A Consolidation Problem and its Solution

By

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(Received June 30, 1978)

Abstract

As we know in civil engineering, it is the first task for constructing banks or building structures to predict the subsidence of the underlying ground. When ground of clay saturated with pore water is compressed, the water flows out and the total volume decreases. Consequently, the ground subsides and the strength of the clay increases. We call this phenomena consolidation.

For the simplicity of analysis, we usually assume that the thickness of the clay stratum will not vary during consolidation and neglect the effect by its decrease.

Alternatively in this paper, we formulate a mathematical model of one dimensional consolidation problem, taking the decrease of the thickness of the clay stratum into consideration. And we show the existence and uniqueness of the exact solution of the problem under the assumption of small initial data.

1. Introduction

The consolidation of clay is expressed by using the deformation resistance of the skeleton structure of the clay and the permeability of the pore water. In compressing the clay, (for example, in the case of reclamation, putting a sand pile on the clay stratum) the total compressive stress is supported by the effective compressive stress acting among the clay particles and also the increment of the pore water pressure which we call the excess pore water pressure. The compressive strain does not arise right after compressing, so the total compressive stress is first supported by the excess pore water pressure. When the water pressures happen to differ between the inside and the outside of the stratum, the pore water flows out and the volume of the clay stratum decreases. As the consolidation advances, the effective compressive stress increases gradually, and the total stress is supported by both the effective compressive stress and the excess pore water pressure. Finally it follows that the total compressive stress is supported only by

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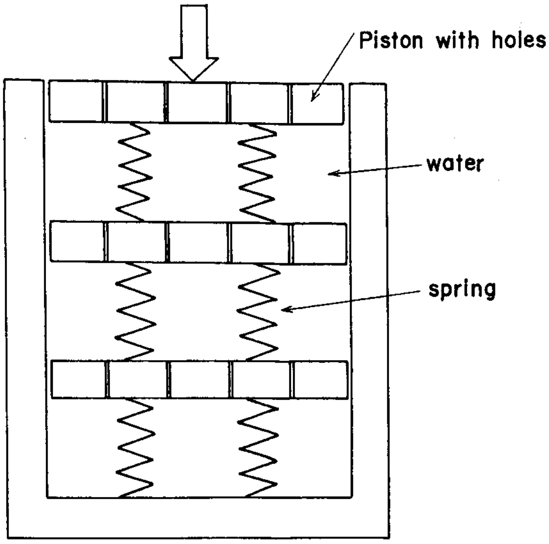

Fig. 1. Model of consolidation.

the skeleton structure, the excess pore water pressure goes down to zero and the pore water ceases to flow out. That means the end of subsidence due to consolidation.

This phenomenon is illustrated by the schematic model shown in Fig. 1. In Fig. 1 the spring corresponds to the skeleton structure and some holes made in the pistons are analogous to percolative paths in the clay.

This model was considered by Terzaghi, who established the one dimensional basic equation which the excess pore water pressure satisfies. He assumed that both the compression of the skeleton structure and the flow of pore water arise only toward the direction of the load. In the analysis of consolidation based on the equation, usually the decrement of the thickness of the clay stratum is neglected because of its smallness. But in practice, as in the case of weak ground, it happens that the stratum decreases fairly much in thickness.

In this paper we consider such cases and, for simplicity, one dimensional consolidation problem of the clay stratum lying between the upper and lower drainage strata, taking the decreasing thickness of the clay stratum into consideration.

First we formulate the problem mathematically in §2. In §3, using Green's function we represent the solution of the problem by the initial condition and the gradients of the excess pore water pressure on the boundary. By using it, in §4 we reduce the original problem to solve a certain system of nonlinear integral equations with respect to the gradients of the excess pore water pressure on the boundary and the amount of subsidence. Then, in §5 we show the existence and uniqueness of the solution of the system of the integral equations by Picard's method of successive approximation.

2. Mathematical Formulation of the Problem

We consider the following clay stratum of thickness H lying between the upper and lower drainage strata shown in Fig. 2. Let us take the ordinate z toward the gravitational direction, its origin being at the top of the clay stratum. Suppose that the position of the bottom of the clay stratum is fixed during consolidation. Let y denote the amount of subsidence from the initial position. Now we will take the following assumptions, that is, the clay stratum is homogeneous and is completely saturated, the load put on is constant during consolidation, the stress by self-weight of the stratum can be neglected and the compression of the skeleton structure and the flow of the pore water arise only toward the direction of the z-axis, not toward the other direction.

It is well known that the excess pore water pressure $u(z, t)$ then, satisfies the so-called diffusion equation;

$$u_t = c u_{zz} \tag{2·1}$$

(See 1)). Here c, which is called the coefficient of consolidation and is determined by the volume compressive coefficient m and the coefficient of the percolation of the clay, is assumed to be constant during consolidation.

Next let us introduce a relation which the amount of subsidence $y=y(t)$ satisfies. Let $\varepsilon=\varepsilon(z)$ be the amount of compressive strain per unit volume, which, we assume, is proportional to the effective compressive stress σ';

$$\varepsilon = m\sigma'. \tag{2·2}$$

Then the amount of subsidence is given by the formula

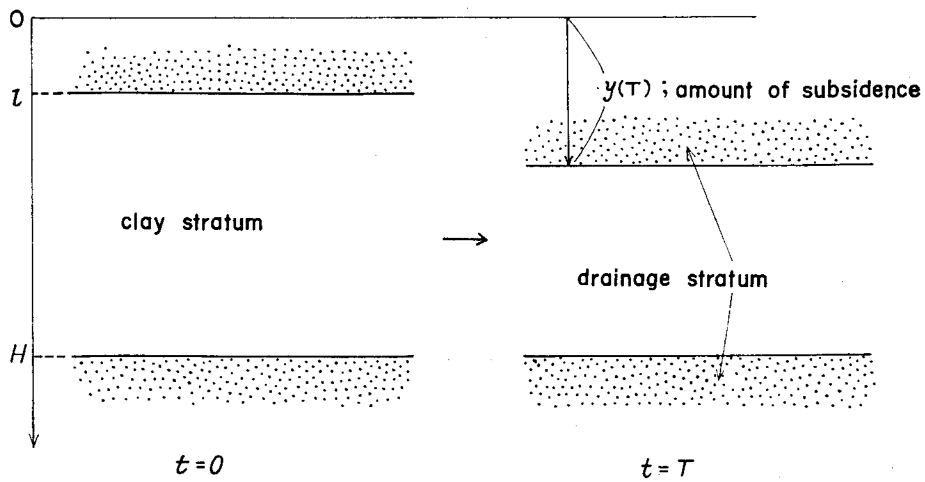

Fig. 2. One dimensional model.

$$y(t)=\int_{y(t)}^{H}\varepsilon(z)dz=m\int_{y(t)}^{H}\sigma' dz. \tag{2·3}$$

Because the effective compressive stress σ' is equal to the difference between the total compressive stress σ and the excess pore water pressure, we get

$$y(t)=m\int_{y(t)}^{H}(\sigma-u(z,t))dz=m(H-y(t))\sigma-m\int_{y(t)}^{H}u(z,t)dz.$$

Hence

$$y(t)=\frac{mH\sigma}{1+m\sigma}-\frac{m}{1+m\sigma}\int_{y(t)}^{H}u(z,t)dz. \tag{2·4}$$

This is the desired equation which relates the amount of the subsidence and the excess pore water pressure.

