A Network Analogy for Three-dimensional Eddy Current Problems

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Abstract

We present a numerical method for the solution of eeddy current problems arising in the analysis of electric machines. The method is based on a network analogy which is equivalent to a differential-integral equation formulation of Maxwell's equations, using the Biot-Savart Law. The present approach is intended mainly to give an intuitive picture rather than to be used as a general method for solving practical problems. A more effective method is presented in a companion article, where we start from a direct mathematical formulation.

1. Introduction

The recent advances in computing and numerical analysis has made it possible to solve efficiently a variety of classical problems arising in physics and engineering. Among them are partial differential equations describing e.g. the electromagnetic field distribution in an electrical machine. This kind of computation is very sensitive (both in cost and accuracy) on the particular algorithm used. For three-dimensional field distributions, yielding a large number of unknown variables after discretization, the numerical solution is often economically unfeasible. In such cases the engineer still has to resort to physical modelling or crude analytical approximations, and those are timeconsuming or unreliable. For this reason there is a constant search for new and better numerical techniques for field calculations.

Integral equation methods have been advocated for several reasons. They provide a natural way to deal with boundaries in infinity. Also in cases of stationary three dimensional magnetic fields and two-dimensional TM and TE-mode induction problems, it was shown how to reduce the dimensionality (and thus the number of unknowns)

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using the boundary-type Green functions for the kernel of the integral equation.[1),2),3),4)] The possibility of reducing a three-dimensional problem to a two-dimensional (surface-type) one becomes attractive, however, only when dealing with relatively large scale problems. For example, if the number of subdivisions of a solid cube is n in all directions, then the number of surface elements will be lower than the number of volume elements. Only when n is greater than 7, that is, for the case of 882 ($=3\times 6\times 7^2$) or more complex unknowns (vecotr potential components), will the shells (at most two volume elements think) have the same number of surface and volume elements. In that case the surface equation does not mean reduction of unknowns. Besides this, it is easy to find an intuitive picture for the volume type integral equations, as will be shown by the following. In contrast with this, it is not easy to give a simple physical interpretation for the mathematical manipulations, by which a surface-type integral equation is derived. With the volume-type approach we can still reduce the number of unknowns (compared with the finite difference or the finite element methods) by confining the calculation to the region of interest, that is, to the conducting bodies.

2. A Continuous Analogy

We can get an integral equation for the eddy current problem for non-ferromagnetic media using the equations of Maxwell for a time harmonic field with negligible displacement currents. The magnetic field strength is given by

$$\boldsymbol{H}(\boldsymbol{r})=1/(4\pi)\int_v \boldsymbol{J}(\boldsymbol{r})(\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^3 dv+\boldsymbol{H}(\boldsymbol{r}) \tag{1}$$

where integration is over all space. We can now substitute $(j/\omega\mu_0))\nabla E$ for H, and the resulting differential-integral equation is easy to transform into an integral equation by using Green's theorem

$$\nabla(\rho \boldsymbol{J})=j\omega\mu_0/(4\pi)\int_v \boldsymbol{J}(\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^3 dv-j\omega\mu\boldsymbol{H}_0 \tag{2}$$

$$\int_p \boldsymbol{J}/\sigma d\boldsymbol{p}=-j\omega\mu/(4\pi)\int_s\left(4\pi\boldsymbol{H}_0+\int_v \boldsymbol{J}\Delta\boldsymbol{r}/|\Delta\boldsymbol{r}|^3 dV\right)dS \tag{3}$$

Here ρ and S are the perimeter and the surface of the same (arbitrary) surface element.

Where

$$\Delta r=r-r'$$

It is possible to use eq. (3) directly for a numerical solution by transforming it into an algebraic system using an appropriate approximation space and numerical integration formula. We choose instead an indirect way and first establish a network analogy.

3. A Network Analogy

We can visualize the current density distribution inside a conducting body as discrete currents flowing in the branches of a resistance network.[5),6)] The currents generate a magnetic field which in turn induces electromotive forces in the meshes of the network. Notice that the resistance network has to be created only inside the conducting body since there are no currents in insulators. If we are to find an equivalent circuit for the finite difference/finite element approach, it is also necessary to approximate the magnetic field distribution by discrete magnetic fluxes flowing in the branches of a magnetic resistance network interconnected or inteılinked with the electric network. This magnetic network, however, encompasses all space (at least inside of a well described closed boundary with sufficient boundary conditions).

