.

3. 2. Modified Interaction-Coordination Algorithm

By introducing a vector variable $w=(y', q')'$, called an interaction variable, we replace the performance index of (24) by the following one:

$$\hat{J}=\frac{1}{2}\int_{t_0}^{t_f}[\kappa x'Qx-(\kappa-1)y'Qy+(1/\beta)u'Ru+(1-1/\beta)q'Eq]dt \tag{26}$$

where both y and q are n-dimensional variables, $E\triangleq BR^{-1}B'$, and κ and β are positive scalar parameters called the weights. The choice of the values of the weights affects significantly the convergence rate of the algorithm; and hence it is one of the major factors determining the total computing time (Ref. 8).

A necessary condition for the optimality of the problem is derived by using the variational principle. Define the Hamiltonian of the problem of (23) and (26):

$$\begin{aligned}H\triangleq&(1/2)[\kappa x'Qx-(\kappa-1)y'Qy+(1/\beta)u'Ru+(1-1/\beta)q'Eq]\\&+p'(Ax+Bu+\varepsilon f)+\lambda'(x-y)+\pi'(u+R^{-1}B'q)\end{aligned} \tag{27}$$

where p is the n-dimensional costate variable, λ and π are Lagrange multipliers. For optimality, the following relations are to be satisfied:

$$H_u=H_y=H_q=0' \tag{28}$$

$$\dot{p}=-(\partial H/\partial x)',\ p(t_f)=0 \tag{29}$$

The solution to the nonlinear TPBVP given by (23), (28), and (29) coincides with the solution of the original problem (Ref. 8). However, for problems with strong interactions and/or nonlinearities, the choice of the weights is somewhat limited because of the numerical errors.

3. 3. Three-Level Solution Procedure

In order to overcome the disadvantage noticed above, a computational algorithm with both the time-decomposition and the modified interaction-coordination is considered. The algorithm constructs essentially a three-level computational scheme.

(a) The First-Level Calculation

By substituting (28) into (23) and (29), and then by replacing x and p in the nonlinear terms by y and q, respectively, we obtain the following linear TPBVP for Subsystem i:

$$\begin{bmatrix}\dot{x}_i(t)\\ \dot{p}_i(t)\end{bmatrix}=\begin{bmatrix}A_{ii}(t), & -\beta E_{ii}(t)\\ -\kappa Q_{ii}(t), & -A'_{ii}(t)\end{bmatrix}\begin{bmatrix}x_i(t)\\ p_i(t)\end{bmatrix}+\begin{bmatrix}h_{1i}(t,y,q)\\ h_{2i}(t,y,q)\end{bmatrix} \tag{30}$$

$$x_i(t_0)=x_{i0},\quad p_i(t_f)=0$$

where

$$\begin{aligned}\begin{bmatrix}h_{1i}\\ h_{2i}\end{bmatrix}=&\begin{bmatrix}0, & (\beta-1)E_{ii}(t)\\ (\kappa-1)Q_{ii}(t), & 0\end{bmatrix}\begin{bmatrix}y_i(t)\\ q_i(t)\end{bmatrix}\\&+\varepsilon\begin{bmatrix}A_{ij}(t), & -E_{ij}(t)\\ -Q_{ij}(t), & -A'_{ji}(t)\end{bmatrix}\begin{bmatrix}y_j(t)\\ q_j(t)\end{bmatrix}\\&+\varepsilon\begin{bmatrix}f_i(t,y_1,y_2)\\ -(\partial f_i/\partial y_i)'q_i-(\partial f_i/\partial y_j)'q_j\end{bmatrix}(i,j=1,2;i\neq j)\end{aligned} \tag{31}$$

Once y and q are given from the higher level, the linear system (30) can be

solved by the time-decomposition algorithm in Section 2. In addition, we assume here that the boundary condition of x at the torn time, that is, ${}^k\hat{x}_i(t_1)$ $(i=1,2)$ is also given from the higher level. Then the boundary conditions for Subsystem i in each subarc are given by

$$\begin{aligned} &{}^kx_i^{(1)}(t_0)=x_{i0}, \quad {}^kx_i^{(1)}(t_1)={}^k\hat{x}_i(t_1), \quad \text{for Subarc 1} \\ &{}^kx_i^{(2)}(t_1)={}^k\hat{x}_i(t_1), \quad {}^kp_i(t_f)=0, \quad \text{for Subarc 2} \end{aligned} \tag{32}$$

where k denotes the iteration number.