Since the clay stratum is in contact with the upper and lower drainage strata at the top and bottom respectively, we have the boundary condition

$$u(y(t),t)=u(H,t)=0. \tag{2·5}$$

It is natural to assume that the initial data

$$u(z,0)=\phi(z) \qquad (y(0)\leqq z\leqq H) \tag{2·6}$$

is non-negative and smooth, and satisfies the compatibility condition

$$\phi(y(0))=\phi(H)=0 \tag{2·7}$$

and

$$\int_{y(0)}^{H}\phi(z)dz=H\sigma-\frac{1+m\sigma}{m}y(0) \qquad \text{(see (2·4))}. \tag{2.8}$$

Consequently the problem is as follows.

Find the pair of functions $\{u(z,t), y(t)\}$ that satisfy the following equations and conditions:

$$[\mathrm{A}]\left\{\begin{array}{lr}
Lu\equiv u_{zz}-u_t=0 \qquad (y(t)<z<H,\ 0<t\leqq T) & (2·9)\\
y(t)=a-b\int_{y(t)}^{H}u(z,t)dz \qquad (0<t\leqq T) & (2·10)\\
y(0)=l>0 & \\
u(y(t),t)=0 \qquad (0<t\leqq T) & (2·11)\\
u(H,t)=0 \qquad (0<t\leqq T) & (2·12)\\
u(z,0)=\phi(z)\geqq 0 \qquad (l<z<H) & (2·13)\\
\quad\phi(l)=\phi(H)=0 & \\
\quad\int_{l}^{H}\phi(z)dz=H\sigma-l/b &
\end{array}\right.$$

where

$$a=\frac{mH\sigma}{1+m\sigma}>0, \quad b=\frac{m}{1+m\sigma}>0$$

and the coefficient of consolidation that appears in (2.1) is eliminated by the change of the variable $ct \to t$. In addition, we assume that $\phi(z)$ is twice differentiable on the interval $[l, H]$. Here, to be exact, l must be equal to zero, but because of the difficulty of defining the initial time strictly when the load is put down in the consolidation problem, we take the origin of time a little later, after the subsidence occurs, and so let $l>0$.

Definition. *The pair of functions* $\{u(z, t), y(t)\}$ *is the solution of system* $[A]$ *if the following condition are satisfied*;

1) $y(0)=l$, $y(t)>0$, $y(t)$ *is continuous for* $0 \leqq t \leqq T$, *continuously differentiable for* $0< t \leqq T$ *and satisfies* (2.10).
2) *In the domain* $D=\{(z, t) \mid y(t)<z<H,\ 0<t \leqq T\}$ u_{zz} *and* u_t *are continuous and satisfy the equation* (2.9).
3) *On the closed domain* $\bar{D}=\{(z, t) \mid y(t) \leqq z \leqq H,\ 0 \leqq t \leqq T\}$ u *is continuous and satisfies* (2·11), (2·12) *and* (2·13) *on the boundary.*

Our aim is to prove the following theorem.

Main Theorem. *If the initial data* $\phi(z)$ *is twice continuously differentiable for* $l \leqq z \leqq H$ *and satisfies* $\max_{l \leq z \leq H} \phi(z) < \frac{9\pi}{64\sqrt{2}b}$, *then there exists the unique global solution* $\{u(z, t), y(t)\}$ *of system* $[A]$.

3. Representation of the Solution

Here we give a representation of the solution $\{u(z, t), y(t)\}$ of system [A] by using some data on the boundary which are, in fact, themselves unknown. Suppose now that $u(\xi, \tau)$ and $v(\xi, \tau)$ satisfy the equations $u_{\xi\xi}-u_\tau=0$ and $v_{\xi\xi}+v_\tau=0$ in the domain $D_t= \{(\xi, \tau) \mid y(\tau)<\xi<H, 0<\tau<t\}$ (see Fig. 3). Then

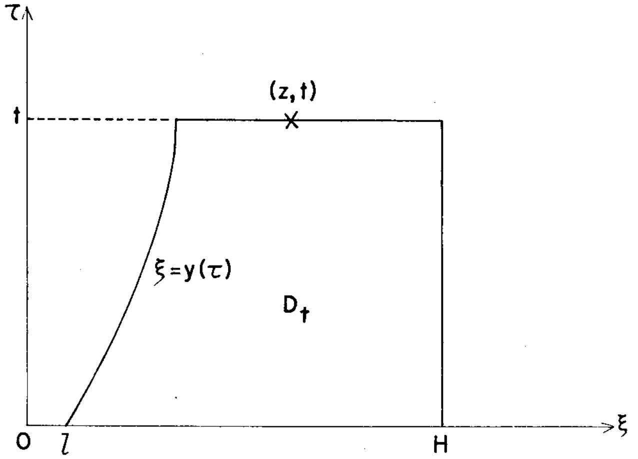

Fig. 3. Domain D_t.

$$\iint_{D_t} \{v(u_{\xi\xi}-u_\tau)-u(v_{\xi\xi}+v_\tau)\}\,d\xi d\tau=0. \tag{3·1}$$

Hence

$$\iint_{D_t} \{(vu_\xi-uv_\xi)_\xi-(uv)_\tau\}\,d\xi d\tau=0.$$

Using Green's formula we obtain

$$\int_{\partial D_t} uv d\xi+(vu_\xi-uv_\xi)d\tau=0. \tag{3·2}$$

Here we will take for u the solution $u(z, t)$ of system [A] and for v

$$v(\xi, \tau)=g(z, t+\varepsilon;\ \xi, \tau) \qquad (y(t)<z<H,\ 0<t\leqq T,\ \varepsilon>0) \tag{3·3}$$

where $g(z, t+\varepsilon;\ \xi, \tau)$ is the function defined by using the fundamental solution U to the heat equation $LU\equiv U_{zz}-U_t=0$ as follows:

$$g(z, t+\varepsilon;\ \xi, \tau)\equiv U(H-z, t+\varepsilon;\ H-\xi, \tau)-U(H-z, t+\varepsilon;\ -(H-\xi), \tau) \tag{3.4}$$

$$U(z, t;\ \xi, \tau)=\begin{cases} \dfrac{1}{2\sqrt{\pi(t-\tau)}}\exp\left\{-\dfrac{(z-\xi)^2}{4(t-\tau)}\right\} & (t>\tau) \\ 0 & (t\leqq\tau). \end{cases} \tag{3.5}$$

