Minimal Dimension Realization for Stochastic Discrete Time Systems

by

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Abstract

This paper is concerned with the problem of minimal dimension realization for stochastic systems. The innovation representation is adopted as the model of the system. Two algorithms are presented which yield the minimal dimension realization from the given finite length of output covariance data. One is for the scalar output system, the other is for the vector output system.

1. Introduction

Recently the study of the realization problem, or the problem of obtaining the system representation in a state equation form from the input-output relation, is investigated widely. Input-output relation is usually given in the form of a transfer function matrix or an impulse response matrix. In this case, several alogrithms are already given by using the Hankel matrix.[1)-3)] It is known that the minimal dimension realizations are observable and controllable, and algebraically equivalent to each other. A slightly different problem is the realization from an input-output sequence. This problem is also investigated in several papers. Among these, Gopinath,[4)] Budin[5)] and Lieu and Suen,[6)] have investigated the multi-input multi-output case. The key to their success is the introduction and the application of selector matrices.

In stochastic systems, Akaike[7)-9)] pointed out that the Auto-regressive Moving-average model and the Markovian representation (state equation form with a white noise input) are equivalent, and the state is the basis of the predictor space (the space spanned by the components of the prediction of the stationary time series). In his

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model, the input to the system is the innovation of the time series. He also suggested a maximum liklihood ratio method to determine the coefficient matrices by using the information criterion. Kailath and Gevers[10] investigated the problem in which the covariances of the output time series at time m and n are given in the form of the product of the matrices which are dependent only on m or n. They considered the innovation representation of the system, applying it to the filtering problem. However, the problem is not solved yet for the case where only the output covariance data are given. Therefore in this paper this problem is considered. The innovation representation is adopted as the model of the system, and selector matrices are introduced in the realization algorithm. It is also assumed that the system is observable, since the unobservable part of the state in no way influences the output. In section 2, the problem of the scalar output system is considered and in section 3, the problem of the vector output system is investigated.

2. Scalar Observation

2-1. Derivation of Innovation Representation

In this section the system with a scalar valued observation is considered for simplicity. Let y_n, $n=0, 1, \cdots, N$ be the observation data from the system

$$\begin{aligned} x_{n+1} &= Ax_n + Bu_{n+1}, \qquad x_0 = \xi. \\ y_n &= c'x_n, \end{aligned} \tag{2-1}$$

where A, B and c' are $d\times d$, $d\times r$ and $1\times d$ matrices respectively, x_n is the d-dimensional state vector, y_n is the scalar valued observation, and $\{\xi, u_n\}$ are the Gaussian stochastic processes with zero mean with the properties*

$$E[u_m u_n'] = I\delta_{mn}, \qquad E[\xi u_n'] = 0, \qquad E[\xi\xi'] = \Xi.$$

Let Π_n be the variance of the state x_n, then

$$\Pi_n = E[x_n x_n'], \qquad n=0, 1, \cdots, N$$

satisfy the relations

$$\Pi_{n+1} = A\Pi_n A' + BB', \qquad \Pi_0 = \Xi. \tag{2-2}$$

Let the covariance of the output y_n be $r_y(m, n) = E[y_m y_n]$, then

$$r_y(m, n) = \begin{cases} c'A^{m-n}\Pi_n c & \text{for } m>n \\ c'\Pi_m A^{n-m} c & \text{for } m<n \\ c'\Pi_n c & \text{for } m=n. \end{cases} \tag{2-3}$$

The Kalman-Bucy filter[11] of system (2-1) is

* Here, $E[.]$ means the expectation, $'$ the transpose, I the identity matrix, 0 the zero matrix, and δ Kronecker's delta.

$$\hat{x}_n=\hat{x}_n^-+k_n\nu_n, \qquad \hat{x}_0^-=0, \tag{2-4a}$$

$$\hat{x}_{n+1}^-=A\hat{x}_n, \tag{2-4b}$$

$$k_n=P_n^-c(c'P_n^-c)^{-1}, \tag{2-4c}$$

$$P_n=P_n^--k_nc'P_n^-, \tag{2-4d}$$

$$P_{n+1}^-=AP_nA'+BB', \tag{2-4e}$$

where*

$$\hat{x}_n=E[x_n|Y_n], \qquad \hat{x}_n^-=E[x_n|Y_{n-1}],$$
$$P_n=E[(x_n-\hat{x}_n)(x_n-\hat{x}_n)'], \qquad P_n=E[(x_n-\hat{x}_n^-)(x_n-\hat{x}_n^-)']$$

and

$$\nu_n=\nu_n-E[y_n|Y_{n-1}]=y_n-c'\hat{x}_n^-, \qquad \nu_0=y_0 \tag{2-5}$$

is the innovation process of the observation which has the properties

$$E[\nu_n]=0, \qquad E[\nu_m\nu_n]=c'P_n^-c\delta_{mn}.$$

Now let

$\boldsymbol{H}_n(X)$: the Gaussian Space spanned by x_i, $i=0, 1, \cdots, n$
$\boldsymbol{H}_n(Y)$: the Gaussian Space spanned by y_i, $i=0, 1, \cdots, n$

then

$$\boldsymbol{H}_n(Y)\subset\boldsymbol{H}_n(X).$$

However, if we observe the output for some more time, we may estimate the state x_i, $i=0, 1, \cdots, n$, that is, there may exist a certain natural number q such that $\boldsymbol{H}_{n+q}(Y)\supset \boldsymbol{H}_n(X)$. This is the definition of the observability in stochastic systems. The necessary and sufficient condition that system (2–1) is observable is

$$\text{rank} \quad (c, A'c, \cdots, A'^{d-1}c)=d.$$

Since the purpose of this paper is to represent the system in a state space form from the output data, we may consider the system observable, and hence it is so assumed.

