A New Method of Ranking: The Relative Position Ranking

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Abstract

A new method of ranking based on an indicator estimating the relative position of an element determined from the preference relation on the elements to be classified is proposed.

1. Introduction

Although it is commonly thought that decision problems are naturally formulated as choosing one alternative action or a set of actions considered to be the best among those studied, Roy [8] gives two other equally important and useful formulations, i.e., the sorting of all the elements that seem good among those studied, and the ranking of the alternatives in a decreasing order of preference. Some practical decision situations give some merit to the ranking formulation of the decision problem. The selection of research and development projects, the planning on national, regional, and urban programs, the choice of candidates to fill similar posts and other similar problems, due to the nature of the selection process, are oftentimes best formulated as a ranking problem. Moreover, it is often the case that it is difficult to convince decision makers that the optimal solution with respect to the model should be adopted, and consequently they often request for other solutions to consider.

In this case, a ranking of the solutions would provide them more confidence and satisfaction in their subsequent choices. In addition, it seems that the ranking formulation is more general than the other two in the sense that with a solution to the ranking formulation of the problem, the solutions required by the former two can be easily provided, i.e., the action ranked first and the first k ranked actions may be considered as the best choice and the k acceptable good actions, respectively. In spite of the above

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mentioned merits of such a formulation, the ranking problem seems to have attracted little attention among researchers.

Under the utility theory, the ranking of the elements of a set is studied as a consequence of the existence of utility functions on a set of alternatives that preserves the ordering of the alternatives determined by the individual's preference relation [2]. However, due to the restrictiveness of the assumptions associated with such functions, the results in the utility theory have a limited value in applications. Since the ranking of the elements of a set requires only an ordinal output, the results in the utility theory can be extended for this special case by relaxing the restrictive assumptions. Some methods arising from this observation are the semi-order method of Jacquet-Lagreze [3], Electre II [7], and many of the methods based on the outranking concept introduced by Roy [7]. Another approach concerns the idea of compromise ranking, where the elements are classified on the basis of a ranking most compatible to the given information on the preference of the decision maker [1], [4], [6].

This paper aims to explore some heuristic ideas which can be developed for application. A new method of ranking is proposed, based on the estimate of the relative position of an element in the set determined from the preference relations on the elements of the set to be classified. The ranking situation is described in Section 2, together with the observations which lead to this new approach. In Section 3, the method is presented as an algorithm. A brief summary together with a short discussion indicating possible developments and extensions of the method concludes the paper.

2. The Ranking Problem

There are many actual decision making cases where the decision maker is not constrained to accept only the best alternative actions, and yet cannot accept all the good ones, or does not even know 'a priori' how many of the alternatives he is going to accept. In these cases, the set of alternatives has to be classified in such a way that these actions have to be arranged systematically so that the decision maker can pick out the desired or preferred choices conveniently, according to his preference priorities. Thus, the alternatives need to be classified into a decreasing order of desirability or preference —a process which we call ranking.

Specifically, the alternative considered best is corresponded with the integer 1, the next with 2, the following with 3, and so on. This classification of elements is done sequentially in a decreasing order of preference. Thus all the alternatives corresponding to the same number are approximately equally desirable to the decision maker. For alternatives corresponding to different integers, one alternative is more desirable than the other if the former is associated with an integer less than that corresponding to the latter.

In a strict sense, ranking [9] is an association, say r, of a positive integer with each element of a set, say A, which is assumed to have a given weak order relation R (i.e., R is complete and transitive in A), such that for every element a and b of A, aRb holds if and only if $r(a) \leq r(b)$. This definition of ranking, however, has a limited value in practical applications due to the restrictiveness of the assumptions. Indeed, the weak order relation R imposed on the elements of set A is seldom verified without difficulties in the assessment of the decision maker's preferences, especially when the consequences of the alternatives involve uncertainties, partial information, multiple attributes, group and multiple decision maker considerations, etc. [8]. Moreover, under such assumptions the problem of ranking can be handled conveniently by the utility theory [2], [5].

On the other hand, a more practical approach to the ranking problem involves the idea of compromise ranking. The elements of set A are classified on the basis of a ranking most compatible to the given information on the preference of the decision makers. The methods used in the construction of such rankings are usually based on the comparison of feasible rankings and on a paired comparison of elements. A more sophisticated approach is to define a distance on the set of relations in A, and find a weak order S from the set of all weak orders W defined on A, minimizing the distance from W to the given relation R assessed from the decision maker. This weak order S is then used to construct a ranking in the strict sense. Other approaches based on the graph theory [1], [6] have also been proposed.

