Design of Dynamic Deadbeat Controllers Using an Observer or a Dual Observer in Discrete Time Linear Multivariable Systems

By

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Abstract

This paper considers the problem of designing the minimum time dynamic deadbeat controllers, using an observer or a dual observer. As a preliminary, optimal controller and observer are defined and obtained. Then, the existence of a dynamic deadbeat controller is examined, and a separation theorem is proved. This theorem states that the minimum time dynamic deadbeat controller is given by the optimal controller combined with the optimal observer. These results are dualized to yield the corresponding result in the case of using the dual observer, i. e., the result on the minimum time dual deadbeat controller. Finally, a method is presented for finding reduced order controllers by applying the results on linear function observers.

1. Introduction

In recent years, much attention has been paid to the deadbeat control problem for discrete time linear multivariable systems: namely, the problem of designing a controller which drives the system state to zero in a finite number of steps. In 1960, Kalman[1] solved this problem for single-input systems. The controller obtained there can be realized by a constant state feedback, and it drives the state to zero in a minimum number of steps. This work was extended to multi-input systems by several authors[2-5]. The connection between deadbeat control strategies and quadratic optimal control policies is discussed by Leden[6].

Those works postulate the use of a state feedback control law. In many applications, however, it is not possible to measure all the state variables. Very often, only the output can be measured, which consists of a linear combination of the state variables. In such a case, an approach to deadbeat control is to utilize the constant output feedback. This has been studied by Seraji[7]. An alternate approach is to utilize ob-

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servers or dynamic controllers. Porter & Bradshow presented methods of designing deadbeat controllers, using full-order and reduced order observers[8),9)]. In their works, however, the discussions are not satisfactory as regards the minimality of the control steps. Akashi & Adachi[10)] defined and obtained a minimum time dynamic controller with time variable parameters. However, from a practical point of view, time invariant controllers are more desirable.

In this paper, we consider the problem of designing a minimum time dynamic deadbeat controller with constant parameters for general multi-input multi-output discrete time linear systems. As preliminaries, the class of optimal deadbeat controllers via state feedback is defined, and a method of obtaining an optimal controller is shown. Also, the class of optimal observers is defined, and an optimal observer is obtained. Then, in section 4, the necessary and sufficient condition is derived for the existence of dynamic deadbeat controllers, and a separation theorem is proved. This theorem states that the optimal controller combined with the optimal observer yields the minimum time dynamic deadbeat controller. Reduction of the order of controllers is a matter of practical importance. For this purpose, the use of dual observers[11)] is interesting. This is considered in section 5. Finally, a method is given for designing reduced order controllers by applying the results on reduced order linear function observers[15)].

Notation: The set of all the positive integers including zero is denoted by $\boldsymbol{i}$. If $k \in \boldsymbol{i}$, $\underline{k}$ denotes the set of integers $\{0, 1, \dots, k\}$. $\boldsymbol{R}^n$ denotes an n-dimensional vector space defined over the field of real numbers. Subspaces of $\boldsymbol{R}^n$ are denoted by script capitals, e. g., $\mathscr{A}$, $\mathscr{B}$, ... The italic capitals denote linear maps or their matrix representations. The image (kernel) of a map A is written Im A (Ker A). The set of all the n×m matrices is denoted by $M_{n,m}$. The (Moore-Penrose) pseudo-inverse of a matrix A is written A^+. Let $\mathscr{A}$ and $\mathscr{B}$ be such that $\mathscr{A} \oplus \mathscr{B} = \boldsymbol{R}^n$. Then, $P_{\mathscr{A},\mathscr{B}}$ denotes the projector on $\mathscr{A}$ along $\mathscr{B}$. The symbol ' is used for denoting dual spaces, dual maps, or transformed matrices. Let $A, B \in M_{n,n}$ and $\mathscr{D} \subset \boldsymbol{R}^n$. Then we write $A = B \pmod{\mathscr{D}}$ if $Ax = Bx \pmod{\mathscr{D}}$ for all $x \ni \boldsymbol{R}^n$.

2. Optimal Deadbeat Controller

2.1 Problem Statement

Consider the system:

$$x(i+1) = Ax(i) + Bu(i) \tag{2, 1}$$

where $x(\cdot) \in \boldsymbol{R}^n$ is the state of the system and $u(\cdot) \in \boldsymbol{R}^r$ is the control input to the system. It is assumed that B is monic, i. e., rank $B = r$, and that rank $[A : B]=n$.

A linear map $K : \boldsymbol{R}^n \to \boldsymbol{R}^r$ is called a controller. The set of all the controllers is denoted by $\underline{K}$. The purpose of control is to drive the state to zero as soon as possible.