Let Σ_n^- and Σ_n be the variance of $\hat{x}_n^-$ and $\hat{x}_n$ respectively, i.e.,
$\Sigma_n^-=E[\hat{x}_n^-\hat{x}_n^{-\prime}]$, and $\Sigma_n=E[\hat{x}_n\hat{x}_n']$,

then

$$P_n^-=\Pi_n-\Sigma_n^-, \qquad P_n=\Pi_n-\Sigma_n. \tag{2-6}$$

From equation (2–3),

$$\Pi_nc=N_d^{-1}Z_d(n, n), \tag{2-7}$$

* $E[\cdot\mid \boldsymbol{Y}_{n-1}]$ means the conditional expectation with respect to the observation $y_0, y_1, \cdots, y_{n-1}$.

where

$$Z_d(n, n)=\begin{bmatrix} r_y(n, n) \\ r_y(n+1, n) \\ \vdots \\ r_y(n+d-1, n) \end{bmatrix}, \quad N_d=\begin{bmatrix} c' \\ c'A \\ \vdots \\ c'A^{d-1} \end{bmatrix}$$

Using the above results, two theorems with respect to the innovation representation are given as follows.

[Theorem 2–1] (Innovation representation 1)

If (A, c') is an observable pair, then the system has the innovation representation 1 as

$$\hat{x}_{n+1}^{-}=A\hat{x}_n^{-}+Ak_n\nu_n, \qquad \hat{x}_0^{-}=0, \tag{2–8a}$$

$$y_n=c'\hat{x}_n^{-}+\nu_n, \tag{2–8b}$$

$$k_n=(N_d^{-1}Z_d(n, n)-\Sigma_n^{-}c)(r_y(n, n)-c'\Sigma_n^{-}c)^{-1}, \tag{2–8c}$$

$$\begin{aligned}\Sigma_{n+1}^{-}=&A\Sigma_n^{-}A'+A(N_d^{-1}Z_d(n, n)-\Sigma_n^{-}c)(r_y(n, n)\\ &-c'\Sigma_n^{-}c)^{-1}(N_d^{-1}Z_d(n, n)-\Sigma_n^{-}c)'A', \qquad \Sigma_0^{-}=0,\end{aligned} \tag{2–8d}$$

$$E[\nu_n]=0, \qquad E[\nu_m\nu_n]=(r_y(n, n)-c'\Sigma_n^{-}c)\delta_{mn}. \tag{2–8e}$$

(*proof*)

Equations (2–8a) and (2–8b) are obtained from (2–1), (2–4a), (2–4b) and the definition of ν_n. From equations (2–3), (2–5) and (2–6),

$$E[\nu_m\nu_n]=c'P_n^{-}c=(c'\Pi_nc-c'\Sigma_n^{-}c)\delta_{mn}=(r_y(n, n)-c'\Sigma_n^{-}c)\delta_{mn}.$$

Using equations (2–4c), (2–6), (2–7) and (2–3), we obtain (2–8c). Eliminating P_n^{-} from eqns. (2–4d) and (2–4e), we have

$$P_{n+1}^{-}=AP_n^{-}A'-Ak_nc'P_n^{-}A'+BB'. \tag{2–8f}$$

Rearranging (2–8f) by (2–2) and (2–6),

$$\Sigma_{n+1}^{-}=A\Sigma_n^{-}A'+Ak_nc'P_n^{-}A'.$$

Noting (2–3) and (2–8c), we have (2–8d).

[Theorem 2–2] (Innovation representation 2)

If (A, c') is an observable pair, then the system has the innovation representation 2 as

$$\begin{aligned}&\hat{x}_{n+1}=A\hat{x}_n+k_{n+1}\nu_{n+1}, \qquad \hat{x}_{-1}=0,\\ &y_n=c'\hat{x}_n,\\ &k_n=(N_d^{-1}Z_d(n, n)-A\Sigma_{n-1}A'c)(r_y(n, n)-c'A\Sigma_{n-1}A'c)^{-1},\\ &\Sigma_{n+1}=A\Sigma_nA'+(N_d^{-1}Z_d(n+1, n+1)\\ &\qquad -A\Sigma_nA'c)(r_y(n+1, n+1)-c'A\Sigma_nA'c)^{-1}\\ &\qquad\qquad *(N_d^{-1}Z_d(n+1, n+1)-A\Sigma_nA'c)'\\ &\Sigma_{-1}=0,\end{aligned} \tag{2–9}$$

$$E[\nu_n]=0, \qquad E[\nu_m\nu_n]=(r_y(n, n)-c'A\Sigma_{n-1}A'c)\delta_{mn}.$$

(*proof*)

The proof of Theorem 2–2 can be made in a way similar to Theorem 2–1.