This function is called Green's function for the first boundary value problem of the heat equation in the half space $z<H$, and has the property;

$$g_{\xi\xi}+g_\tau=0, \quad g|_{z=H}=g|_{\xi=H}=0.$$

Evidently, v satisfies the equation $v_{\xi\xi}+v_\tau=0$ and the condition

$$v(H, \tau)=0 \qquad (0<\tau<t). \tag{3·6}$$

Hence, from the conditions (2.11)∼(2.13) in [A] and (3.6), (3.2) turns into the equation

$$\int_l^H g(z, t+\varepsilon;\ \xi, 0)\phi(\xi)d\xi-\int_{y(t)}^H g(z, t+\varepsilon;\ \xi, t)u(\xi, t)d\xi$$
$$-\int_0^t g(z, t+\varepsilon;\ y(\tau), \tau)u_\xi(y(\tau), \tau)d\tau=0$$

or equivalently

$$\int_{y(t)}^H g(z, t+\varepsilon;\ \xi, t)u(\xi, t)d\xi=\int_l^H g(z, t+\varepsilon;\ \xi, 0)\phi(\xi)d\xi$$
$$-\int_0^t g(z, t+\varepsilon;\ y(\tau), \tau)u_\xi(y(\tau), \tau)d\tau. \tag{3.7}$$

Now passing to the limit for $\varepsilon\rightarrow+0$, we find that the left side of (3.7) is equal to $u(z, t)$; for in the expression

$$\lim_{\varepsilon\to 0}\int U(z, t+\varepsilon;\ \xi, t)u(\xi, t)d\xi-\lim_{\varepsilon\to 0}\int U(2H-z, t+\varepsilon;\ \xi, t)u(\xi, t)d\xi$$

the first term is equal to $u(z, t)$, and the second vanishes because $2H-z$ is out of the interval $y(t)<\xi<H$. On the other hand, since the point (z, t) is away from the boundary of the domain D_t, the right side of (3.7) is a continuous function of ε when ε is sufficiently small. Consequently, passing to the limit for $\varepsilon\to+0$ in (3.7), we obtain

$$u(z, t)=\int_l^H g(z, t; \xi, 0)\phi(\xi)d\xi-\int_0^t g(z, t; y(\tau), \tau)u_z(y(\tau), \tau)d\tau. \tag{3.8}$$

This shows that the value of the solution u is represented by using the data of u at $t=0$, the amount of subsidence and the value of the function u_z on the boundary $z=y(t)$.

Next, differentiating (2.10) with respect to t, we obtain

$$\dot{y}(t)=-bu(y(t), t)\dot{y}(t)-b\int_{y(t)}^H u_t(z, t)dz.$$

Using (2.9) and (2.11)

$$\dot{y}(t)=-b\int_{y(t)}^H u_{zz}(z, t)dz=-b\{u_z(H, t)-u_z(y(t), t)\}.$$

Integrating both sides, we find that the amount of subsidence y is also represented by using the values of the function u_z on the boundaries $z=y(t)$ and $z=H$;

$$y(t)=l+b\int_0^t \{u_z(y(\tau), \tau)-u_z(H, \tau)\}\, d\tau. \tag{3.9}$$

4. Reduction to the System of Integral Equations

If $y(t)$ and $v(t)=u_z(y(t), t)$ are determined, we find also the desired solution u easily from (3.8). In fact, we now suppose the following are known: $y(t)$ is continuous on $[0, T]$, contuniously differentiable on $(0, T]$, $\dot{y}(t)$ is bounded on $(0, T]$ and $v(t)$ is bounded and continuous on $(0, T]$. Then we can easily show that the function defined by (3.8) satisfies the equation (2.9), the boundary conditions (2.11), (2.12) and the initial condition (2.13).

On the other hand, we can determine the amount of subsidence y if $v(t)$ and $w(t)=u_z(H, t)$ are known. In fact, if $v(t)$ and $w(t)$ are both bounded and continuous on $(0, T]$, it is obvious that the function y defined by (3.9) is continuously differentiable and $\dot{y}(t)$ is bounded on $(0, T]$. Furthermore, we find that it satisfies (2.10) by pursuing inversely the process which led to (3.9) and also using the conditions $u(y(t), t)=0$, $y(0)=l$ and $\int_l^H u(z, 0)dz=\int_l^H \phi(z)dz=H\sigma-l/b$.

Therefore, if we could determine $v(t)$, $w(t)$ such that

$$\lim_{z\to y(t)+0} u_z(z, t)=v(t) \qquad (0<t\leqq T) \tag{4.1}$$

$$\lim_{z\to H-0} u_z(z, t)=w(t) \qquad (0<t\leqq T) \tag{4.2}$$

we would conclude that system [A] is solved completely.

Let us write down conditions (4.1) and (4.2) explicitly. From (3.8)

$$u(z,t)=\int_l^H \{U(z,t;\xi,0)-U(z-2H,t;\ -\xi,0)\}\phi(\xi)d\xi$$
$$-\int_0^t \{U(z,t;y(\tau),\tau)-U(z-2H,t:\ -y(\tau),\tau)\}v(\tau)d\tau \qquad (4.3)$$

Differentiating with respect to z, we obtain

$$u_z(z,t)=\int_l^H \{U_z(z,t;\xi,0)-U_z(z-2H,t;\ -\xi,0)\}\phi(\xi)d\xi$$
$$-\int_0^t \{U_z(z,t;y(\tau),\tau)-U_z(z-2H,t;\ -y(\tau),\tau)\}v(\tau)d\tau \qquad (4.4)$$

Passing to the limit for $z\to y(t)+0$, from the well known property of the double layer potential we find that

$$\lim_{z\to z(t)+0} u_z(z,t)=\int_l^H \{U_z(y(t),t;\xi,0)-U_z(y(t)-2H,t;\ -\xi,0)\}\phi(\xi)d\xi$$
$$-\int_0^t \{U_z(y(t),t;y(\tau),\tau)-U_z(y(t)-2H,t;\ -y(\tau),\tau)\}v(\tau)d\tau$$
$$+\frac{1}{2}v(t).$$

Hence the condition (4.1) means that

$$v(t)=2\int_l^H \{U_z(y(t),t;\xi,0)-U_z(y(t)-2H,t;\ -\xi,0)\}\phi(\xi)d\xi$$
$$-2\int_0^t \{U_z(y(t),t;y(\tau),\tau)-U_z(y(t)-2H,t;\ -y(\tau),\tau)\}v(\tau)d\tau. \qquad (4.5)$$

Next, passing to the limit for $z\to H-0$ in (4.4), the condition (4.2) turns to be

$$w(t)=2\int_l^H U_z(H,t;\xi,0)\phi(\xi)d\xi-2\int_0^t U_z(H,t;y(\tau),\tau)v(\tau)d\tau. \qquad (4.6)$$

Rewriting (3.9), we obtain

$$y(t)=l+b\int_0^t \{v(\tau)-w(\tau)\}\,d\tau. \qquad (4.7)$$

Thus we have reduced our original problem to that of finding the functions $\{v(t), w(t), y(t)\}$ that satisfy (4.5), (4.6) and (4.7).