Returning to our electric network, we can take the divergence-free nature of eddy currents into account simply by transforming the branch currents into the mesh currents. The transformation is singular, reflecting the fact that we took a new condition into account. The main difficulty with the equivalent circuit formulation is to determine the value of the circuit elements. The program we used is a simple ALGOL 60 program based on the network analogy, which generates the resistance and inductance matrices together with the excitation vector at first, and then solves the resulting system of equations for various frequency values.

4. Generation Of The Discrete System

Using uniform cubic subdivisions and permitting only homogeneous conductivity (also excluding ferromagnetic materials), it is a relatively straightforward computing taken to calculate the resistance matrix. We first set up the transformation matrix between the branch and mesh quantities, and then the diagonal matrix of branch resistances can easily be transformed into the mesh resistance matrix.

$$R_{(m)}=C^{t}R_{(b)}C=C^{t}\langle R_0\rangle C \tag{4}$$

Hear R_0 is the resistance of a branch. In the first approximation,

$$R_0=\rho/\Delta h \tag{5}$$

The inductance matrix is directly calculated from the geometric date. Although the mutual inductance of two rectangular current loops can be given explicitly, the self-inductance of a current loop is not defined. To simplify the program we use the same formula for self and mutual inductances:

$$L_{ij}=\mu_0 \boldsymbol{H}_{ij}\boldsymbol{n}_i I_i^{-1}(\Delta h)^2 \tag{6}$$

where H_{ij} is the magnetic field at the center of the *i-th* loop induced by the *j-th* current loop. (i and j can be equal).

The effect of the original field is a set-induced electromotive force in the meshes, given in a way similar to (6):

$$U_i = -j\omega\mu_0 \boldsymbol{H}_i \boldsymbol{n}_i (\Delta h)^2 \tag{7}$$

The mesh currents are calculated from

$$I_{(m)} = (R_{(m)} + jwL_{(m)})^{-1} U_{(m)} \tag{8}$$

The total eddy current loss is given by

$$P = \frac{1}{2} Re\{R_{(m)} I_{(m)} I_{(m)}{}^*\} \qquad \text{*complex conjugate} \tag{9}$$

As it can be seen from the simple approximations and the many restrictions, the aim of our program is to demonstrate the feasibility of a numerical IE-type formulation based on the network theory principles, rather than to develoep an effective general procedure for practical applications. In a program for "production runs", higher order approximations would be advantageous, using possibly a coordinate system conforming with the problem, in a manner similar to that of.[7]). However, since the extension of the above expressions for the cases of non-uniform discretization, inhomogenous conductivity, multiple body problems etc. appears almost trivial, we mention here only the difficulties arising from the presence of ferromagnetic bodies. In principle, forromagnetic effects can be represented with the appropriate modification of the inductance values. In practice, however, it would seem better to retain a set of auxiliary variables (e.g. magnetization vector components) inside the magnetic bodies, except possibly for some simple cases of magnetic reflection.

5. Computational Aspects

The system matrix resulting from (8) is full, if we regard it as a single complex matrix. This is in contrast with the FFM/FDM-type formulations where the sparse algebraic systems are usually well suited for solution by special numerical methods. Our compact system, however, can be regarded as a partially eliminated form of e.g. an FDM matrix, so we can expect a better performance, especially in not too "dense" situations. If active (conductive) material occupies only a small part of the region of interest, the FFM/FDM makes it necessary to calculate a large number of (most probably) irrelevant field values (in the air etc). The drastic reduction in the number of unknowns, reached by our method, is expected to be rewarding, even if we cannot use effective sparse equation solvers.

Since the impedance matrix is only a weakly diagonally dominant, we could not use standard algebraic equation solvers. The algorithm used in the test runs was a very strongly convergent, tough and consequently particularly slow interative procedure. For this reason the run times are not included, being irrelevant. In practice an orthogonalization based algorithm would seem to be useful.

