Numerical Solution of Three-dimensional Eddy Current Problems

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Abstract

A general numerical method is given for the solution of quasi-stationary electromagnetic field problems. The integral equation-type formulation used permits the elimination of field quantities of inactive regions (air etc.) from the calculation. Some difficulties present in other methods (like surface charges etc.) are also avoided. The use of global variables in connection with the coordinate-independent (tensorial) form of the basic equations is likely to be helpful for an easy definition of a class of practical problems for a general computer program. A numerical example is included.

1. Introduction

1.1

The applicability of the generally used numerical methods for the calculation of general three-dimensional quasi-stationary electromagnetic fields is restricted by the great number of the field variables (e.g. node potentials). Although we are usually not interested in the values of the field quantities in inactive regions (like the air or other non-conductive materials), they are necessary for the correct formulation of the finite difference or finite element equations. It seems natural to ask for an alternate formulation of the basic equations by which we can eliminate the "superfluous" variables from the numerical calculation.

1.2

There is a problem concerning the "infinite boundaries". Usually we are forced to define a finite region with closed boundaries and boundary conditions having a unique solution for the algebraic system derived from the partial differential equations. In some cases these finite boundaries can be interpreted in physical terms. In other cases, how-

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ever, the more or less arbitrary boundaries introduce an error which is difficult to estimate. While some efforts have been made to find ways to deal with this problem, we would prefer a method in which the boundaries in infinity are inherent, since this is the physical reality.

1.3

We would like to have an easy way to deal with all kinds of symmetry conditions which we can find in a physical problem. A really general method should be able to take advantage of the symmetries to reduce the number of unknowns. For example, if the solution is an even or odd function of a variable, the usual procedure is to define a fictive, reflective boundary. In case of more general symmetries, however, the introduction of fictive boundaries with complicated reflecting properties could be both tiresome and aesthetically unpleasing.

1.4

The point-by point description of inner material interfaces (body surfaces inside the region of interest) can be so cumbersome that considerable efforts have been made to write computer programs for the automatic generation of the coefficients of the finite difference/finite element equations, asking for only a little geometric data as imput. It could be better in some cases to start from a global approximation which makes it possible to give the geometric data in functional form. The body of Fig. 1. is completely defined e.g. by the Algol 60 function

real procedure body (r, phi, z); body:=(if $Z\geq 0\Lambda Z<h\Lambda$ ($r\leq r2v(r>r2\Lambda$mod
(phi, alfa 1)$\geq$alfa 2))$\Lambda r\leq r1$ *then* 1.0 *else* 0.0)

Much more complicated examples can also be given. The use of global variables with the Ritz (Galerkin)-type formulation is theoretically possible but not widely used.

1.5

We have attempted to find a method of calculation that satisfies the conditions given in 1.1-4. In the following we will discuss the physical base and the mathematical formulation of the method.

2. derivation of the volume integral equation method

2.1

It is easy to satisfy 1.1 by choosing the current density (J) as a dependent variable. The current density is zero everywhere except in the conducting bodies, so we can leave these already known (0) values out of the calculation. According to 1.2, we take the Biot-Savart law as a start, in which way we get a solution for all space. (Another possibility could be to use a 'fast poisson-solver' like [1], but it would give a solution for a finite region.) To conform with 1.3 and 1.4 we approximate the current density distribution by one global function inside each conductive body. The approximating function can be, for example, a complete power series of the order n. Known properties of the solution can be directly taken into account by deleting terms from the series, or expressing some coefficients with others. The fact that the approximating function is defined only inside the conducting body is taken into account by introducing a contour function Y in a way similar to the example in 1.4

$$J(x_1, x_2, x_3)=r(x_1, x_2, x_3)\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} c_{ijk}\theta_{ijk}(x_1, x_2, x_3) \tag{2.1.1}$$

where

$r=1$: if (x_1, x_2, x_3) is inside the conducting body
 0: otherwise

and for example

$$\theta_{ijk}=x_1{}^i x_2{}^j x_3{}^k \tag{2.1.2}$$

or other similar function.