(b) The Second-Level Calculation

After the first-level calculation, the continuity condition for the corresponding costate variable at the torn time t_1, that is, ${}^k\xi_i(t_1) \triangleq {}^k\hat{p}_i^{(1)}(t_1) - {}^k\hat{p}_i^{(2)}(t_1)=0$ $(i=1,2)$, is not always satisfied. Consequently, the task of the second level is to reduce the cost function ${}^kG_2$ defined by

$${}^kG_2 \triangleq \left[(1/n)\sum_{i=1}^{2} {}^k\xi_i'(t_1)\,{}^k\xi_i(t_1)\right]^{1/2} \tag{33}$$

to zero or to a sufficiently small unit. To this end, the algorithm of (21) is adopted to obtain the exact solution ${}^kx_i(t_1)$ at the torn time, i.e.,

$${}^kx_i(t_1)={}^k\hat{x}_i(t_1)-\Gamma_i^{-1}\left[\hat{p}_i^{(1)}(t_1)-\hat{p}_i^{(2)}(t_1)\right] \quad (i=1,2) \tag{34}$$

(c) The Third-Level Calculation

After the calculation in the first and the second levels, the subproblem solutions do not necessarily satisfy the interaction balance:

$${}^kr_i(t) \triangleq \begin{bmatrix} {}^kx_i(t)-{}^ky_i(t) \\ {}^kp_i(t)-{}^kq_i(t) \end{bmatrix} = 0, \quad \begin{matrix} t\in[t_0,t_f] \\ (i=1,2) \end{matrix} \tag{35}$$

The task of the third level is to correct the interaction variables y_i and q_i so as to satisfy the interaction balance (35).

To this end, the cost function at the third level is introduced:

$$\begin{aligned} {}^kG_3 \triangleq \Big\{(1/2n)\sum_{i=1}^{2}\Big[&(1/(t_1-t_0))\int_{t_0}^{t_1} {}^kr_i^{(1)\prime}(t)\,{}^kr_i^{(1)}(t)\,dt \\ &+(1/(t_f-t_1))\int_{t_1}^{t_f} {}^kr_i^{(2)\prime}(t)\,{}^kr_i^{(2)}(t)\,dt\Big]\Big\}^{1/2} \end{aligned} \tag{36}$$

The goal is to adjust y_i and q_i so as to reduce (36) to zero or a sufficiently small unit. A gradient algorithm is adopted to obtain ${}^{k+1}y_i$ and ${}^{k+1}q_i$, the $(k+1)$-th iterates:

$$\begin{bmatrix} {}^{k+1}y_i(t) \\ {}^{k+1}q_i(t) \end{bmatrix} = \begin{bmatrix} {}^ky_i(t) \\ {}^kq_i(t) \end{bmatrix} + \alpha \begin{bmatrix} {}^kx_i(t)-{}^ky_i(t) \\ {}^kp_i(t)-{}^kq_i(t) \end{bmatrix} \quad (i=1,2) \tag{37}$$

where $\alpha(>0)$ is an appropriate constant step size. The initial estimate ${}^0y_i(t)$ and ${}^0q_i(t)$ is set to ${}^0y_i(t)={}^0q_i(t)=0$, $t\in[t_0,t_f]$ $(i=1,2)$.

3. 4. Computational Algorithm

We now summarize the above discussion in the form of an algorithm.