Thus two different types of innovation representations are obtained from the same system (2–1). However, both types of innovation representations give the same impulse response:

$$y_n=\sum_{i=1}^{n} c'A^{n-1}k_i\nu_i.$$

It can be shown by simple calculation that both give the same covariances with respect to the output y_n as that of (2–1).

2–2. Transformation to Canonical Form

Let $\tilde{x}_n$ be a new state which is transformed from x_n by multiplying the observation matrix N_d, that is,

$$\tilde{x}_n=N_dx_n.$$

Then, noting that N_d has an inverse since system (2–1) is observable, we have the following equations with respect to $\tilde{x}_n$.

$$\tilde{x}_{n+1}=\tilde{A}\tilde{x}_n+\tilde{B}u_{n+1}, \tag{2–10a}$$

$$y_n=\tilde{c}'\tilde{x}_n, \tag{2–10b}$$

where

$$\tilde{A}=N_dAN_d^{-1}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_d \end{pmatrix}, \tag{2–10c}$$

$$\tilde{c}'=c'N_d^{-1}=[1\ 0\cdots 0], \tag{2–10d}$$

$$\tilde{B}=N_dB, \tag{2–10e}$$

and $a_1, a_2, \cdots, a_d$ are the coefficients of the characteristic equation of A, i.e.,

$$\det\,[\lambda I-A]=\lambda^d-(a_1+a_2\lambda+\cdots+a_d\lambda^{d-1}).$$

We call system (2–10) the canonical form. The innovation representation of this system is as follows.

Innovation representation 1

$$\hat{\tilde{x}}_{n+1}=\tilde{A}\,\hat{\tilde{x}}_n^-+\tilde{A}\,\tilde{k}_n\nu_n, \qquad \hat{\tilde{x}}_0=0, \tag{2–11a}$$

$$y_n=\tilde{c}'\hat{\tilde{x}}_n^-+\nu_n, \tag{2–11b}$$

$$\tilde{k}_n=(Z_d(n, n)-\tilde{\Sigma}_n^-\tilde{c})(r_y(n, n)-\tilde{c}'\tilde{\Sigma}_n^-\tilde{c})^{-1}, \tag{2–11c}$$

$$\begin{aligned}\tilde{\Sigma}_{n+1}^-=&\tilde{A}\,\tilde{\Sigma}_n^-\tilde{A}'+\tilde{A}(Z_d(n, n)-\tilde{\Sigma}_n^-\tilde{c})(r_y(n, n)\\ &-\tilde{c}'\tilde{\Sigma}_n^-\tilde{c})^{-1}(Z_d(n, n)-\tilde{\Sigma}_n^-\tilde{c})'\tilde{A}',\end{aligned} \tag{2–11d}$$

$$\tilde{\Sigma}_0^-=0,$$

$$E[\nu_n]=0, \qquad E[\nu_m\nu_n]=(r_y(n, n)-\tilde{c}'\tilde{\Sigma}_n\tilde{c})\delta_{mn}. \tag{2–11e}$$

Innovation representation 2

$$\left.\begin{aligned}&\hat{\tilde{x}}_{n+1}=\tilde{A}\hat{\tilde{x}}_n+\tilde{k}_{n+1}\nu_{n+1}, \qquad \hat{\tilde{x}}_{-1}=0,\\ &y_n=\tilde{c}'\hat{\tilde{x}}_n,\\ &\tilde{k}_n=(Z_d(n, n)-\tilde{A}\,\tilde{\Sigma}_{n-1}\tilde{A}'\tilde{c})(r_y(n, n)-\tilde{c}'\tilde{A}\,\tilde{\Sigma}_{n-1}\tilde{A}'\tilde{c})^{-1}\\ &\tilde{\Sigma}_{n+1}=\tilde{A}\,\tilde{\Sigma}_n\tilde{A}'+(Z_d(n+1, n+1)-\tilde{A}\,\tilde{\Sigma}_n\tilde{A}'\tilde{c})^{-1}(r_y(n+1, n+1)\\ &\qquad -\tilde{c}'\tilde{A}\,\tilde{\Sigma}_n\tilde{A}'\tilde{c})^{-1}(Z_d(n+1, n+1)-\tilde{A}\,\tilde{\Sigma}_n\tilde{A}'\tilde{c})'\\ &\tilde{\Sigma}_{-1}=0,\\ &E[\nu_n]=0, \qquad E[\nu_m\nu_n]=(r_y(n, n)-\tilde{c}'\tilde{A}\,\tilde{\Sigma}_{n-1}\tilde{A}'\tilde{c})\delta_{mn}.\end{aligned}\right\} \tag{2–12}$$

We adopt this type of representation of the system, since the systems which are algebraically equivalent* have the same covariance with respect to output y_n.

2–3. Derivation of the Algorithm of Minimal Dimension Realization

The covariance of the output of systems will be the same as long as BB' is the same, even if u_n or B is different by the systems. It is usual in the real complicated systems that all the variables influence each other, i.e., every variable has a feedback loop and we cannot distinguish the input from the output. Under these circumstances, it is natural to take the innovation process $y_n-E[y_n|Y_{n-1}]$ as the input. For these reasons and the usefulness of the innovation representation in application, it is better to represent the systems in innovation forms rather than in the form of (2–1). Thus, if we adopt the canonical form to represent a system, it is enough to identify the coefficient matıix A.