In a set of elements ranked according to a descending order of desirability or preference, the element ranked first, i.e., the most desirable element, dominates all the others in the sense that it is preferred to the rest, and no other element in the set dominates it. The second in rank is dominated only by the first in the ranking, and dominates the remainder of the set. The third is dominated only by the first and the second and in turn dominates the others, and so on.

In this way, the rank of the elements of a set can be characterized by its relative position in the ordering of the elements of the set, i.e., by the relation between the numbers of elements dominating it and of those it dominates. Thus, a numerical valued function of a particular element, constructed in terms of the number of elements dominating it and of the number of elements it dominates, may be defined as an indicator of its relative position among the elements of the set, and consequently may be used to indicate its rank in the set. One such indicator, for instance, may be defined as the difference between the number of dominated elements and the number of dominating elements in the set, i.e., $V(a)=p(a)-q(a)$, where $p(a)$ is the number of elements a dominates and $q(a)$ the number of elements dominating a. Using this indicator, the elements of the set can be ranked accordingly.

This is particularly easy to accomplish when the relation defined on the elements of the set is a weak order. On the other hand, when such strong assumptions cannot be guaranteed, such indicators can still be defined with respect to the available information at hand, and a ranking may still be achieved. In this case, depending on the quantity and quality of the information used, the ranking obtained in general will not be uniquely determined, giving the decision maker a chance for gathering more relevant information and for a more careful assessment of his preference (if he can or is still willing to do so), now concentrating only on the elements where the previous information was insufficient. This observation gives an interesting approach to the ranking problem, especially when the restrictive assumptions of the ranking formulation are not necessarily guaranteed by the present information available.

3. The Relative Position Ranking

In this paper we shall let A denote a set whose elements are to be classified in terms of the preference of the decision maker in a particular situation. It is important to note that A has a general interpretation, depending on the context, that is, the elements of A may be called alternatives, allocations, projects, systems, strategies, decision rules, etc. However, for simplicity, we will be primarily concerned with cases where A is a finite set. In fact, most of the problems associated with the ranking formulation involve only a finite number of alternatives to choose from. To be specific, the set A will be represented as follows: $A=\{a, b, \ldots, z\}$.

In order to compare the elements of A with respect to the decision maker's preference we will be comparing pairs of alternatives, since it is an easier and simpler process than making a systematic comparison of all the alternatives simultaneously. For this reason we consider a binary relation R on A reflecting the information with respect to the decision maker's preference. We shall express such preference pairwise, that is, for any two elements a and b of A the relation aRb will be used to express the preference of the element a over the element b. This preference signifies that in the decision maker's point of view, after considering sufficient information related to the relevant aspects of the problem, he considers that the element a is at least preferred to b or at least as good as b (without precluding the possibility of the other way).

Indeed, sometimes the decision maker considers the elements a and b more or less equally desirable and we express this by the relation aIb. To express this in terms of the relation R we adopt the convention that aIb is equivalent to that aRb and bRa hold at the same time, although other definitions are possible. Moreover, due to a lack of relevant sufficient information or for some reason or another, the decision maker may consider the possibility of incomparability between two elements of A, i.e., there is no sufficient relevant information to consider which of the elements is preferred to the other. In this situation, we regard the relation R does not hold for either case, i.e., not aRb and not bRa at the same time. To be realistic, we will not assume the transitivity and

$$R = \{a_1Ra_1, a_1Ra_4, a_2Ra_2, a_2Ra_3, a_3Ra_1, a_3Ra_3, a_3Ra_4, a_4Ra_2, a_4Ra_4, a_5Ra_1, a_5Ra_4, a_5Ra_5\}$$

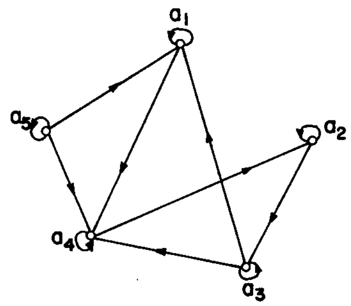

Fig. 1. An Example of a Preference Relation R and its Associated Graph.

the completeness of R. Instead, in the light of the information available, the relation R will be determined between any of the two elements as far as the information and the preference assessment allow, without forcing the decision maker to decide, when he is truly not able to do so (thus distorting his preference).