5. Existence and Uniqueness of the Solution

Here we will go to the proof of the main theorem. For that we consider only the following equivalent system;

$$[\mathrm{B}]\begin{cases} v(t)=2\int_l^H \{U_z(y(t),t;\xi,0)-U_z(y(t)-2H,t;\ -\xi,0)\}\phi(\xi)d\xi \\ \qquad -2\int_0^t \{U_z(y(t),t;y(\tau),\tau)-U_z(y(t)-2H,t;\ -y(\tau),\tau)\}v(\tau)d\tau \\ w(t)=2\int_l^H U_z(H,t;\xi,0)\phi(\xi)d\xi-2\int_0^t U_z(H,t;y(\tau),\tau)v(\tau)d\tau \\ y(t)=l+b\int_0^t \{v(\tau)-w(\tau)\}\,d\tau. \end{cases}$$

First we prove the existence of the local solution of system [B] by Picard's method of successive approximation. Let $v_0(t)$ and $w_0(t)$ be continuously differentiable functions for $t>0$, and $y_0(t)$ be a function which is continuous for $t\geqq 0$ and continuously differentiable for $t>0$ such that

$$\begin{aligned}&|v_0(t)|<N_1,\ |w_0(t)|<N_2,\ |y_0(t)|<N_3=H-h(h>0),\ y_0(0)=l\\ &|\sqrt{t}\,\dot{v}_0(t)|<M_1,\ |\sqrt{t}\,\dot{w}_0(t)|<M_2,\ |\dot{y}_0(t)|<M_3,\end{aligned}\tag{5.1}$$

where N_1, N_2, N_3, M_1, M_2, M_3 are some constants which are determined later.

We now define the sequences of functions $\{v_n(t)\}$, $\{w_n(t)\}$ and $\{y_n(t)\}$ $(n=1, 2, 3, \cdots)$ by the scheme

$$\begin{cases} v_n(t)=2\int_l^H \{U_x(y_{n-1}(t), t; \xi, 0)-U_x(y_{n-1}(t)-2H, t; -\xi, 0)\}\phi(\xi)d\xi \\ \qquad -2\int_0^t \{U_x(y_{n-1}(t), t; y_{n-1}(\tau), \tau)-U_x(y_{n-1}(t)-2H, t; -y_{n-1}(\tau), \tau)\} \\ \qquad \times v_{n-1}(\tau)d\tau \\ w_n(t)=2\int_l^H U_x(H, t; \xi, 0)\phi(\xi)d\xi-2\int_0^t U_x(H, t; y_{n-1}(\tau), \tau)v_{n-1}(\tau)d\tau \\ y_n(t)=l+b\int_0^t \{v_{n-1}(\tau)-w_{n-1}(\tau)\}\, d\tau \end{cases}\tag{5.2}$$

In the following, we prove the existence of a time $T^*>0$ such that for $0\leqq t\leqq T^*$ the sets of functions $\{v_n(t)\}$, $\{w_n(t)\}$ and $\{y_n(t)\}$ $(n=0, 1, 2, \cdots)$ are uniformly bounded and equicontinuous. If there exists such a $T^*>0$, we find from the theorem of Ascoli and Arzelà that the sequences of functions $\{v_n(t)\}$, $\{w_n(t)\}$ $\{y_n(t)\}$ $(n=0, 1, 2,\cdots)$ contain subsequences that converge uniformly. Then the limiting functions $v(t)$, $w(t)$ and $y(t)$ of the subsequences turn to be the solution of system [B]. This means the existence of the solution in the small interval.

Lemma 1. *There exists a $T_1>0$ such that for $0\leqq t\leqq T_1$ the following inequalities are valid*;

$$\begin{aligned}&|v_n(t)|<N_1,\ |w_n(t)|<N_2,\ |y_n(t)|<N_3=H-h \quad (h>0)\\ &|\dot{y}_n(t)|<M_3 \qquad (n=0, 1, 2, \cdots)\end{aligned}\tag{5.3}$$

Proof. We will prove by mathematical induction. For $n=0$ it is obvious from (5.1). We show it for $n=k+1$ under the assumption that it is valid for $n=k$.

First we consider $v_{k+1}(t)$;

$$\begin{aligned} v_{k+1}(t)=2\int_l^H &\{U_x(y_k(t), t; \xi, 0)-U_x(y_k(t)-2H, t; -\xi, 0)\}\phi(\xi)d\xi \\ &-2\int_0^t \{U_x(y_k(t), t; y_k(\tau), \tau)-U_x(y_k(t)-2H, t; -y_k(\tau), \tau)\} \\ &\times v_k(\tau)d\tau. \end{aligned}$$

Setting

$$v_{k+1}(t)=I_1+I_2+I_3+I_4,$$

$$I_1=2\int_l^H U_x(y_k(t),t;\xi,0)\phi(\xi)d\xi,$$

$$I_2=-2\int_l^H U_x(y_k(t)-2H,t;-\xi,0)\phi(\xi)d\xi,$$

$$I_3=-2\int_0^t U_x(y_k(t),t;y_k(\tau),\tau)v_k(\tau)d\tau$$

and

$$I_4=2\int_0^t U_x(y_k(t)-2H,t;-y_k(\tau),\tau)v_k(\tau)d\tau.$$

We estimate each integral.

Let $\max_{0\le\xi\le H}|\phi'(\xi)|=A_1$, $\sup_{\substack{x>2h\\t>0}}\frac{x}{t}e^{-x^2/4t}=C_1$. By calculation, we obtain

$$|I_1|<A_1, \tag{5.4}$$

$$|I_2|<A_1, \tag{5.5}$$

$$|I_3|<\frac{M_3N_1}{\sqrt{\pi}}\sqrt{t} \tag{5.6}$$

and

$$|I_4|<\frac{C_1N_1}{\sqrt{\pi}}\sqrt{t}\,. \tag{5.7}$$

From the inequalities (5.4)~(5.7) we find that

$$|v_{k+1}(t)|<2A_1+\frac{N_1}{\sqrt{\pi}}(M_3+C_1)\sqrt{t}\,. \tag{5.8}$$

Next we consider $w_{k+1}(t)$.