Let

$$Z_d(n+i, n)=\begin{bmatrix} r_y(n+i, n) \\ r_y(n+i+1, n), \\ \vdots \\ r_y(n+i+d-1, n) \end{bmatrix},$$

then, from (2–3),

$$Z_d(n+i, n)=N_dA^i\Pi_nc.$$

Therefore

$$Z_d(n+i+1, n)=AZ_d(n+i, n),$$

* Two systems (A_1, B_1, C_1) and (A_2, B_2, C_2) are algebraically equivalent if and only if there exists a normal matrix T such that $A_2=TA_1T^{-1}$, $B_2=TB_1$, $C_2=C_1T^{-1}$.

where

$$\tilde{A}=N_d A N_d^{-1}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_d \end{pmatrix}. \qquad \text{(canonical form)}$$

Hence the algorithm of minimal dimension realization is stated as follwos.

(Step 1) (Determine the dimension of the state d)

Suppose that the covariances of the output $r_y(i, j)$, $i, j=0, 1, \cdots, N$ are given. Let

$$z'(0, i)=[r_y(0, 0)\ r_y(1, 0) \cdots r_y(N-i, 0) \vdots r_y(1, 1)\ r_y(2, 1) \cdots$$
$$r_y(N-i, 1) \vdots \cdots \vdots r_y(N-i, N-i)],$$

$$z'(j, i)=[r_y(j, 0)\ r_y(j+1, 0) \cdots r_y(N-i+j, 0) \vdots r_y(j+1, 1) \cdots$$
$$r_y(N-i+j, 1) \vdots \cdots \vdots r_y(N-i+j, N-i)], \quad j=1, 2, \cdots, i-1,$$

$$z'(i, i)=[r_y(i, 0)\ r_y(i+1, 0) \cdots r_y(N, 0) \vdots r_y(i+1, 1) \cdots r_y(N, 1) \vdots$$
$$\cdots \vdots r_y(N, N-i)],$$

$$Q^*(i)=\begin{bmatrix} z'(0, i) \\ z'(1, i) \\ \vdots \\ z'(i, i) \end{bmatrix}, \qquad Q_*(i)=\begin{bmatrix} z'(0, i) \\ z'(1, i) \\ \vdots \\ z'(i-1, i) \end{bmatrix}.$$

Calculate

$$\alpha_i=\text{rank } Q^*(i)-\text{rank } Q_*(i)$$

for $i=0, 1, \cdots, N$. Let d be the first natural number such that $\alpha_i=0$, then this is the dimension of the state.

(Step 2) (Determine the coefficient matrix $\tilde{A}$)

$\alpha_d=0$ implies that $z'(d, d)$ can be expressed as a linear combination of $z'(0, d), \cdots,$ $z'(d-1, d)$, that is,

$$z'(d, d)=a_1 z'(0, d)+a_2 z'(1, d)+ \cdots +a_d z'(d-1, d).$$

Then

$$\tilde{A}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_d \end{pmatrix}.$$

By using this matrix $\tilde{A}$, the innovation representation of the system is given as (2–11) and (2–12).

Remark 1: It is obvious that, if $a_d=0$ for a certain number d, then $a_i=0$ for all i, $i \geqq d$.
Remark 2: $d \leqq N$ since $a_N=0$.

[Theorem 2-3]

The realization which is obtained by the above algorithm is good in the sense that the covariance of the output is equal to the given covariance data.

(*proof*)

We will prove only innovation representation 1 in canonical form, since for innovation representation 2, the proof can be given in a similar way. If $m>n$, then, from equations (2-11a) and (2-11b),

$$\begin{aligned} y_m &= \tilde{c}'\hat{\tilde{x}}_m^- + \nu_m \\ &= \tilde{c}'(\tilde{A}\hat{\tilde{x}}_{m-1} + \tilde{A}\tilde{k}_{m-1}\nu_{m-1}) + \nu_m \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= \tilde{c}'A^{m-n}\hat{\tilde{x}}_n^- + \sum_{i=0}^{m-n} \tilde{c}'A^{m-n-i}\tilde{k}_{n+i}\nu_{n+i}. \end{aligned}$$

Let $r_y{}^*(m, n)$ be the covariance of the output of the innovation representation 1. It is assumed $m \geqq n$ without loss of generality since $r_y{}^*(m, n)=r_y{}^*(n, m)$. Then,

$$r_y(m, n)=E[y_m y_n]=\tilde{c}'\tilde{A}^{m-n}\tilde{\Sigma}_n^-\tilde{c}+\tilde{c}'\tilde{A}^{m-n}\tilde{k}_n E[\nu_n\nu_n].$$

By using equations (2-11e) and (2-11c),

$$r_y{}^*(m, n)=\tilde{c}'\tilde{A}^{m-n}Z_d(n, n). \tag{2-13}$$

If $m-n \leqq d-1$, then

$$\tilde{c}'\tilde{A}^{m-n}=[\overbrace{0 \cdots 0}^{m-n}\ 1\ 0 \cdots 0]$$

since $\tilde{c}=[1\ 0 \cdots 0]$ and $\tilde{A}$ is the companion form as in (2-10c).
Thus

$$r_y{}^*(m, n)=\tilde{c}'\tilde{A}^{m-n}\begin{bmatrix} r_y(n, n) \\ r_y(n+1, n) \\ \vdots \\ r_y(n+d-1, n) \end{bmatrix}=r_y(m, n), \quad \text{for } m-n \leqq d-1,$$

that is, the output covariance of the innovation representation 1 is equal to the given output covariance data for $m-n \leqq d-1$. If $m-n \geqq d$, from Step 2 of the algorithm,

$$\begin{aligned} r_y(n+d, n) &= a_1 r_y(n, n)+a_2 r_y(n+1, n)+\cdots+a_d r_y(n+d-1, n) \\ &= [a_1\ a_2 \cdots a_d]Z_d(n, n). \end{aligned}$$