Step 0-1: Set $h_{ji}(t,y,q)=0$, $t\varepsilon[t_0,t_f]$ $(i,j=1,2)$ in (30). Select the values of κ and β.

Step 0-2: Compute (30) with the boundary conditions

$$\begin{aligned} &x_i^{(1)}(t_0)=0, \quad x_i^{(1)}(t_1)=e_\nu, \qquad \text{for Subarc 1} \\ &x_i^{(2)}(t_1)=e_\nu, \quad p_i^{(2)}(t_f)=0, \qquad \text{for Subarc 2} \quad (i=1,2) \end{aligned} \tag{38}$$

where e_ν is the ν-th unit vector ($\nu=1,2,\ldots,n_i$). Then let the obtained difference $\hat{p}_i^{(1)}(t_1)-\hat{p}_i^{(2)}(t_1)$ be the ν-th column of the matrix Γ_i $(i=1,2)$.

Step 0-3: Calculate Γ_i^{-1}.

Step 1: Set $k=0$, ${}^0y_i^{(l)}(t)={}^0q_i^{(l)}(t)=0\,(l=1,2)$, and ${}^0\hat{x}_i(t_1)=0\,(i=1,2)$.

Step 2: Compute the subarc solution $\hat{p}_i^{(l)}(t_1)$ to the problem (30) with ${}^ky_i^{(l)}(t)$ and ${}^kq_i^{(l)}(t)$ $(i,l=1,2)$, and with the boundary conditions (32).

Step 3: For ${}^ky_i^{(l)}(t)$ and ${}^kq_i^{(l)}(t)$ given, obtain from (34) the exact solution ${}^kx_i(t_1)$ at the torn time t_1.

Step 4: For ${}^ky_i^{(l)}(t)$ and ${}^kq_i^{(l)}(t)$ given, obtain from (30) the exact solutions ${}^kx_i^{(l)}(t)$ and ${}^kp_i^{(l)}(t)$, $t\varepsilon[t_{l-1},t_l]$ $(i,l=1,2\,;\,t_2=t_f)$, with the boundary conditions

$$\begin{aligned} &{}^kx_i^{(1)}(t_0)=x_{i0}, \quad {}^kx_i^{(1)}(t_1)={}^kx_i(t_1), \qquad \text{for Subarc 1} \\ &{}^kx_i^{(2)}(t_1)={}^kx_i(t_1), \quad {}^kp_i^{(2)}(t_f)=0, \qquad \text{for Subarc 2} \quad (i=1,2) \end{aligned} \tag{39}$$

Step 5: Compute ${}^kG_3$ defined by (36). If ${}^kG_3\leqq\sigma$ (σ: a small positive number prescribed), then the calculation is terminated. If ${}^kG_3>\sigma$, set ${}^{k+1}\hat{x}_i(t_1)={}^kx_i(t_1)$ $(i=1,2)$, adjust y and q by (37), replace k by $k+1$, and then proceed to Step 2.

The three-level computational structure is illustrated in Fig. 2. The algorithm is