$$w_{k+1}(t)=2\int_0^H U_x(H,t;\xi,0)\phi(\xi)d\xi-2\int_0^t U_x(H,t;y_k(\tau),\tau)v_k(\tau)d\tau$$
$$=I_5+I_6,$$

where

$$I_5=2\int_0^H U_x(H,t;\xi,0)\phi(\xi)d\xi,$$

$$I_6=-2\int_0^t U_x(H,t;y_k(\tau),\tau)v_k(\tau)d\tau.$$

Let $\sup_{\substack{x\geq h\\t>0}}\frac{x}{t}e^{-x^2/4t}=C_2$. We obtain the estimates

$$|I_5|<A_1 \tag{5.9}$$

and

$$|I_6|<\frac{C_2N_1}{\sqrt{\pi}}\sqrt{t} \tag{5.10}$$

From (5.9) and (5.10) we find that

$$|w_{k+1}(t)|<A_1+\frac{C_2N_1}{\sqrt{\pi}}\sqrt{t}\,. \tag{5.11}$$

Finally we consider $y_{k+1}(t)$. From the formula

$$y_{k+1}(t)=l+b\int_0^t \{v_k(\tau)-w_k(\tau)\}\,d\tau,$$

we obtain

$$|y_{k+1}(t)|<l+b(N_1+N_2)t. \tag{5.12}$$

Since $\dot{y}_{k+1}(t)=b\{v_k(t)-w_k(t)\}$, we find that

$$|\dot{y}_{k+1}(t)|<b(N_1+N_2). \tag{5.13}$$

Here we can take the constants N_1, N_2 and M_3 so large that $2A_1<\frac{1}{2}N_1$, $A_1<\frac{1}{2}N_2$, $b(N_1+N_2)<M_3$ and take h so small that $l<\frac{1}{2}(H-h)$. Then we find from (5.8) and (5.11)~(5.13) that there exists a number $T_1>0$ which satisfies the following inequalities;

$$2A_1+\frac{N_1}{\sqrt{\pi}}(M_3+C_1)\sqrt{T_1}<N_1,$$

$$A_1+\frac{C_2N_1}{\sqrt{\pi}}\sqrt{T_1}<N_2$$

and

$$l+b(N_1+N_2)T_1<N_3.$$

Hence it is valid for $n=k+1$. This completes the proof.

Lemma 2. *There exists a $T_2>0$ such that for $0<t<T_2$ the following inequalities are valid*;

$$|\sqrt{t}\,\dot{v}_n(t)|<M_1,\ |\sqrt{t}\,\dot{w}_n(t)|<M_2 \qquad (n=0, 1, 2, \cdots). \tag{5.14}$$

Proof. We assume that (5.14) is valid for $n=k$. First we consider $v_{k+1}(t)$. Using the same notation as in Lemma 1,

$$v_{k+1}(t)=I_1+I_2+I_3+I_4.$$

Differentiating with respect to t, we get

$$\dot{v}_{k+1}(t)=\dot{I}_1+\dot{I}_2+\dot{I}_3+\dot{I}_4.$$

Let us estimate each integral. Introducing the notation

$$\max_{0\le\xi\le H}|\phi''(\xi)|=A_2,\ \sup_{\substack{x>h\\ t>0}}\frac{x}{t^{3/2}}e^{-x^2/4t}=C_3,\ \sup_{x>H-h}\frac{x}{t^{3/2}}e^{-x^2/4t}=C_4,$$

$$\sup_{t>0} \frac{1}{t} e^{-h^2/t} = C_5, \quad \sup_{\substack{x>2h \\ t>0}} \frac{x}{t^2} e^{-x^2/4t} = C_6, \quad \sup_{\substack{x>2h \\ t>0}} \frac{x^2}{t^2} e^{-x^2/4t} = C_7,$$

$$\sup_{\substack{x>2h \\ t>0}} \frac{x^3}{t^3} e^{-x^2/4t} = C_8$$

we obtain the following estimates

$$|\dot{I}_1| < \frac{3M_3A_1 + 2A_2}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}} + \frac{C_3A_1}{2\sqrt{\pi}}, \tag{5.15}$$

$$|\dot{I}_2| < \frac{M_3A_1 + A_2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}} + \frac{A_1}{2\sqrt{\pi}}(C_3 + C_4), \tag{5.16}$$

$$|\dot{I}_3| < \frac{N_1M_3}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}} + \frac{3bN_1}{\sqrt{\pi}}(M_1 + M_2) + \frac{3}{2}\sqrt{\pi}\, M_3M_1 + \frac{17M_3{}^3N_1}{2\sqrt{\pi}}\sqrt{t}$$
$$+ \frac{3\sqrt{\pi}\, M_3{}^3}{4} t + \frac{9M_3{}^5N_1}{16\sqrt{\pi}} t^{3/2} \tag{5.17}$$

and

$$|\dot{I}_4| < \frac{N_1}{4\sqrt{\pi}}(4M_3C_5 + 6C_6 + 2C_7 + C_8)\sqrt{t}\,. \tag{5.18}$$

Hence, from (5.14)~(5.17) we find that

$$|\sqrt{t}\, \dot{v}_{k+1}(t)| < \frac{5M_3A_1 + N_1M_3 + 4A_2}{2\sqrt{\pi}} + \left\{ \frac{A_1(2C_3 + C_4)}{2\sqrt{\pi}} + \frac{3bN_1}{\sqrt{\pi}}(M_1 + M_2) \right.$$
$$\left. + \frac{3}{2}\sqrt{\pi}\, M_3M_1 \right\} \sqrt{t} + \frac{N_1}{4\sqrt{\pi}}(34M_3{}^3 + 4M_3C_5 + 6C_6 + 2C_7 + C_8)t$$
$$+ \frac{3\sqrt{\pi}\, M_3{}^5}{4} t^{3/2} + \frac{9M_3{}^5N_1}{16\sqrt{\pi}} t^2. \tag{5.19}$$

Next we consider $w_{k+1}(t)$.

$$w_{k+1}(t) = I_5 + I_6.$$

Differentiating with respect to t, we get

$$\dot{w}_{k+1}(t) = \dot{I}_5 + \dot{I}_6.$$

Introducing the notation

$$\sup_{t>0} \frac{1}{t^{3/2}} e^{-H^2/4t} = C_9, \quad \sup_{\substack{x>h \\ t>0}} \frac{x}{t^2} e^{-x^2/4t} = C_{10}, \quad \sup_{\substack{x>h \\ t>0}} \frac{x^3}{t^3} e^{-x^2/4t} = C_{11}$$

we obtain the estimates

$$|\dot{I}_5| < \frac{A_1 + A_2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}} + \frac{HC_9}{2\sqrt{\pi}} \tag{5.20}$$

and

$$|\dot{I}_6| < \frac{N_1}{4\sqrt{\pi}}(6C_{10}+C_{11})\sqrt{t}\,. \tag{5.21}$$

Hence, from (5.20) and (5.21) we find that

$$|\sqrt{t}\,\dot{w}_{k+1}(t)| < \frac{A_1+A_2}{\sqrt{\pi}} + \frac{HC_9}{2\sqrt{\pi}}\sqrt{t} + \frac{N_1}{4\sqrt{\pi}}(6C_{10}+C_{11})t. \tag{5.22}$$

Here we can take constants M_1 and M_2 so large that $\frac{5M_3A_1+N_1M_3+4A_2}{2\sqrt{\pi}} < \frac{1}{2}M_1$, $\frac{A_1+A_2}{\sqrt{\pi}} < \frac{1}{2}M_2$. Then from (5.19) and (5.22) we find that there exists a number $T_2(\leqq T_1)>0$ independent of k such that for $0<t\leqq T_2$ (5.14) is also valid for $n=k+1$. This completes the proof.