Therefore

$$Z_d(n+1, n)=\tilde{A}Z_d(n, n).$$

Similarly,

$$Z_d(m, n)=\tilde{A}^{m-n}Z_d(n, n). \tag{2-14}$$

Substituting (2-14) into (2-13),

$$r_y^*(m, n)=\tilde{c}'\tilde{A}^{m-n}Z_d(n, n)=\tilde{c}'Z_d(m, n)=r_y(m, n), \qquad \text{for } m-n\geqq d.$$

Thus the output covariance of the innovation representation 1 is equal to the given output covariance data for all values of m and n.

Remark 3: From Step 1, it is obvious that this is the minimal dimension realization. A necessary and sufficient condition in order that $\tilde{A}$ can be determined uniquely is

$$\text{rank}\begin{bmatrix} z'(0, d) \\ z'(1, d) \\ \vdots \\ z'(d-1, d) \end{bmatrix}=d.$$

3. Vector Observation

Let y_n be the vector valued output from the system

$$\begin{aligned} x_{n+1}&=Ax_n+Bu_{n+1}, \qquad x_0=\xi, \\ y_n&=Cx_n \end{aligned} \tag{3-1}$$

where y_n and C are p-dimensional vectors and $p\times d$ matrix, respectively, and the other parameters are the same as in the case of scalar observation. Let $R_y(m, n)$ be the output covariance matrix, or

$$R_y(m, n)=E[y_m y_n'],$$

then

$$R_y(m, n)=\begin{cases} CA^{m-n}\Pi_n C' & \text{for } m>n \\ C\Pi_m A^{n-m}C' & \text{for } m<n \\ C\Pi_n C' & \text{for } m=n. \end{cases} \tag{3-2}$$

The variance matrix of state x_n satisfies equation (2-2). It is assumed that system (3-1) is observable as in the case of scalar output, that is,

$$\text{rank } [C'\ A'C'\ A'^2C' \cdots A'^{d-1}C']=d.$$

Now, let q be the fiıst natural number such that

$$\text{rank } N_i=\text{rank } N_d=d,$$

where

$$N_i=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}.$$

Let

$$Z_q(n, n) = \begin{bmatrix} R_y(n, n) \\ R_y(n+1, n) \\ \vdots \\ R_y(n+q-1, n) \end{bmatrix},$$

then,

$$Z_q(n, n) = N_q \Pi_n C'. \tag{3-3}$$

There exists a $d \times pq$ matrix N_q^- such that $N_q^- N_q = I_d$ since rank $N_q = d$. (For example, $N_q^- = (N_q' N_q)^{-1} N_q'$). Then,

$$\Pi_n C = N_q^- Z_q(n, n).$$

Using the Kalman filter of system (3–1):

$$\begin{aligned}
&\hat{x}_n = \hat{x}_n^- + K_n \nu_n, \qquad \hat{x}_0 = 0' \\
&\hat{x}_{n+1}^- = A\hat{x}_n \\
&K_n = P_n^- C'(CP_n^- C')^{-1} \\
&P_n = P_n^- - K_n C P_n^- \\
&P_{n+1}^- = AP_n A' + BB' \\
&\nu_n = y_n - C\hat{x}_n^- \qquad \text{(Innovation process)},
\end{aligned}$$

From (2–2) and (3–3), we obtain the following theorem with respect to the innovation representation in a similar manner as in the previous section.

[Theorem 3–1] (Innovation representations)

If (A, C) is an observable pair, then system (3–1) has the innovation representations 1 and 2 as follows.

Innovation representation 1

$$\hat{x}_{n+1}^- = A\hat{x}_n^- + AK_n \nu_n, \qquad \hat{x}_0^- = 0. \tag{3-4a}$$

$$y_n = C\hat{x}_n^- + \nu_n \tag{3-4b}$$

$$K_n = (N_q^- Z_q(n, n) - \Sigma_n^- C')(R_y(n, n) - C\Sigma_n^- C')^{-1} \tag{3-4c}$$

$$\begin{aligned}\Sigma_{n+1}^- = A\Sigma_n^- A' + A(N_q^- Z_q(n, n) - \Sigma_n^- C')(R_y(n, n) - C\Sigma_n^- C')^{-1} \\ *(N_q^- Z_q(n, n) - \Sigma_n^- C')' A',\end{aligned} \tag{3-4d}$$

$$\Sigma_0^- = 0,$$

$$E[\nu_n] = 0, \qquad E[\nu_m \nu_n] = (R_y(n, n) - C\Sigma_n^- C')\delta_{mn}. \tag{3-4e}$$