By substituting $u(i)=Kx(i)$ into (2, 1) we obtain

$$x(i+1)=(A+BK)x(i). \tag{2, 2}$$

It is clear that Ker $(A+BK)^i$ denotes the set of all $x(\cdot)$ which can be driven to zero in i-steps by the controller K. We introduce a sequence:

$$\underline{\underline{\mathscr{X}}}=\{\mathscr{X}_0, \mathscr{X}_1, \dots\}$$

with

$$\mathscr{X}_0=0 \tag{2, 3a}$$

$$\mathscr{X}_i=A^{-1}(\mathscr{X}_{i+1}+\mathscr{B}), \quad i=1,2,\dots \tag{2, 3b}$$

where

$$\mathscr{B} \triangleq \mathrm{Im}B.$$

The sequence $\underline{\underline{\mathscr{X}}}$ is non-decreasing, so that we can define

$$q \triangleq min\{k : \mathscr{X}_i=\mathscr{X}_k, \forall i \geq k \in \boldsymbol{i}\} \tag{2, 4}$$

It is readily verified that $\mathscr{X}_i$ is an upper bound for Ker $(A+BK)^i$: i. e.,

$$\mathrm{Ker}(A+BK)_i \subset \mathscr{X}_i, \quad \forall i \in \underline{q}.$$

DEFINITION 1 : A controller K is said to be optimal if

$$(A+BK)^i=\mathscr{X}_i, \quad \forall i \in \underline{q}.$$

We write $\underline{K}^\circ$ for the class of optimal controllers.

The problem to be treated in this section is to find the optimal controller for the system (2,1).

2.2 Solution of the Problem

First we show that the optimal controllers are closely related to projector matrices. This property will be utilized for obtaining the optimal controller.

LEMMA 1 : If $K \in \underline{K}^\circ$, there exists a complementary subspace $\mathscr{Y}$ to $\mathscr{B}$ such that

$$(A+BK)=P_{\mathscr{Y},\mathscr{B}}A.$$

PROOF : If $K \in \underline{K}^\circ$, by definition

$$\mathrm{Ker}\ (A+BK)=A^{-1}\mathscr{B}.$$

Since

$$A^{-1}\mathscr{B}=\text{Ker }(I-BB^{+})A,$$

there exists $X\in M_{n,n}$ such that

$$A+BK=X(\text{I}-BB^{+})A.$$

Since K must solve this equation, and since B is monic, we obtain

$$K=B^{+}[X(I-BB^{+})-I]A.$$

It is readily verified that $-B^{+}[X(I-BB^{+})-I]\in \underline{B}^{-}$, where $\underline{B}^{-}$ denotes the set of all the generalized inverses of B, and therefore the assertion follows. □

The problem now reduces to one of finding the direction of projection such that

$$\text{Ker}\,(P_{\mathscr{Y}.\mathscr{B}}A)^{i}=\mathscr{X}_{i}, \quad \forall i\in\underline{q}. \tag{2, 5}$$

The following lemma provides a sufficient condition for Y to have the property (2,5).

LEMMA 2 : Let $\mathscr{Y}$ be a complementary subspace to $\mathscr{B}$, and suppose that $\mathscr{Y}$ has the property:

$$\mathscr{X}_{i}=X_{i}\cap\mathscr{B}\oplus\mathscr{X}_{i}\cap\mathscr{Y}, \quad \forall i\in\underline{q}, \tag{2, 6}$$

then $\mathscr{Y}$ achieves (2, 5).

PROOF : Clearly, (2, 5) holds for $i=0$, and if (2, 5) holds for $i=k$, it follows by LEMMA A (see Appendix) and assumption that

$$\begin{aligned}\text{Ker}\,(P_{\mathscr{Y}.\mathscr{B}}A)^{k+1}&=A^{-1}(P_{\mathscr{Y}.\mathscr{B}})^{-1}\mathscr{X}_{k}\\&=A^{-1}(\mathscr{Y}\cap\mathscr{X}_{k}\oplus\mathscr{B})\\&=A^{-1}(\mathscr{Y}\cap\mathscr{X}_{k}+\mathscr{X}_{k}\cap\mathscr{B}+\mathscr{B})\\&=A^{-1}(\mathscr{X}_{k}+\mathscr{B})=\mathscr{X}_{k+1}.\end{aligned}$$

Thus the lemma follows by induction. □

We can utilize LEMMA 2 to solve the problem. For this, we decompose every $\mathscr{X}_{i}\in\underline{\underline{\mathscr{X}}}$, $i\in\underline{q}$, as follows:

$$\mathscr{X}_{i}=\mathscr{X}_{i}\cap\mathscr{B}\oplus\mathscr{Y}_{i} \tag{2, 7}$$

$$\mathscr{Y}_{0}\subset\mathscr{Y}_{1}\subset\ldots\subset\mathscr{Y}_{q}. \tag{2, 8}$$