Therefore setting $T^*=T_2$, we conclude from Lemma 1 and Lemma 2 that the set of functions $\{v_n(t)\}$, $\{w_n(t)\}$ and $\{y_n(t)\}$ $(n=0, 1, 2, \cdots)$ are all uniformly bounded and equi-continuous for $0\leqq t\leqq T^*$. This leads to the existence of the solution $\{v(t), w(t), y(t)\}$ of system [B] for $0\leqq t\leqq T^*$.

Next we will show that the result obtained above can be extended to an arbitrary interval $[0, T]$ if the initial data is sufficiently small, namely that the solution of the system exists globally. It is sufficient to prove the following assertion:

"For any positive number t_0, there exists an $\varepsilon>0$ independent of t_0 such that if the solution of system [B] exists for $0<t\leqq t_0$, it also exists for $0<t\leqq t_0+\varepsilon$."

Examining the discussion about the existence of the local solution, we find that we have only to show that $\sup\limits_{y(t_0-\eta)<z<H}|u_z(z, t_0-\eta)|$ and $\sup\limits_{y(t_0-\eta)<z<H}|u_{zz}(z, t_0-\eta)|$ are uniformly bounded with respect to any sufficiently small $\eta>0$. In order to show this we will prove several lemmas.

Lemma 3. *If the free boundary $z=y(t)$ in [A] is given, then $v(t)>0$ and $w(t)<0$.*

This can be easily seen from the strong maximum principle and Friedman's lemma (see 2)).

From this lemma we find that $\dot{y}(t)>0$, that is, $y(t)$ is strictly monotonously increasing.

Lemma 4. *The uniform boundedness of $\sup\limits_{y(t_0-\eta)<z<H}|u_z(z,t_0-\eta)|$ and $\sup\limits_{y(t_0-\eta)<z<H}|u_{zz}(z, t_0-\eta)|$ with respect to any sufficiently small $\eta>0$ follows from the boundedness of $v(t)$ for $0<t\leqq t_0$.*

Proof. It is easily seen that the uniform boundedness of $\sup\limits_{y(t_0-\eta)<z<H}|u_z(z, t_0-\eta)|$ follows from the expression (4.4) of $u_z(z, t)$. We now examine the function $s(z, t)=u_{zz}(z, t)$ in the domain $D_{t_0-\eta}=\{(z, t)|y(t)<z<H,\ 0<t\leqq t_0-\eta\}$. It is clear that this function satisfies the equation $Ls\equiv s_{zz}-s_t=0$ in the domain $D_{t_0-\eta}$ and it is continuous on the closed domain $\bar{D}_{t_0-\eta}$, except for the points $(l, 0)$ and $(H, 0)$. Moreover, it satisfies the initial and boundary conditions:

$$s(z, 0)=\phi''(z) \qquad (l<z<H)$$
$$s(H, t)=u_t(H, t)=0 \qquad (0<t\leqq t_0-\eta)$$
$$s(y(t), t)=-bv(t)\{v(t)-w(t)\} \qquad (0<t\leqq t_0-\eta)$$

In fact, since $y(t)$ is strictly monotonously increasing, passing to the limit for $\Delta t\to 0$ in

$$\frac{u(y(t), t)-u(y(t), t-\Delta t)}{\Delta t}=-\frac{u(y(t), t-\Delta t)-u(y(t-\Delta t), t-\Delta t)}{\Delta y}\frac{\Delta y}{\Delta t}$$
$$(\Delta y=y(t)-y(t-\Delta t))$$

we obtain

$$u_t(y(t), t-0)=-u_z(y(t)+0, t)\dot{y}(t)$$

Consequently,

$$u_{zz}(y(t), t-0)=-bv(t)\{v(t)-w(t)\}$$

Here we note that under the assumption that $v(t)$ is bounded for $0<t\leqq t_0$, $w(t)$ is also bounded for $0<t\leqq t_0$ from the expression (4.6).

Therefore $s(z, t)$ is bounded on $t=t_0-\eta$ by the maximum principle. Since this boundedness is uniform with respect to η, the uniform boundedness of $\sup_{y(t_0-\eta)<z<H}|u_{zz}(z, t_0-\eta)|$ with respect to η also follows from the boundedness of $v(t)$ for $0<t\leqq t_0$. This completes the proof.

From this lemma we find that in order to prove our assertion it is sufficient to show the boundedness of $v(t)$ for $0<t\leqq t_0$.

Now let us show this. Since $v(t)>0$, it suffices to estimate its supremum. Let $\mu=2\eta$. We introduce the functions

$$p(t)=\sup_{t_0-\mu\leq\tau<t} v(\tau)$$

and

$$q(t)=\inf_{t_0-\mu\leq\tau<t} w(\tau).$$

For $t_0-\mu<t\leqq t_0$ we obtain from (4.5)

$$v(t)=2\int_{y(t_0-\mu)}^{H}\{U_z(y(t), t; \xi, t_0-\mu)-U_z(y(t)-2H, t; -\xi, t_0-\mu)\}u(\xi, t_0-\mu)d\xi$$
$$-2\int_{t_0-\mu}^{t}\{U_z(y(t), t; y(\tau), \tau)-U_z(y(t)-2H, t; -y(\tau), \tau)\}v(\tau)d\tau \qquad (5.23)$$

We denote this as follows;

$$v(t)=I_1'+I_2'+I_3'+I_4',$$
$$I_1'=2\int_{y(t_0-\mu)}^{H}U_z(y(t), t; \xi, t_0-\mu)u(\xi, t_0-\mu)d\xi,$$

$$I_2'=-2\int_{y(t_0-\mu)}^{H} U_x(y(t)-2H, t;\ -\xi, t_0-\mu)u(\xi, t_0-\mu)d\xi,$$

$$I_3'=-2\int_{t_0-\mu}^{t} U_x(y(t), t; y(\tau), \tau)v(\tau)d\tau,$$

$$I_4'=2\int_{t_0-\mu}^{t} U_x(y(t)-2H, t;\ -y(\tau), \tau)v(\tau)d\tau.$$

Let us estimate each integral. First we consider I_1'.