Innovation representation 2

$$\begin{aligned}
&\hat{x}_{n+1} = A\hat{x}_n + K_{n+1}\nu_{n+1}, \qquad \hat{x}_{-1} = 0, \\
&y_n = C\hat{x}_n
\end{aligned}$$

$$
\begin{aligned}
K_n &= (N_q{}^- Z_q(n, n) - A\Sigma_{n-1}{}^- A'C')(R_y(n, n) - CA\Sigma_{n-1}A'C')^{-1} \\
\Sigma_{n+1} &= A\Sigma_n A' + (N_q{}^- Z_q(n+1, n+1) - A\Sigma_n A'C')(R_y(n+1, n+1) \\
&\quad - C\Sigma_n A'C')^{-1}(N_q Z_q(n+1, n+1) - A\Sigma_n A'C')', \\
\Sigma_{-1} &= 0, \\
E[\nu_n] &= 0, \qquad E[\nu_m \nu_n'] = (R_y(n, n) - CA\Sigma_{n-1}A'C')\delta_{mn}.
\end{aligned}
\tag{3–5}
$$

From (3–3),

$$Z_q(n+i, n) = N_q A^i \Pi_n C \tag{3–6}$$

where

$$Z_q(n+i, n) = \begin{bmatrix} R_y(n+i, n) \\ R_y(n+i+1, n) \\ \vdots \\ R_y(n+i+q-1, n) \end{bmatrix}.$$

Let S_q be a $d \times pq$ selector matrix which selects d independent row vectors from the pq rows of N_q, then $S_q N_q$ has an inverse matrix. Multiplying (3–6) by S_q from the left,

$$S_q Z_q(n+i, n) = S_q N_q A^i \Pi_n C = (S_q N_q) A (S_q N_q)^{-1} (S_q N_q) A^{i-1} \Pi_n C.$$

Therefore,

$$S_q Z_q(n+i+1, n) = \tilde{A} S_q Z_q(n+i, n),$$

where

$$\tilde{A} = (S_q N_q) A (S_q N_q)^{-1}. \tag{3–7}$$

We may choose S_q arbitrarily. In this paper we choose S_q, as follows, according to R. Liu and L. C. Suen[6]. Let

$$m_i = \text{rank } N_i - \text{rank } N_{i-1}, \qquad i = 2, 3, \cdots, q,$$

where

$$N_i = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}.$$

Let us choose the $m_i \times m_{i-1}$ selector matrices $S(i)$ such that

$$
\begin{aligned}
&\text{rank } S(1)C = \text{rank } C, \\
&\text{rank } \begin{bmatrix} S(1)C \\ S(2)S(1)CA \end{bmatrix} = \text{rank} \begin{bmatrix} S(1)C \\ S(1)CA \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\text{rank } \begin{bmatrix} S(1)C \\ S(2)S(1)CA \\ \vdots \\ S(q)S(q-1)\cdots S(1)CA^{q-1} \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}
\end{aligned}
$$

where

$$m_0=p,\ m_1=\text{rank } C.$$

Then we define the selector matrix S_q as

$$S_q=\begin{bmatrix} S(1) & & \\ & S(2)S(1) & 0 \\ & 0 \quad \ddots & \\ & & S(q)S(q-1)\cdots S(1) \end{bmatrix}. \tag{3-8}$$

It is obvious that

$$m_1+m_2+\cdots+m_q=d.$$

To multiply N_q by S_q from the left means that first, m_1 independent rows are selected from the rows of C, second, m_2 independent rows are selected from m_1 rows of CA which correspond to m_1 rows of C, and so forth until CA^{q-1}. Then $\tilde{A}$ in (3–7) will have the form of

$$\tilde{A}=\begin{pmatrix} A_{11} & A_{12} & 0 \cdots\cdots\cdots\cdots 0 \\ A_{21} & A_{22} & A_{23} \quad 0 \cdots\cdots 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\ A_{q-1,1} & A_{q-1,2} & \cdots\cdots\cdots\cdots A_{q-1,q} \\ A_{q1} & A_{q2} & \cdots\cdots\cdots\cdots A_{qq} \end{pmatrix}, \tag{3-9a}$$

where

$$A_{ij}:\quad m_i\times m_j \text{ block matrices, } i,j=1,2,\cdots,q,$$

$$A_{ij}=0 \qquad \text{for } j>i+1, \tag{3-9b}$$

$$S(i+1)A_{i,i+1}=I_{m_{i+1}}, \tag{3-9c}$$

$$S(i+1)A_{ij}=0 \qquad \text{for } j\neq i+1. \tag{3-9d}$$

After transforming system (3–1) by the normal matrix S_qN_q, A becomes $\tilde{A}$ in (3–9) and C becomes

$$\tilde{C}=C(S_qN_q)^{-1}=[C_1\,0\cdots 0], \qquad C_1:\ p\times m_1 \text{ matrix}, \tag{3-10}$$

$$C_1=S(1)'+S^c(1)'M, \tag{3-11}$$

where $S^c(1)$ is a $(p-m_1)\times p$ selector matrix which selects the rows of C such that $S(1)$ does not select, and M is a $(p-m_1)\times m_1$ matrix and is a coefficient matrix when $S^c(1)C$ is expressed as a linear combination of the rows of $S(1)C$, or $S^c(1)C=MS(1)C$. Furthermore, $\tilde{A}$, $\tilde{C}$, and S_q have the relations

$$\begin{aligned} &S(1)\tilde{C}=[I_{m_1}\,\vdots\,0\cdots 0];\ m_1\times d \text{ matrix} \\ &S(2)S(1)\tilde{C}\tilde{A}=[0_{m_2\times m_1}\,\vdots\,I_{m_2}\,\vdots\,0\cdots 0];\ m_2\times d \text{ matrix} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &S(q)S(q-1)\cdots S(1)\tilde{C}\tilde{A}^{q-1}=[0\cdots 0\,\vdots\,I_{m_q}];\ m_q\times d \text{ matrix.} \end{aligned} \tag{3-12}$$

Note that the above relation is independent of M.