Since $\mathscr{Y}_{q}\cap\mathscr{B}=0$, we can choose $\text{Y}\in M_{r,n}$ such that

$$\text{Ker }Y\supset\mathscr{Y}_{q} \tag{2, 9}$$

and

$$\text{Ker }Y\oplus\mathscr{B}=\boldsymbol{R}^{n}. \tag{2, 10}$$

It is noted that $YB\in M_{r,r}$ is invertible. On this basis we obtain

THEOREM 1 : Let $Y\in\text{M}_{r,n}$ be defined by (2, 9)—(2, 10). Then

$$K=-(YB)^{-1}YA \tag{2, 11}$$

is an optimal controller for the system (2, 1).

PROOF : It follows from (2, 7)—(2, 10) and the modular distributive rule that

$$\mathrm{Ker}\ Y \cap \mathscr{X}_i = \mathscr{Y}_i, \quad i \in \underline{q},$$

and so we can write

$$\mathscr{X}_i = \mathscr{B}_i \cap \mathscr{B} \oplus \mathscr{X}_i \cap \mathrm{Ker}\ Y, \quad i \in \underline{q}. \tag{2, 12}$$

Substitution of (2, 11) into $A+BK$ yields

$$A+BK=RA$$

where

$$R \triangleq I-B(YB)^{-1}Y.$$

It is readily verified that R is the projector on Ker Y along $\mathscr{B}$. Thus the theorem follows from (2, 12) and LEMMA 2. □

REMARK 1 : The statement of THEOREM 1 involves the existence of the optimal controller, because the choice of Y having the properties (2, 7)—(2, 10) is possible at any time.

REMARK 2 : The pair (A, B) is said to be controllable[12] if

$$\mathrm{Im}\ A^n \subset \sum_{i=0}^{n-1} A^i \mathscr{B},$$

It is easy to verify that the pair (A, B) is controllable if and only if $\mathscr{X}_k = \boldsymbol{R}^n$ for some $k \in \underline{n}$, where $\mathscr{X}_k$ is computed by (2, 3). We call $K \in \underline{K}$ a deadbeat controller if there exists $k \in \underline{n}$ such that $x(k)=0$ for any $x(0)$. A deadbeat controller can be obtained, obviously, if and only if the pair (A, B) is controllable, in which case the controller given by (2, 11) becomes an optimal deadbeat controller.

3. Optimal Observer

3.1 Problem Statement

Consider the system (2, 1) and suppose that the state of the system is measured only through the system output:

$$y(i)=Cx(i) \tag{3, 1}$$

where $y(\cdot) \in \boldsymbol{R}^m$. It is assumed that rank $C=m$. To reconstruct the state we use an observer of a rather specific form:

$$z(i+1)=TA\bar{x}(i)+TBu(i) \tag{3, 2}$$

$$\bar{x}(i)=Sz(i)+Vy(i) \tag{3, 3}$$

where $\bar{x}(\cdot) \in \boldsymbol{R}^n$ and $z(\cdot) \in \boldsymbol{R}^l$ with $l \leq n$. It is assumed that both S and T are of full rank and that the observer is unbiased[13]. Since rank $[A : B]=n$, it is readily verified that the observer is unbiased if and only if

$$I-VC=ST. \tag{3, 4}$$

Denote the estimation error by $e(i) \overset{\Delta}{=} x(i)-\hat{x}(i)$. Then the error dynamics is given by

$$e(i+1)=(I-VC)Ae(i) \tag{3, 5a}$$

and

$$e(0)=(I-VC)x(0)-Sz(0). \tag{3, 5b}$$

Since T is of full rank, (3, 4) implies Im $(I-VC)=\operatorname{Im} S$. Thus all the possible $e(0)$ span Im $(I-VC)$. In view of this, we write

$$\mathscr{E}_i(V) \overset{\Delta}{=} [(I-VC)A]^i(I-VC)\boldsymbol{R}^n. \tag{3, 6}$$

The subspace $\mathscr{E}_i(V)$ denotes the set of all the possible $e(i)$ when an observer V is used for estimating $x(i)$.