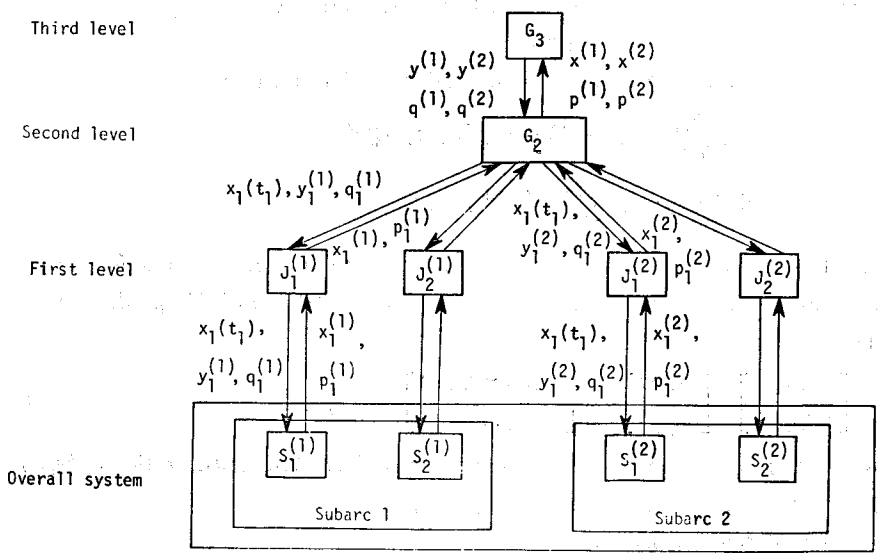

Fig. 2. Three-level computational structure with time- and state-decompositions.

especially effective for a problem of strongly interconnected systems with strong nonlinearities, or for a problem with a long control interval.

4. An Illustrative Example

In this section, we examine the numerical solution of a physical problem in order to illustrate the application of the present algorithm. For the numerical integration of the differential equations, the fourth-order Runge-Kutta-Gill method is adopted, where use is made of one hundred grid points on the interval $[0, \pi]$.

The following equations, as discussed in the previous paper (Ref. 8), approximately describe the minimum-fuel transfer of a low-thrust propulsion vehicle between circular orbits:

$$\left.\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=x_1-\varepsilon[2x_4+x_1/(x_1^2+x_3^2+x_5^2)^{3/2}]+u_1,\end{aligned}\right\}\ \text{Subsystem 1}$$

$$\left.\begin{aligned}\dot{x}_3&=x_4,\\ \dot{x}_4&=x_3+\varepsilon[2x_2-x_3/(x_1^2+x_3^2+x_5^2)^{3/2}]+u_2,\end{aligned}\right\}\ \text{Subsystem 2}\qquad(40)$$

$$\left.\begin{aligned}\dot{x}_5&=x_6,\\ \dot{x}_6&=-\varepsilon[x_5/(x_1^2+x_3^2+x_5^2)^{3/2}]+u_3,\end{aligned}\right\}\ \text{Subsystem 3}$$

where ε is a parameter introduced for convenience. The performance index is taken to be

$$J=\frac{1}{2}\int_0^{\pi}(u_1^2+u_2^2+u_3^2)\,dt\qquad(41)$$

Corresponding to (30), the following linear TPBVP is obtained for Subsystem 1 in Subarc $l\,(l=1,2)$:

$$\begin{aligned}\dot{x}_1^{(l)}&=x_2^{(l)}\\ \dot{x}_2^{(l)}&=x_1^{(l)}-\beta p_2^{(l)}+(\beta-1)\,q_2^{(l)}-\varepsilon[2y_4^{(l)}+y_1^{(l)}/(y_1^{(l)2}+y_3^{(l)2}+y_5^{(l)2})^{3/2}]\\ \dot{p}_1^{(l)}&=-\kappa x_1^{(l)}-p_2^{(l)}+\kappa y_1^{(l)}+\varepsilon\{[q_2^{(l)}(-2y_1^{(l)2}+y_3^{(l)2}+y_5^{(l)2})\\ &\qquad-3y_1^{(l)}(y_3^{(l)}q_4^{(l)}+y_5^{(l)}q_6^{(l)})]/(y_1^{(l)2}+y_3^{(l)2}+y_5^{(l)2})^{5/2}\}\\ \dot{p}_2^{(l)}&=-\kappa x_2^{(l)}-p_1^{(l)}+\kappa y_2^{(l)}-2\varepsilon q_4^{(l)}\end{aligned}\qquad(42)$$

where $\kappa x_1^{(l)}$ and $\kappa x_2^{(l)}$ at the right sides of the third and the fourth equations, respectively, are artificially added terms to accelerate the convergence, and $y_i^{(l)}$ $(i=1\sim5)$ and $q_i^{(l)}$ $(i=2,4,6)$ are interaction variables standing for $x_i^{(l)}$ and $p_i^{(l)}$, respectively. Similar problems which are obtained for Subsystems 2 and 3 in Subarc l are omitted here.