After normally transforming system (3–1) by S_qN_q, the new coefficient matrices of the system are $\tilde{A}$, $(S_qN_q)B$, and $\tilde{C}$. We call this a canonical form. As mentioned in the scalar output case, we represent the system in the form of an innovation representation for this canonical form, since every algebraically equivalent system gives the same output covariance matrix. To represent the system in this form, it is enough to identify only $\tilde{A}$, $\tilde{C}$ and S_q. Thus, the algorithm of Minimal Dimension Realization is given as follows.

The algorithm of Minimal Dimension Realization

(Step 1) (Determine q, $m_1, m_2, \cdots, m_q$)

Suppose that the output covariance matrices $R_y(i,j)$, $i,j=0, 1, \cdots, N$ are given. Let

$$\begin{aligned}Z(0,i)=[&R_y(0,0)\,R_y(1,0)\cdots R_y(N-1,0)\,\vdots\,R_y(1,1)\,R_y(2,1)\cdots\\&R_y(N-i,1)\,\vdots\cdots\vdots\,R_y(N-i,N-i)]\end{aligned}$$

$$\begin{aligned}Z(j,i)=[&R_y(j,0)\,R_y(j+1,0)\cdots R_y(N-i+j,0)\,\vdots\,R_y(j+1,1)\cdots\\&R_y(N-i+j,1)\,\vdots\cdots\vdots\,R_y(N-i+j,N-i), \qquad j=1,2,\cdots,i-1,\end{aligned}$$

$$Z(i,i)=[R_y(i,0)\cdots R_y(N,0)\,\vdots\,R_y(i+1,1)\cdots R_y(N,1)\,\vdots\cdots\vdots\,R_y(N,N-i)]$$

$$\underline{Q}^*(i)=\begin{bmatrix}Z(0,i)\\Z(1,i)\\\vdots\\Z(i,i)\end{bmatrix},\qquad \underline{Q}_*(i)=\begin{bmatrix}Z(0,i)\\Z(1,i)\\\vdots\\Z(i-1,i)\end{bmatrix}$$

Evaluate

$$\underline{\alpha}_i=\text{rank }\underline{Q}^*(i)-\text{rank }\underline{Q}_*(i)$$

for $i=0, 1, \cdots, N$. Let q be the first natural number such that $\underline{\alpha}_i=0$, then $m_i=\underline{\alpha}_{i-1}$, $i=1, 2, \cdots, q$).

(Step 2) (Determine $S(1), S(2), \cdots, S(q), S_q$)

First choose an $m_1\times p$ selector matrix $S(1)$ for which

$$\text{rank }S(1)Z(0,0)=\text{rank }Z(0,0). \tag{3–13}$$

Next choose $m_i\times m_{i-1}$ selector matrices $S(i)$, $i=2, 3, \cdots, q$ for which

$$\text{rank }\underline{S}(i)\underline{Q}_*(i)=\text{rank }\underline{Q}_*(i)$$

where

$$\underline{S}(i)=\begin{bmatrix}S(1) & & & \\ & S(2)S(1) & & 0\\ & & \ddots & \\ 0 & & & S(i)S(i-1)\cdots S(1)\end{bmatrix}\quad (m_1+m_2+\cdots+m_i)\times pi$$

(Step 3) (Construct $\tilde{A}$, $\tilde{C}$)

Equation (3–13) implies that $S^c(1)Z(0, 0)$ can be represented as a linear combination of the rows of $S(1)Z(0, 0)$, that is, there exists a $(p-m_1)\times m_1$ matrix $\bar{M}$ such that

$$S^c(1)Z(0, 0)=\bar{M}S(1)Z(0, 0). \tag{3–14}$$

There exists a $p\times d$ matrix $\bar{A}$ such that

$$Z(q, q)=\bar{A}S_q\begin{bmatrix} Z(0, q) \\ Z(1, q) \\ \vdots \\ Z(q-1, q) \end{bmatrix},$$

since $\underline{a}_q=0$. From Step 2, $S(i)S(i-1)\cdots S(1)Z(i-1, q)$ can be represented as a linear combination of $S(1)Z(0, q)$, $S(2)S(1)Z(1, q), \cdots, S(i+1)S(i)\cdots S(1)Z(i, q)$, $i=1, 2, \cdots, q-1$. Therefore, there exists a $d\times d$ matrix $\bar{\bar{A}}$ such that

$$S_q\begin{bmatrix} Z(1, q) \\ Z(2, q) \\ \vdots \\ Z(q, q) \end{bmatrix}=\bar{\bar{A}}S_q\begin{bmatrix} Z(0, q) \\ Z(1, q) \\ \vdots \\ Z(q-1, q) \end{bmatrix}. \tag{3–15}$$

Thus

$$\tilde{A}=\bar{\bar{A}},$$

and

$$\tilde{C}=[S(1)'+S^c(1)\bar{M} \,\vdots\, 0 \cdots 0]. \tag{3–16}$$

Substituting the above $\tilde{A}$ and $\tilde{C}$ in (3–4) and (3–5) for A and C, we can express the system in the form of an innovation representation.