We introduce the sequence $\underline{\underline{\mathscr{S}}}$:

$$\mathscr{S}_0=\boldsymbol{R}^n \tag{3, 7a}$$

$$\mathscr{S}_i=A(\operatorname{Ker} C\cap\mathscr{S}_{i-1}). \tag{3, 7b}$$

Since $\underline{\underline{\mathscr{S}}}$ is non-increasing, we can define

$$p \overset{\Delta}{=} min\{k : \mathscr{S}_i=\mathscr{S}_k,\ \forall i\geq k\in\underline{n}\}. \tag{3, 8}$$

It is easy to verify that Ker $C\cap\mathscr{S}_i$ is a lower bound for $\mathscr{E}_i(V)$: i. e.,

$$\mathscr{E}_i(V)\supset \operatorname{Kre} C\cap\mathscr{S}_i,\ \ \forall i\in\underline{p}\ ;$$

for, clearly $\mathscr{E}_0(V)=(I-VC)\boldsymbol{R}^n\supset$ Ker $C\supset$ Ker $C\cap\mathscr{S}_0$, and if $\mathscr{E}_i(V)\supset$ Ker $C\cap\mathscr{S}_i$, then

$$\begin{aligned}\mathscr{E}_{i+1}(V) &= (I-VC)A\mathscr{E}_i(V)\supset(I-VC)A(\operatorname{Kre} C\cap\mathscr{S}_i)\\ &=(I-VC)\mathscr{S}_{i+1}\supset(I-VC)(\operatorname{Kre} C\cap\mathscr{S}_{i+1})=Ker\, C\cap\mathscr{S}_{i+1}.\end{aligned}$$

DEFINITION 2 : An observer is said to be optimal if

$$\mathscr{E}_i(V)=\operatorname{Kre} C\cap\mathscr{S}_i,\ \ \forall i\in\underline{p}. \tag{3, 9}$$

We write $\underline{V}^{\circ}$ for the class of optimal observers. The problem to be treated in this section is to find an optimal observer for the system described by (2, 1) & (3, 1).

3.2 Solution

Corresponding to LEMMA 1 we have the following:

LEMMA 3 : If $V\in\underline{V}^{\circ}$, there exists a complementary subspace $\mathscr{X}$ to Ker C such that

$$I-VC=P_{Ker C,\mathscr{X}}.$$

PROOF : If $V\in\underline{V}^{\circ}$, it follows by definition that Im $(I-VC)=$Ker C. Hence $C(I-VC)=0$, so that $V\in\underline{C}^{-}$. Thus, we have shown that $(I-VC)$ is the projector on Ker C along $\mathscr{X}\overset{\Delta}{=}\text{Im}\, VC$. □

The following result is corresponding to LEMMA 2.

LEMMA 4 : Let $\mathscr{X}$ be a complementary subspace to Ker C. Then

$$\text{Im}\,(P_{\text{Ker}\,C,\mathscr{X}}A)^{i}P_{\text{Ker}\,C,\mathscr{X}}=\text{Ker}\,C\cap\mathscr{S}_{i},\quad \forall i\in\underline{p}, \tag{3, 10}$$

if and only if

$$\mathscr{S}_{i}=\mathscr{S}_{i}\cap\text{Ker}\,C\oplus\mathscr{S}_{i}\cap\mathscr{X},\quad \forall i\in\underline{p}. \tag{3, 11}$$

PROOF : (If) Obviously, (3, 10) holds for $i=0$. Suppose that (3, 10) holds for $i=k-1$. Then it follows from LEMMA A and (3, 11) that

$$\begin{aligned}(P_{\text{Ker}\,C,\mathscr{X}}A)^{k}P_{\text{Ker}\,C,\mathscr{X}}\boldsymbol{R}^{n} &= (P_{\text{Ker}\,C,\mathscr{X}}A)\,(\mathscr{S}_{k-1}\cap\text{Ker}\,C)\\ &=P_{\text{Ker}\,C,\mathscr{X}}\mathscr{S}_{k}=(\mathscr{X}+\mathscr{S}_{k})\cap\text{Ker}\,C\\ &=(\mathscr{X}+\mathscr{S}_{k}\cap\text{Ker}\,C)\cap\text{Ker}\,C=\mathscr{S}_{k}\cap\text{Ker}\,C.\end{aligned}$$

Thus, (3, 10) follows by induction.

(Only if) If (3, 10) holds, we oftain

$$\begin{aligned}(P_{\text{Ker}\,C,\mathscr{X}}A)^{i}P_{\text{Ker}\,C,\mathscr{X}} &= P_{\text{Ker}\,C,\mathscr{X}}A\,(\mathscr{S}_{i-1}\cap\text{Ker}\,C)\\ &=P_{\text{Ker}\,C,\mathscr{X}}\mathscr{S}_{i}=\text{Ker}\,C\cap(\mathscr{X}+\mathscr{S}_{i}).\end{aligned}$$

Therefore,

$$\text{Ker}\,C\cap(\mathscr{X}+\mathscr{S}_{i})=\text{Ker}\,C\cap\mathscr{S}_{i}. \tag{3, 12}$$