6. Computational Results

Unfortunately, we were not able to find any published computational and test results for truly three-dimensional problems, suitable to compare with our method. For a test problem, we have chosen the case of a conductive sphere placed into an homogenous time-harmonic magnetic field. The analytic solution can be found in e.g.[8]). The calculated power loss for three discretizations is shown in Fig. 2 together with the exact analytic solution. The sphere is the "worst case" for our method in several senses. Our volume-type formulation compares least favourably with the surface methods in the case of a sphere for the number of subdivisions. Also, the rectangular elements are clearly better suited for straight boundaries. The method presented here can be reasonably expected to perform better in other cases. From Fig. 1, a discrepancy of 0 (1) can be seen between the numerical and the analytical results. The primitive

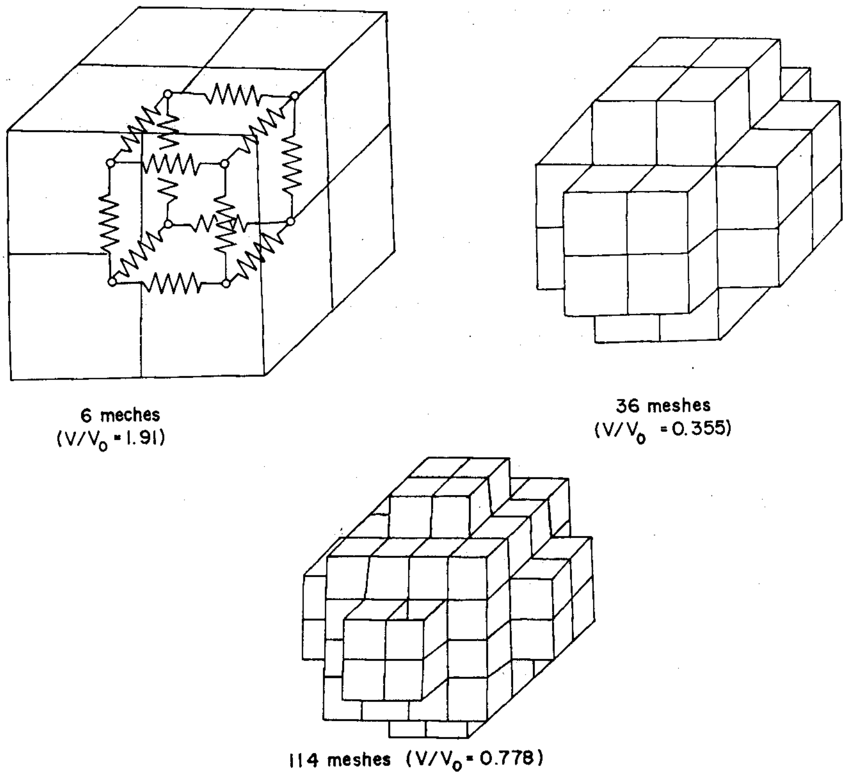

Fig. 1. Subdivisions for a Sphere.

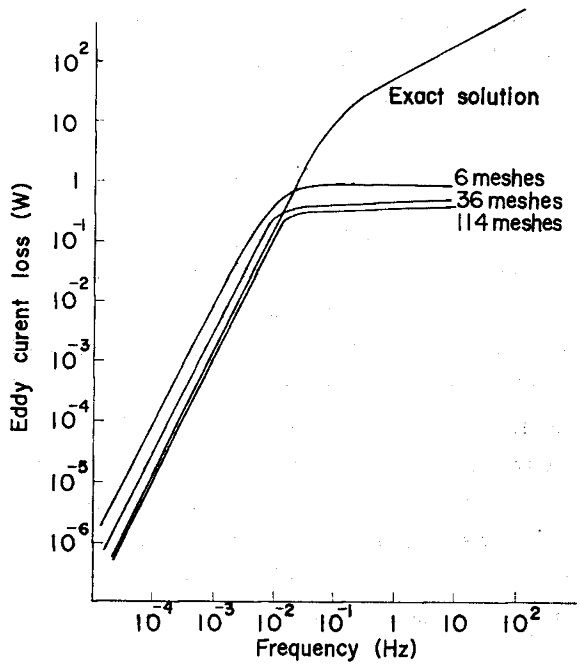

Fig. 2. Eddy Current Losses in a Sphere.

approximations for the values of the circuit elements could be responsible for this. The results indicate generally the possibility of using the volume integral equation method for eddy current calculations.

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