Such preference relations can be conveniently represented by a directed graph $G(A, R)$, where the set of vertices is the set of alternatives A and the set of arcs is the preference relation R, i.e., aRb is represented by an arc emanating from vertex a and ending at vertex b. It is noted here that due to the convention we adopted, the relation aIb is represented by two arcs connecting in both directions the vertices a and b, and two incomparable elements by two vertices without any connecting arc between them. An example of such a graph representing a preference relation is shown in Fig. 1. It would help us to define the degree of the vertices of G. The in-degree $d^-(a)$ of vertex a is the number of arcs ending at vertex a and the out-degree $d^+(a)$ of vertex a is the number of arcs emanating from vertex a.

For any element a of A we define the following sets:

$$P(a) = \{b \mid b \in A,\ aRb,\ \text{not } bRa\},$$

$$Q(a) = \{b \mid b \in A,\ bRa,\ \text{not } aRb\},$$

$$I(a) = \{b \mid b \in A,\ aIb\}, \quad \text{and}$$

$$U(a) = \{b \mid b \in A,\ a \text{ is incomparable with } b\}.$$

In addition, we also denote by $p(a)$, $q(a)$, $i(a)$ and $u(a)$ the numbers of elements in the sets $P(a)$, $Q(a)$, $I(a)$, and $U(a)$, respectively.

From these quantities we can define indicators from which we can base our ranking. One such indicator was already mentioned in the previous section—the index of the relative position $V(a)$ of an element a which is defined as the difference between the numbers of the elements dominated by a and the number of the elements dominating a, i.e., in the notation just defined, $V(a)=p(a)-q(a)$. Due to the simplicity of the form of this index we choose this as the indicator for our method. Other indices using any of the quantities defined above can also be constructed, depending on the particular application situation involved.

The following is a description of the simplest form of an algorithm implementing the method. Other possible variants, modifications, and extensions of it are indicated in the conclusion.

Algorithm

Step 1. *Assessment of the Preference Relation*

As much as the information available in the decision maker's preference allows, assess the preference relation R on set A. From the decision maker's preference, determine whether aRb or bRa holds for as many pairs of elements a and b as possible. Recall that from our assumption aIb is equivalent to that aRb and bRa hold simultaneously.

Step 2. *Construction of the Ranking Indicator*

For every element a of A determine $V(a)=p(a)-q(a)$, where $p(a)$ is the number of elements dominated by a and $q(a)$ is the number of elements dominating a. It is noted that $V(a)$ can also be determined equivalently from the associated graph $G(A, R)$ by $V(a)=d^{+}(a)-d^{-}(a)$, where $d^{+}(a)$ and $d^{-}(a)$ are the out-degree and in-degree of a, respectively. In fact, in most cases the graph approach provides a more practical and efficient scheme than by directly enumerating and counting the dominating and dominated elements.

Step 3. *Classification of Set A Using the Ranking Indicator*

Using the values of V we can partition the set A in the following manner: two elements a and b of A belong to the same class if and only if $V(a)=V(b)$. From this, partition the order of the classes in such a way that the class having the highest value of V is assigned to rank 1, the next to rank 2, and so on.

Thus, set A is classified and ranked using the relative position indicator V. It is noted that when R is a weak order, i.e., transitive and complete, the ranking obtained by our method is similar to that resulting from the utility theory in the sense that there exists an isomorphism between the two rankings. Indeed, if R is transitive and complete, the indicator V has the property that aRb holds if and only if $V(a) \geq V(b)$. In addition, whenever aIb (i.e., aRb and bRa simultaneously) holds, $V(a)=V(b)$. Thus our method, formulated in a general environment, gives results similar to that of the utility theory under the restrictive conditions of a weak order.

4. Conclusion

We have presented a new method of ranking, based on an indicator estimating the relative position of an element in the ordering of the elements of the set, induced by the preference relation among the elements of the set to be classified. Possible developments regarding variants, modifications, and extensions of our method can be constructed in order to attain a better performance relative to the information available on particular application situations, and in general to cope with various decision environments.

In these circumstances, different types of information structures may be present in the relevant aspects of the problem. Thus, the simple preference relation assumed in the method may not be appropriate for some decision problems. If, for instance, in a particular situation comparison can be made at different risk levels, a family of nested relations reflecting this property can be constructed. These relations (at different risk levels) will then be used to reach a ranking classification of the alternatives reflecting the multiple risk levels. In a fuzzy environment, fuzzy relations may be used instead of the simple preference relations, and an extension of our method using fuzzy preference relations can be developed. As there seems to be a large number of possibilities in developing and exploring this method and other related methodologies, a further study and analysis of these methodologies offer opportunities for future research.

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