To show that (3, 12) implies (3, 11), let $s\in\mathscr{S}_{i}$. Then, there exist $z\in\mathscr{X}$ and $x\in$Ker C such that

$$s=z+x,$$

because $\mathscr{X}\oplus\text{Ker}\,C=\boldsymbol{R}^{n}$. It follows from (3, 12) that

$$x=s-z\in(\mathscr{S}_{i}+\mathscr{X})\cap\text{Ker}\,C=\mathscr{S}_{i}\cap\text{Ker}\,C\subset\mathscr{S}_{i}.$$

Thus,

$$z=s-x\in\mathscr{S}_{i}\cap\mathscr{X},$$

and therefore we have

$$s=x+z\in\mathscr{S}_{i}\cap\text{Ker}\,C\oplus\mathscr{S}_{i}\cap\mathscr{X}$$

as claimed. □

To give the main result of this section, we decompose every $\mathscr{S}_{i}\in\underline{\underline{\mathscr{S}}}$ as follows:

$$\mathscr{S}_{i}=\mathscr{S}_{i}\cap\text{Ker}\,C\oplus\mathscr{X}_{i} \tag{3, 13}$$

$$\mathscr{X}_0 \supset \mathscr{X}_1 \supset \ldots \supset \mathscr{X}_p. \tag{3, 14}$$

Clearly such a decomposition is not unique. So, we denote by $\underline{\mathscr{X}}$ the set of all $\mathscr{X}_0$ having the properties (3, 13) – (3, 14), and define a set of matrices $\underline{Z}$ according to

$$\underline{Z} = \{Z: \text{Im } Z \stackrel{\Delta}{=} \mathscr{X}_0, \mathscr{X}_0 \in \underline{\mathscr{X}}, \text{ and } Z \in M_{n,m}\}. \tag{3, 15}$$

It is noted that if $Z \in \underline{Z}$, CZ is invertible.

THEOREM 2 : $V \in M_{n,m}$ is optimal if and only if

$$V = Z(CZ)^{-1} \tag{3, 16}$$

for some $Z \in \underline{Z}$.

PROOF : (Only if) If $V \in \underline{V}^{\circ}$, $(I - VC)$ is the projector on Ker C along Im $VC \stackrel{\Delta}{=} Z$, as noted in the proof of LEMMA 3. Therefore, it follows from LEMMA 4 and DEFINITION 2 that

$$\mathscr{S}_i = \mathscr{S}_i \cap \text{Ker } C \oplus \mathscr{S}_i \cap \mathscr{X}, \quad i \in \underline{p}.$$

Identifying $\mathscr{S}_i \cap \mathscr{X}$ with $\mathscr{X}_i$ for every $i \in \underline{p}$, we see that $\mathscr{X}_0 \in \underline{\mathscr{X}}$. Since

$$\text{Ker } (I - VC) = \mathscr{X} = \mathscr{X}_0,$$

V must solve the equation

$$(I - VC) Z = 0.$$

Thus, (3, 16) follows. The proof of the converse follows the same lines as for THEOREM 1. □

REMARK : The pair (C, A) is said to be reconstructible if the pair (A', C') is controllable. The concept of reconstructibility can be characterized in terms of $\underline{\underline{\mathscr{S}}}$: i. e., the pair (C, A) is reconstructible if and only if $\mathscr{S}_p = 0$[14].

4. Dynamic Deadbeat Controller

In this section we consider the problem of designing a deadbeat controller for the system (2, 1), whose state is reconstructed by the observer (3, 2) – (3, 3). Thus, the postulated control law is of the form.

$$u(i) = K\hat{x}(i) = KSz(i) + KVy(i) \tag{4, 1}$$

$$z(i+1) = TASz(i) + TAVy(i) + TBu(i). \tag{4, 2}$$

It is assumed that the observer is unbiased. The dynamic system (4, 1) – (4, 2) is called a dynamic controller and denoted by $D(K, V)$. In the sequel, we write $x(i : K, V)$ for $x(i)$ to indicate the dependence of $x(i)$ on $D(K, V)$. The purpose of control is the same as in section 2.

DEFINITION 3 : A controller $D(K, V)$ is called a dynamic deadbeat controller, if there exists $\rho \in \boldsymbol{i}$ such that

$$x(i;K,V)=0, \quad \forall i \geq \rho \tag{4, 3}$$

for any $x(0)$ and $z(0)$.

We denote by t the minimum ρ for which (4, 3) holds. Clearly, t depends on K and V. To indicate this dependence explicitly, we may write $t(K, V)$ for t. We call $D(K, V)$ a minimum time controller if it minimizes $t(K, V)$. The problem to be treated in this section is to find the minimum time deadbeat controller.

The main result of this section is the following.

THEOREM 3 : (I) There exists a dynamic deadbeat controller if and only if the pair (A, B) is controllable, and the pair (C, A) is reconstructible.