2.2

The basic equations are

$$H(r)=1/(4\pi)\int_v J(r')(r-r')/|r-r'|^3 dv \tag{2.2.1}$$

$$\text{curl}\ \bar{E}(r)=-j\omega\bar{B}(r) \tag{2.2.2}$$

$$\text{div}\ J=0 \tag{2.2.3}$$

$$J=\sigma\bar{E} \tag{2.2.4}$$

$$\bar{B}=\mu\bar{H} \tag{2.2.5}$$

Thus, in a homogeneous conductor, it follows

$$\text{curl}\ J=-j\omega\mu\sigma(1/(4\pi)\int_v J'*\Delta r/|\Delta r|^3 dv+\bar{H}_0) \tag{2.2.6}$$

where we have introduced for convenience the 'original magnetic field strength' ($\bar{H}_0$), defined as the field induced by known currents in the absence of eddy currents. Substituting (2.1.1) and (2.1.2) into the above (2.2.3) and (2.2.6) we get two expressions with r as parameter. Let's substitute now the $r_1\cdots r_m$ coordinates of m points into these expressions. We get $6m$ equations in $3(n+1)$ unknowns (n is the order of the approximating series). If $3(n+1)=6m$, then we get the solution using some standard algebraic equation solver like the Gauss elimination. Otherwise we may look for the solution in the least squares sense. In any case, the current distribution is found by substituting the solution (the coefficients) into the (2.1.1) series. The eddy current loss is given by

$$P=\int_v J.J/(2\sigma)dv \tag{2.2.7}$$

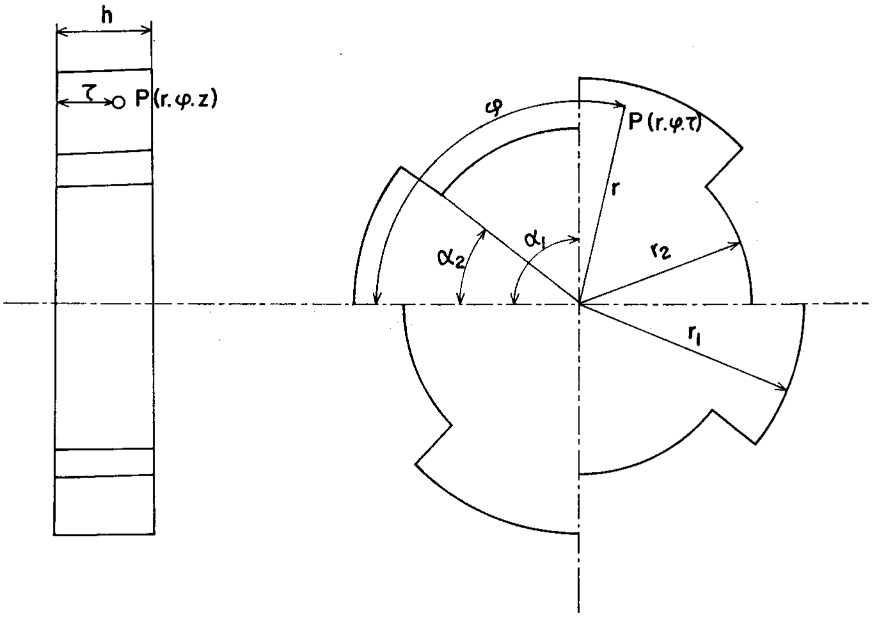

Fig. 1. Functional form Body.

The kernel of the (2.2.6) equation is singular. The (1) integral, however, is convergent as it can be seen considering the physical meaning. Care should be taken, however, not to attempt the evaluation of the kernel at the point of singularity when calculating the coefficients of the algebraic system.

2.3

We take the case of a homogeneous metal sphere placed into a pulsating (time-harmonic) magnetic field for an illustrating example. The exact solution is found in the textbook (e.g. [2]), and the asymptotic solution for the low or high frequency domains is also easy to find. The full power of the method presented here cannot be shown by this simple example. In a later article, a practical application will be given. The purpose of this example is to show the fast convergence and the easy applicability of the algorithm.

In the calculation we have taken into account the symmetry conditions possessed by the problem. The dependent variable (current density) has only one component and it is independent of one of the coordinates, if we use cylindrical or spherical coordinates. It means a 1/6 reduction in the number of unknowns compared with the general case. The derivation of the numerical formula for the case of circular cylindrical coordinates is given in the Appendix. The computed results together with the exact solution are shown in Fig. 2. For this simple case even a first order approximation gave satisfactory redults.