Remark 4: It is obvious that the above $\tilde{A}$, $\tilde{C}$, S_q have the properties (3–9a), (3–9b) and (3–12).

Remark 5: It is also obvious that, if $\underline{a}_q=0$ for a certain number q, then $\underline{a}_i=0$ for all i, $i\geqq q$ as in the case of scalar output.

Remark 6: $q\leqq N$ since $\underline{a}_N=0$.

[Theorem 3–2]

The realization which is obtained by the above algorithm is good in the sense that the covariance of the output is equal to the given covariance data.

(*proof*)

We prove only the innovation representation 1 in a canonical form, since the proof can be made similarly for the other representation. Let $R_y^*(m, n)$ be the output covariance matrix of the innovation representation 1. It is assumed that $m\geqq n$ without loss of generality. Then

$$R_y^*(m, n)=E[y_m y_n']=\tilde{C}\tilde{A}^{m-n}\Sigma_n^-\tilde{C}'+\tilde{C}\tilde{A}^{m-n}\tilde{K}_n E[\nu_n\nu_n'].$$

By using (3–4c) and (3–4e),

$$R_y^*(m, n)=\tilde{C}\tilde{A}^{m-n}S_q Z_q(n, n). \tag{3–17}$$

Equation (3–14) implies that the relation

$$S^c(1)R_y(m, n)=\tilde{M}S(1)R_y(m, n), \qquad 0\leqq n\leqq m\leqq N \tag{3–18}$$

holds. Equation (3–15) also implies the relation

$$S_q\begin{bmatrix} R_y(m+1, n) \\ R_y(m+2, n) \\ \vdots \\ R_y(m+q, n)\end{bmatrix}=\tilde{A}S_q\begin{bmatrix} R_y(m, n) \\ R_y(m+1, n) \\ \vdots \\ R_y(m+q-1, n)\end{bmatrix}, \quad 0\leqq n\leqq m\leqq N-q \tag{3–19}$$

holds. It is obvious that there is an extension of $R_y(m, n)$ for $m=N+1, N+2, \cdots, N+q$ such that $R_y(m, n)$, $m=N+1, \cdots, N+q$ satisfy (3–19).
Using such an extension and noting (3–19), we have

$$\tilde{A}^{m-n}S_q Z_q(n, n)=\tilde{A}^{m-n}S_q\begin{bmatrix} R_y(n, n) \\ R_y(n+1, n) \\ \vdots \\ R_y(n+q-1, n)\end{bmatrix}=S_q\begin{bmatrix} R_y(m, n) \\ R_y(m+1, n) \\ \vdots \\ R_y(m+q-1, n)\end{bmatrix}$$
$$0\leqq n\leqq m\leqq N.$$

Therefore, (3–17) becomes

$$R_y^*(m, n)=\tilde{C}S_q\begin{bmatrix} R_y(m, n) \\ R_y(m+1, n) \\ \vdots \\ R_y(m+q-1, n)\end{bmatrix}, \quad \text{for } 0\leqq n\leqq m\leqq N. \tag{3–20}$$

By multiplying (3–20) by $S(1)$ from the left and using (3–12),

$$S(1)R_y^*(m, n)=S(1)\tilde{C}S_q\begin{bmatrix} R_y(m, n) \\ R_y(m+1, n) \\ \vdots \\ R_y(m+q-1, n)\end{bmatrix}=S(1)R_y(m, n),$$
$$0\leqq n\leqq m\leqq N.$$

This implies that $R_y^*(m, n)$ is equal to $R^c(m, n)$ with respect to the rows selected by $S(1)$.
By multiplying (3–20) by $S^c(1)$ from the left and noting (3–16),

$$S^c(1)R_y(m, n)=\tilde{M}S(1)R_y(m, n) \tag{3–21}$$

since $S^c(1)S(1)'=0$. From (3–20) and (3–21),

$$S^c(1)R_y^*(m, n)=S^c(1)R_y(m, n).$$

This implies that $R_y^*(m, n)$ is also equal to $R_y(m, n)$ with respect to the rows which are not selected by $S(1)$. Therefore, $R_y^*(m, n)=R_y(m, n)$ for all rows.
Remark 6: A necessary and sufficient condition for $\tilde{A}$ to be determined uniquely is

$$\operatorname{rank} S_q \begin{bmatrix} Z(0, q) \\ Z(1, q) \\ \vdots \\ Z(q-1, q) \end{bmatrix} = d.$$

It is obvious that $\bar{M}$ is found uniquely.

4. Conclusion

Algorithms are obtained which yield minimal dimension realizations for stochastic systems from a given finite length of output covariance data. The innovation representation, or the state equation model with the input of the innovation process, is adopted as the model of the system. In real, complex systems, every variable has a feedback and the variables can not be divided into the two groups of input and output. Therefore, it is natural to take the innovation process as the input. This algorithm can be applied to any finite length of output covariance data. Necessary and sufficient conditions for the coefficient matrices to be identified uniquely are given for both scalar output and vector output cases.

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