(II) (Separation Theorem) Under the conditions of (I), every $D(K, V)$ with $K \in \underline{K}^{\circ}$ and $V \in \underline{V}^{\circ}$ is a minimum time controller.

(III) Let $\underline{\underline{\mathscr{X}}}$ and $\underline{\underline{\mathscr{S}}}$ be the sequences already defined by (2, 3) and (3, 7) respectively, and denote by t^* the minimum i for which

$$\mathscr{X}_{i-j} \supset \mathscr{S}_j, \quad \forall j \in \underline{i}. \tag{4, 4}$$

Then,

$$t^* = \min_{K,V} t(K,V) \leq p+q-1. \tag{4, 5}$$

For the proof we need some preliminary results.

LEMMA 5 : Let $i \in \boldsymbol{i}$. Then,

$$x(i : K, V) = 0 \tag{4, 6}$$

for any $x(0)$ and $z(0)$, if and only if

$$(A+BK)^i = 0 \tag{4, 7}$$

and

$$L_i \stackrel{\Delta}{=} \sum_{k=0}^{i-1} (A+BK)^k BK[(I-VC)A]^{(i-1)-k}(I-VC) = 0. \tag{4, 8}$$

PROOF : Substituting (4, 1) into (2, 1) and setting $e(i) \stackrel{\Delta}{=} x(i) - \hat{x}(i)$, we obtain

$$x(i+1) = (A+BK)x(i) - BKe(i). \tag{4, 9}$$

Iteration of (4, 9) and use of (3, 5) gives

$$x(i+1) = \{(A+BK)^{i+1} - \sum_{k=0}^{i} (A+BK)^k BK[(I-VC)A]^{i-k}(I-VC)\}x(0)$$
$$+ \{\sum_{k=0}^{k} (A+BK)^i BK[(I-VC)A]^{i-k}S\}z(0). \tag{4, 10}$$

Thus, (4, 6) holds for any $x(0)$ and $x(0)$ if and only if

$$(A+BK)^i - \sum_{k=0}^{i-1} (A+BK)^k BK[(I-VC)A]^{i-1-k}(I-VC) = 0$$

and

$$\sum_{k=0}^{i-1} (A+BK)^k BK[(I-CV)A]^{i-1-k} S=0.$$

Since Im $S=$ Im $(I-VC)$, the proof follows at once. □

The following result provides a necessary condition for the existence of dynamic deadbeat controllers.

LEMMA 6 : If there exists a dynamic deadbeat controller, then for some $i \in \underline{n}$

$$\mathscr{X}_{i-j} \supset \mathscr{S}_j, \quad \forall j \in \underline{i}.$$

PROOF : First we define $J_j \in M_{n,n}$, $j=1, 2, \ldots, i$, according to

$$J_1 \triangleq A(I-VC) \tag{4, 11}$$

$$J_j \triangleq (A+BK)J_{j-1} - BK(I-VC)[A(I-VC)]^{j-1}, \quad j \geq 2. \tag{4, 12}$$

It is not difficult to check that

$$J_i = (A+BK)^i (I-VC) - L_i. \tag{4, 13}$$

Suppose that (4, 6) holds for any $x(0)$ and $z(0)$. It follows from LEMMA 5 and (4, 13) that Im $J_i = 0 \subset \mathscr{X}_0$, and if Im $J_j \subset \mathscr{X}_{i-j}$, then by (4, 12) we obtain

$$\text{Im } J_{j-1} \subset A^{-1}(\mathscr{X}_{i-j} + \mathscr{B}) = \mathscr{X}_{i-(j-1)}.$$

Therefore, induction proves

$$\text{Im } J_j \subset \mathscr{X}_{i-j}, \quad \forall j, \ 1 \leq j \leq i. \tag{4, 14}$$

Next, we show that

$$\mathscr{S}_j \subset \text{Im } J_j, \quad \forall j, \ 1 \leq j \leq 1. \tag{4, 15}$$

Obviously, Im $J_1 \supset A(I-VC)$ (Ker C) $= A$ (Ker C) $= \mathscr{S}_1$. To prove the assertion for $j \geq 2$, we rewrite (4, 12) as follows.