$$I_1'=\int_{y(t_0-\mu)}^{H} -\frac{y(t)-\xi}{2\sqrt{\pi}\,(t-t_0+\mu)^{3/2}}\exp\left\{-\frac{(y(t)-\xi)^2}{4(t-t_0+\mu)}\right\}u(\xi, t_0-\mu)d\xi.$$

Let $\max_{0\leq\xi\leq H}\phi(\xi)=A_0$ and set $\frac{(y(t)-\xi)^2}{4(t-t_0+\mu)}=\eta$. Since $u(\xi, t_0-\mu)<A_0$ by the maximum principle we obtain

$$0<I_1'<\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{t-t_0+\mu}}\int_0^\infty e^{-\eta}d\eta=\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{t-t_0+\mu}}. \tag{5.24}$$

We now examine I_2'.

$$I_2'=\int_{y(t_0-\mu)}^{H}\frac{y(t)+\xi-2H}{2\sqrt{\pi}\,(t-t_0+\mu)^{3/2}}\exp\left\{-\frac{(y(t)+\xi-2H)}{4(t-t_0+\mu)}\right\}u(\xi, t_0-\mu)d\xi.$$

Since $y(t)+\xi-2H<0$,

$$I_2'<0. \tag{5.25}$$

We now examine I_3'.

$$I_3'=\int_{t_0-\mu}^{t}\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{t-\tau}}\frac{y(t)-y(\tau)}{t-\tau}\exp\left\{-\frac{(y(t)-y(\tau))^2}{4(t-\tau)}\right\}v(\tau)d\tau.$$

Hence,

$$0<I_3'<\frac{1}{\sqrt{\pi}}\sup_{t_0-\mu<\tau<t}\dot{y}(\tau)\cdot p(t)\sqrt{\mu}=\frac{b}{\sqrt{\pi}}\{p(t)-q(t)\}p(t)\sqrt{\mu}. \tag{5.26}$$

Finally we examine I_4'.

$$I_4'=\int_{t_0-\mu}^{t} -\frac{y(t)+y(\tau)-2H}{2\sqrt{\pi}\,(t-\tau)^{3/2}}\exp\left\{-\frac{(y(t)+y(\tau)-2H)^2}{4(t-\tau)}\right\}v(\tau)d\tau.$$

Setting $\sup_{\substack{x>2h\\ t>0}}\frac{x}{t^{3/2}}e^{-x^2/4t}=C_{12}$, we obtain

$$0<I_4'<\frac{C_{12}}{2\sqrt{\pi}}p(t)\mu. \tag{5.27}$$

From the estimates (5.24)～(5.27) we obtain

$$p(t)<\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{t-t_0+\mu}}+\frac{b}{\sqrt{\pi}}\{p(t)-q(t)\}p(t)\sqrt{\mu}+\frac{C_{12}}{2\sqrt{\pi}}p(t). \tag{5.28}$$

Next let us estimate $q(t)$. For $t_0-\mu<t\leqq t_0$, we have from (4.6)

$$w(t)=2\int_{y(t_0-\mu)}^{H} U_x(H,t;\xi,t_0-\mu)u(\xi,t_0-\mu)d\xi$$
$$-2\int_{t_0-\mu}^{t} U_x(H,t;y(\tau),\tau)v(\tau)d\tau. \tag{5.29}$$

Using $w(t)<0$ and

$$-2\int_{t_0-\mu}^{t} U_x(H,t;y(\tau),\tau)v(\tau)d\tau$$
$$=\frac{1}{2\sqrt{\pi}}\int_{t_0-\mu}^{t}\frac{H-y(\tau)}{(t-\tau)^{3/2}}\exp\left\{-\frac{(H-y(\tau))^2}{4(t-\tau)}\right\}v(\tau)d\tau>0,$$

we obtain

$$q(t)>2\int_{y(t_0-\mu)}^{H} U_x(H,t;\xi,t_0-\mu)u(\xi,t_0-\mu)d\xi>-\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{t-t_0+\mu}}. \tag{5.30}$$

Substituting (5.30) into (5.28) we obtain again the estimate for $p(t)$;

$$p(t)<\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{t-t_0+\mu}}+\frac{b\sqrt{\mu}}{\sqrt{\pi}}p(t)^2+\left(\frac{bA_0}{\pi}\sqrt{\frac{\mu}{t-t_0+\mu}}+\frac{C_{12}\mu}{2\sqrt{\pi}}\right)p(t). \tag{5.31}$$

Here restricting t as $t_0-\mu/2=t_0-\eta<t\leqq t_0$, it follows that

$$p(t)<\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{\eta}}+\frac{b\sqrt{2\eta}}{\sqrt{\pi}}p(t)^2+\frac{\sqrt{2}\,bA_0}{\pi}p(t)+\frac{C_{12}}{\sqrt{\pi}}\eta p(t). \tag{5.32}$$

We now assume on the initial data that

$$A_0<\frac{9\pi}{64\sqrt{2}}\cdot\frac{1}{b} \tag{5.33}$$

and take η such that

$$\frac{C_{12}}{\sqrt{\pi}}\eta\leqq\frac{1}{12}.$$

Then (5.32) turns to

$$\frac{3}{4}p(t)<\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{\eta}}+\frac{\sqrt{2}\,b\sqrt{\eta}}{\sqrt{\pi}}p(t)^2.$$

Therefore,

$$\frac{4\sqrt{2}\,b\sqrt{\eta}}{3\sqrt{\pi}}\left(p(t)-\frac{3\sqrt{\pi}}{8\sqrt{2}\,b\sqrt{\eta}}\right)^2-\left(\frac{3\sqrt{\pi}}{16\sqrt{2}\,b\sqrt{\eta}}-\frac{4}{3}\frac{A_0}{\sqrt{\pi}\sqrt{\eta}}\right)>0. \tag{5.34}$$

Combining (5.33) and (5.34), we get

$$p(t)<\frac{3\sqrt{\pi}}{8\sqrt{2}\,b\sqrt{\eta}}\ \text{ or }\ p(t)>\frac{3\sqrt{\pi}}{8\sqrt{2}\,b\sqrt{\eta}}\qquad(t_0-\eta<t\leqq t_0)$$

We now show that the left inequality is valid. Let t_1 be a sufficiently small positive number such that $t_1 < t_0$ and set

$$p_1(t) = \sup_{t_1-\eta\leq\tau<t\leq t_1} v(\tau)$$

Then there exists a $K>0$ such that $p_1(t)<K$. We choose η so small that

$$K < \frac{3\sqrt{\pi}}{8\sqrt{2}\,b\sqrt{\eta}} \qquad \left(\text{and of course } \frac{C_{12}}{\pi}\eta \leqq \frac{1}{12}\right)$$

and fix η hereafter. By the continuity of $v(\tau)$, for t_2 such that $t_1<t_2<t_1+\eta$, we also have

$$\sup_{t_2-\eta\leq\tau<t\leq t_2} v(\tau) = p_2(t) < \frac{3\sqrt{\pi}}{8\sqrt{2}\,b\sqrt{\eta}}.$$

Thus extending an interval successively, we finally obtain

$$\sup_{t_0-\eta\leq\tau<t\leq t_0} v(\tau) = p(t) < \frac{3\sqrt{\pi}}{8\sqrt{2}\,b\sqrt{\eta}}.$$

by the continuity of $v(\tau)$. Since $v(t)$ is evidently bounded for $0<t\leqq t_0-\eta$, the boundedness $v(t)$ for $0<t\leqq t_0$ has thus been verified.