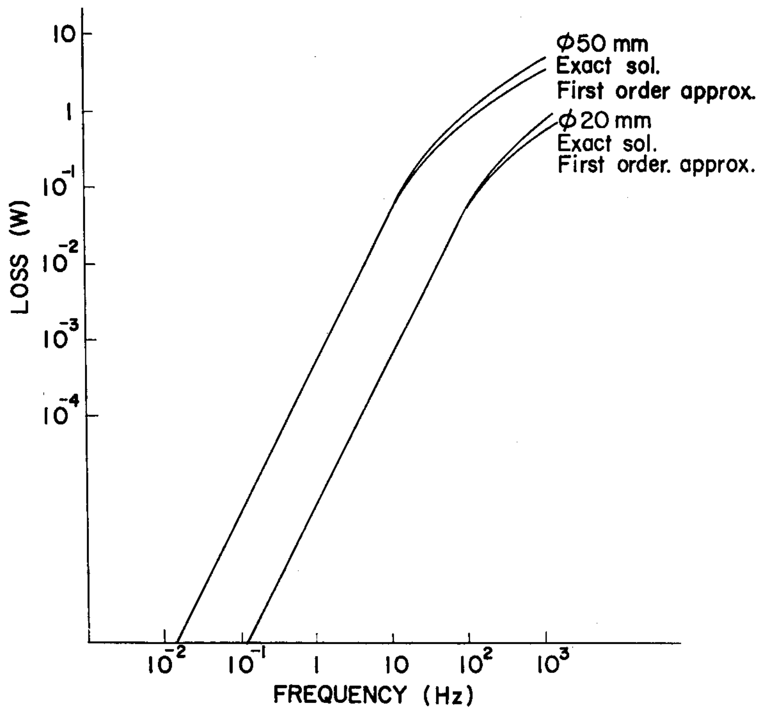

Fig. 2. Eddy Current Losses in a Conductive Spheres.
Original Magnetic Field Strength: $H_0=10^4 \cdot e^{j\omega t}$ A/m.
Specific Conductivity: $\sigma=50 \cdot 10^6$ l/ohm·m.
Magnetic Permeability: $\mu=\mu_0$.

3. discussion

The volume integral equation method, as presented here, is intended for solving some eddy current problems arising in large electric machines like turbine generators and power transformes. In the present form the method can be used only in the case of $\mu=\mu_0$; that is, in the absence of ferromagnetic materials. A simple extension would be to take the presence of an iron core into account approximately by the image method. The extension of the method for more general cases is beyond the scope of this article. We would suggest using a combination of the method presented here and the method for stationary fields given in [3]. For the cases of shielding, inductive heating, and coreless (e.g. superconductive) machines the present form is directly applicable.

Appendix/for "Numerical Solution of Three-dimensional Eddy Current Problems"/ Axially symmetrical TM induction problems are handled most conveniently in cylindrical coordinate systems. An approximating power series expression for the current density/ recognizing that it has only a peripherial component/is

$$J(r, \psi, z)=J(r, z)\bar{e}_\phi=\sum_{i=0}^{m}\sum_{k=0}^{n} c_{ik} r^i z^k$$

where the complex parameters c_{ik} are to be determined/divergence and boundary conditions being apriori satisfied/using/2.2.6/. Substituting the cylindrical coordinates/ r, ψ, z/ of a collocation point and the Cartesian coordinates/ξ, η, ζ/ of the volume elements of the integral into /2.2.6/, the equation is

$$\sum_{i=0}^{m}\sum_{k=0}^{n} c_{ik}\left\{(i+1)r^{i-1}z^{k}+j\frac{\omega\mu\sigma}{4\pi}\int_{(v)}\frac{\rho^{i-1}\zeta^{k}(\rho-r\xi)}{[(r-\xi)^{2}+\eta^{2}+(z-\zeta)^{2}]^{3/2}}d\xi d\eta d\zeta\right\}$$
$$=-j\omega\mu_{0}\sigma H_{0z}$$

where

$$\rho=(\xi^{2}+\eta^{2})^{1/2}$$

All collocation points are in the plane $\psi=0$. An equivalent expression can be given in the r direction, too. The symmetrical conditions of the model problem are taken into account by setting

$$c_{2p,q}=c_{p,2q+1}=0/p,\ q=0,\ 1,\ 2,\ \cdots/$$

Special aspects of choosing the collocation points and the quadrature formula will be treated in a subsequent paper.

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