$$J_j = [(A+BK)^{j-1} VC + J_{j-1}] A(I-VC), \quad j \geq 2.$$

Using this recurrence relation, we can establish by induction that

$$\text{Im } J_j \supset J_{j-k}(\mathscr{S}_k \cap \text{Kre } C), \tag{4, 16}$$

for all $k \in \underline{j-1}$. In fact, (4, 16) holds for $k=0$. Suppose that (4, 16) holds for $k=k$, then we have

$$\begin{aligned}
\text{Im } J_j &\supset [(A+BK)^{j-k-1} VC + J_{j-k-1}] A(I-VC)(\mathscr{S}_k \cap \text{Kre } C) \\
&= [(A+BK)^{j-k-1} VC + J_{j-k+1}] \mathscr{S}_{k+1} \\
&\supset [(A+BK)^{j-k-1} VC + J_{j-k-1}](\mathscr{S}_{k+1} \cap \text{Ker } C) \\
&= J_{j-(k+1)}(\mathscr{S}_{k+1} \cap \text{Ker } C),
\end{aligned}$$

as claimed. Therefore, by putting $k=j-1$ in (4, 16) we obtain

$$\text{Im } J_j \supset J_1(\mathscr{S}_{j-1} \subset \text{Ker } C) = A(\mathscr{S}_{j-1} \cap \text{Ker } C) = \mathscr{S}_j,$$

and this verifies (4, 15). Since $\mathscr{S}_0 = \mathscr{X}_i = \boldsymbol{R}^n$ by (4, 7), the lemma now follows from (4, 14) and (4, 15). □

As a converse to LEMMA 6 we have the following

LEMMA 7 : Let $K \in \underline{K}^\circ$ and $V \in \underline{V}^\circ$, and suppose that there exists $i \in \boldsymbol{i}$ such that

$$\mathscr{X}_{i-j} \supset \mathscr{S}_j, \quad \forall j \in \underline{i}.$$

Then, for any $x(0)$ and $z(0)$

$$x(i\ ;\ K, V) = 0. \tag{4, 17}$$

PROOF : If $K \in \underline{K}^\circ$, by DEFINITION 1

$$\mathscr{X}_{i-j} = \text{Ker } (A+BK)^{i-j},$$

and if $V \in \underline{V}^\circ$, by DEFINITION 2

$$\mathscr{S}_j = A(\text{Ker } C \cap \mathscr{S}_{j-1}) = [A(I-VC)]^j \boldsymbol{R}^n.$$

Therefore, it follows by assumption that

$$(A+BK)^{i-j}[A(I-VC)]^j = 0, \quad \forall j \in \underline{i}. \tag{4, 18}$$

Since

$$(I-VC)[A(I-VC)]^j \boldsymbol{R}^n = \text{Ker } C \cap \mathscr{S}_j \subset \mathscr{S}_j = [A(I-VC)]^j \boldsymbol{R}^n,$$

we have also

$$(A+BK)^{i-j}(I-VC)[A(I-VC)]^j = 0, \quad \forall j \in \underline{i}. \tag{4, 19}$$

obviously (4, 18) implies (4, 7), and it is easy to verify from (4, 18) – (4, 19) that

$$(A+BK)^k BK(I-VC)[A(I-VC)]^{i-1-k} = 0, \quad k \in \underline{i-1}.$$

Consequently, (4, 17) follows immediately from LEMMA 5. □

Now, we are ready to prove THEOREM 3.

PROOF of THEOREM 3 : If there exists a dynamic deadbeat controller, by LEMMA 6 there exists $i \in \underline{n}$ such that

$$\mathscr{X}_{i-j} \supset \mathscr{S}_j, \quad \forall j \in \underline{i}.$$

Since $\mathscr{X}_0 = 0$ and $\mathscr{S}_0 = \boldsymbol{R}^n$, it follows that

$$\mathscr{X}_i = \boldsymbol{R}^n$$

and

$$\mathscr{S}_i = 0.$$

Hence, by REMARK 2 of THEOREM 1, (A, B) is controllable and, by the REMARK of THEOREM 2, (C, A) is reconstructible. Conversely, if (A, B) is controllable, then $\mathscr{X}_i = \boldsymbol{R}^n$, $i \geq q$, and if (C, A) is reconstructible, then $\mathscr{S}_i = 0$, $i \geq p$. Hence, it is readily verified that

$$\mathscr{X}_{i-j} \supset \mathscr{S}_j, \quad \forall j \in \underline{i},$$

if $i \geq p+q-1$. Therefore, it follows from LEMMA 7 that

$$x(i\ ;\ K, V) = 0, \quad \forall i \geq p+q-1$$

for any $x(0)$ and $z(0)$, provided $K \in \underline{K}^\circ$ and $V \in \underline{V}^\circ$. This verifies (I).