By way of an example, let $x_1(0)=x_2(0)=x_4(0)=x_5(0)=x_6(0)=0$, $x_3(0)=1$, $x_2(\pi)=-0.75$, $x_3(\pi)=1.5$, $x_4(\pi)=x_5(\pi)=0$, $x_6(\pi)=\pi/5$, and $x_1(\pi)$ be free. In this case, the terminal condition of p_1 is given by $p_1(\pi)=0$.

For $\varepsilon=4$, Figure 3 shows variations of G with the computing time T. In the

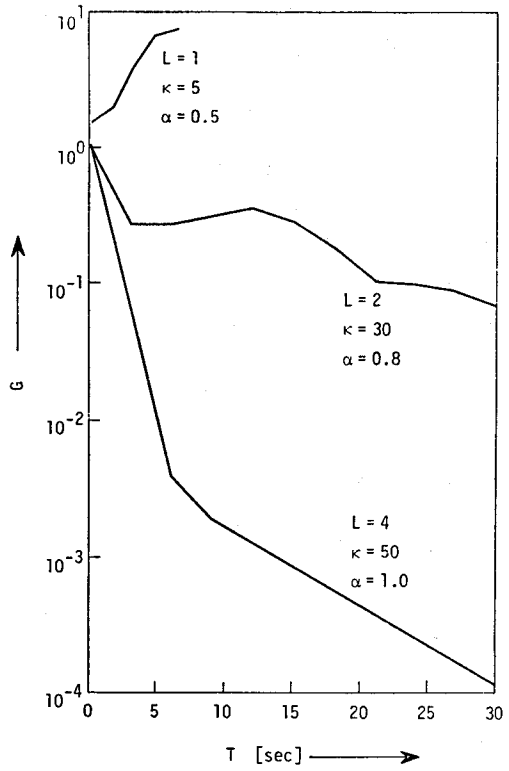

Fig. 3. Variations of G with the computing time T ($\varepsilon=4$, $\beta=1$).

figure, L denotes the number of subarcs, and the length of each subinterval is set equal to π/L.

When $L=1$ (that is, when the time-decomposition is not applied), G is divergent. The reason is that the values of the weights are to be bounded small because of the limitation of numerical accuracy. On the other hand, the variation of G with $L=2$ is oscillatory. Since in this case larger values of the weights can be used than in the previous case, the divergence tendency is suppressed, but convergence is not attained. When $L=4$, since the algorithm can use much larger values of the weights without loss of numerical accuracy, convergence is obtained.

5. Concluding Remarks

In Section 2, a computational technique for the solution of a linear TPBVP, termed time-decomposition method, has been proposed. The technique is to decompose the overall interval into several subintervals, and to construct independent subproblems over the subintervals whose solutions yield the solution of the original

linear problem. Theorems in the section provide with theoretical background to the technique.

In Section 3, a modified interaction-coordination algorithm with the time-decomposition has been proposed for a three-level computation of nonlinear optimal control problems. The modified interaction-coordination algorithm previously proposed by the authors is essentially to solve a linear TPBVP iteratively until the interaction balance is achieved. Thus, the time-decomposition method in Section 2 can be adopted for improvement of the convergence tendency of the iteration. Since the introduction of the time-decomposition to the algorithm results in overcoming difficulties caused by numerical errors, the algorithm can use greater values of the weights, thereby attaining a much faster convergence and a wider convergence region than the previous algorithm.

The example of Section 4 shows the effectiveness of the present algorithm, although it does not take the Jacobian matrix into account.

The algorithm in the present paper is also applicable to multi-point boundary-value problems for ordinary differential equations with strong nonlinearities and/or a long time interval.

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