Now we will prove the uniqueness of the solution of system [B]. We suppose that there exist two solutions, namely $\{v(t), w(t), y(t)\}$ and $\{v^*(t), w^*(t), y^*(t)\}$ of system [B]. Let $\delta v = \max|v^*(t)-v(t)|$, $\delta w = \max|w^*(t)-w(t)|$ and $\delta y = \max|y^*(t)-y(t)|$. We will make estimations of δv, δw and δy.

First we consider δv.

$$\begin{aligned} v^*(t)-v(t) &= 2\int_l^H \{U_z(y^*(t), t; \xi, 0) - U_z(y(t), t; \xi, 0)\}\phi(\xi)d\xi \\ &\quad -2\int_l^H \{U_z(y^*(t)-2H, t; -\xi, 0) - U_z(y(t)-2H, t; -\xi, 0)\}\phi(\xi)d\xi \\ &\quad -2\int_0^t \{U_z(y^*(t), t; y^*(\tau), \tau) - U_z(y(t), t; y(\tau), \tau)\}v^*(\tau)d\tau \\ &\quad -2\int_0^t U_z(y(t), t; y(\tau), \tau)\{v^*(\tau)-v(\tau)\}d\tau \\ &\quad +2\int_0^t \{U_z(y^*(t)-2H, t; -y^*(\tau), \tau) - U_z(y(t)-2H, t; -y(\tau), \tau)\} \\ &\qquad \times v^*(\tau)d\tau \\ &\quad +2\int_0^t U_z(y(t)-2H, t; -y(\tau), \tau)\{v^*(\tau)-v(\tau)\}d\tau \\ &= \Delta I_1 + \Delta I_2 + \Delta I_3 + \Delta I_4 + \Delta I_5 + \Delta I_6 \end{aligned}$$

Setting $\sup_{t>0} \frac{1}{t^{3/2}} e^{-h^2/t} = C_{13}$ and using notations introduced before, by calculation we obtain

$$|\Delta I_1|<\frac{2A_1}{\sqrt{\pi}}\,\delta y\cdot\frac{1}{\sqrt{t}}, \tag{5.35}$$

$$|\Delta I_2|<\frac{2A_1}{\sqrt{\pi}}\,\delta y\cdot\frac{1}{\sqrt{t}}, \tag{5.36}$$

$$|\Delta I_3|<\frac{bN_2}{\sqrt{\pi}}\,(1+M_3N_3)(\delta v+\delta w)\cdot\sqrt{t}, \tag{5.37}$$

$$|\Delta I_4|<\frac{M_3}{\sqrt{\pi}}\,\delta v\cdot\sqrt{t}, \tag{5.38}$$

$$|\Delta I_5|<\frac{N_1}{2\sqrt{\pi}}\,(2C_{12}+C_7)\delta y\cdot t \tag{5.39}$$

and

$$|\Delta I_6|<\frac{C_1}{\sqrt{\pi}}\,\delta y\cdot t. \tag{5.40}$$

Combining (5.35)~(5.40), we obtain

$$\delta v<\frac{4A_1}{\sqrt{\pi}}\,\delta y\cdot\frac{1}{\sqrt{t}}+\frac{bN_2}{\sqrt{\pi}}\,(1+M_3N_3)(\delta v+\delta w)\cdot\sqrt{t}+\frac{M_3+C_1}{\sqrt{\pi}}\,\delta v\cdot\sqrt{t}$$
$$+\frac{N_1}{2\sqrt{\pi}}\,(2C_{12}+C_7)\delta y\cdot t. \tag{5.41}$$

We now consider δw.

$$w^*(t)-w(t)=-2\int_0^t\{U_x(H,t;y^*(\tau),\tau)-U_x(H,t;y(\tau),\tau)\}v^*(\tau)d\tau$$
$$-2\int_0^t U_x(H,t;y(\tau),\tau)\{v^*(\tau)-v(\tau)\}\,d\tau$$
$$=\Delta I_7+\Delta I_8.$$

Setting $\sup\limits_{t>0}\frac{1}{t}e^{-h^2/4t}=C_{14}$, $\sup\limits_{\substack{x>h\\t>0}}\frac{x^2}{t^2}e^{-x^2/4t}=C_{15}$, we obtain

$$|\Delta I_7|<\frac{N_1}{4\sqrt{\pi}}\,(2C_{14}+C_{15})\delta y\cdot t \tag{5.42}$$

and

$$|\Delta I_8|<\frac{C_2}{\sqrt{\pi}}\,\delta v\cdot\sqrt{t} \tag{5.43}$$

Combining (5.42) and (5.43), we get

$$\delta w<\frac{N_1}{4\sqrt{\pi}}\,(2C_{14}+C_{15})\delta y\cdot t+\frac{C_2}{\sqrt{\pi}}\,\delta v\cdot\sqrt{t}. \tag{5.44}$$

Finally we consider δy. From the equation

$$y^*(t)-y(t)=b\int_0^t\{(v^*(\tau)-v(\tau))-(w^*(\tau)-w(\tau))\}\,d\tau$$

we have

$$\delta y < b(\delta v + \delta w)\cdot t. \tag{5.45}$$

Substituting (5.45) into the first term on the right side of (5.41), we obtain

$$\delta v < \frac{b}{\sqrt{\pi}}\{4A_1 + N_2(1 + M_3N_3)\}(\delta v + \delta w)\cdot\sqrt{t} + \frac{M_3 + C_1}{\sqrt{\pi}}\delta y\cdot\sqrt{t}$$

$$+\frac{N_1}{2\sqrt{\pi}}(2C_{12} + C_7)\delta y\cdot t. \tag{5.46}$$

Hence, from (5.44)~(5.46) taking $\delta = max\{\delta v, \delta w, \delta y\}$ we find that there exists a $L>0$ such that the following inequality is valid;

$$\delta < L(t + \sqrt{t})\delta.$$

For any $t>0$, it is true only if δ is equal to zero.

Thus we have proved the uniqueness of the solution, and consequently, the main theorem.

Acknowledgement

We should like to thank Dr. A. Asaoka, who gave us the problem and very useful advice, and also Prof. J. G. Gilmartin who corrected mistakes in the sentences of our manuscript.

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