As for (II), let $D(K^*, V^*)$ be a minimum time controller, and write

$$t^* \triangleq t(K^*, V^*). \tag{4, 20}$$

Then for any $x(0)$ and $z(0)$

$$x(i\ ;\ K^*, V^*) = 0, \quad \forall i \geq t^*,$$

so that by LEMMAs 5 and 6 $\mathscr{X}_{t^*-j} \supset \mathscr{S}_j, \quad \forall j \in \underline{t}.$ Therefore, by LEMMA 7

$$x(i\ ;\ K^\circ, V^\circ) = 0, \quad \forall i \geq t^* \tag{4, 21}$$

for any $x(0)$ and $z(0)$, if $K^\circ \in \underline{K}^\circ$ and $V^\circ \in \underline{V}^\circ$. It follows from (4, 20) & (4, 21) that

$$t(K^\circ, V^\circ) \leq t(K^*, V^*),$$

and since $D(K^*, K^*)$ is a minimum time controller, we conclude that

$$t(K^\circ, V^\circ) = t(K^*, V^*),$$

i. e., $D(K^\circ, V^\circ)$ is a minimum time controller.

The proof of (III) is obvious from the proof of (I) & (II). □

It is apparently sufficient for the existence of a dynamic deadbeat controller that the pair (C, A) is reconstructible modulo Ker K[15] for some $K \in \underline{K}^\circ$, provided the pair (A, B) is controllable. But THEOREM 3 provides a stronger condition that (C, A) is reconstructible. The following corollary states that these two statements are equivalent.

COROLLARY 1 : Let $K \in \underline{K}^\circ$, and suppose that the pair (A, B) is controllable and the pair (C, A) is reconstructible modulo Ker K. Then, (C, A) is reconstructible.

PROOF : Let $\underline{\mathscr{S}}$ be the sequence defined by (3, 7), and suppose that the pair (C, A) is reconstructible modulo Ker K, $K \in \underline{K}^\circ$. Then, there exists $i \in \underline{n}$ such that[15]

$$\text{Ker } K \supset \mathscr{S}_i \cap \text{Ker } C.$$

Since $K \in \underline{K}^\circ$ and the pair (A, B) is controllable, we obtain

$$(A+BK)^n = 0.$$

Thus,

$$0=(A+BK)^n(\mathscr{S}_i\cap\mathrm{Ker}\ C)=(A+BK)^{n-1}A(\mathscr{S}_i\cap\mathrm{Ker}\ C)$$
$$=(A+BK)^{n-1}\mathscr{S}_{i+1}\supset(A+BK)^{n-1}(\mathscr{S}_{i+1}\cap\mathrm{Ker}\ C),$$

so that

$$(A+BK)^{n-1}(\mathscr{S}_{i+1}\cap\mathrm{Ker}\ C)=0.$$

Next, suppose

$$(A+BK)^{n-k}(\mathscr{S}_{i+k}\cap\mathrm{Ker}\ C)=0$$

and note that

$$\mathscr{S}_{i+k}\cap\mathrm{Ker}\ C\subset\mathscr{S}_i\cap\mathrm{Ker}\ K.$$

Then we obtain

$$(A+BK)^{n-(k+1)}A(\mathscr{S}_{+k}\cap\mathrm{Ker}\ C)=(A+BK)^{n-(k+1)}\mathscr{S}_{i+(k+1)}=0,$$

so that

$$(A+BK)^{n-(k+1)}(\mathscr{S}_{i+(k+1)}\cap\mathrm{Ker}\ C)=0.$$

Thus, induction proves $\mathscr{S}_{n+i}\cap\mathrm{Ker}\ C=0$, and so the proposition follows.

The procedure developed in sections 2-4 is illustrated by the following example.

EXAMPLE 1 : Let

$$A=\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\1&-1&2&0&0&0\\1&0&0&1&0&0\\0&0&2&0&0&2\\0&1&0&2&1&1\end{pmatrix},\qquad B=\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$C=\begin{pmatrix}0&1&0&0&0&0\\0&0&0&0&1&0\end{pmatrix}.$$

Optimal Controller: By (2, 3) we obtain

$$\mathscr{X}_1=\mathrm{Im}\begin{pmatrix}0&0&1\\1&0&0\\0&0&0\\0&0&-1\\0&1&0\\0&0&0\end{pmatrix},\qquad \mathscr{X}_2=\boldsymbol{R}^6.$$

so that $q=2$ and the pair (A, B) is controllable. Since $\mathscr{X}_1\cap\mathscr{B}=0$ and $\mathscr{X}_2\cap\mathscr{B}=\mathscr{B}$, (2, 7) yields

$$\mathscr{Y}_1=\mathscr{Y}_2=\mathrm{Im}\begin{pmatrix}0&0&1\\1&0&0\\0&0&0\\0&0&-1\\0&1&0\\0&0&0\end{pmatrix}.$$

Therefore, by (2, 9)-(2, 10) we get

$$Y=\begin{pmatrix}1&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{pmatrix}.$$

Thus, the optimal controller $K\in\underline{K}^{\circ}$ is obtained as follows:

$$K=-\begin{pmatrix}1&1&0&1&0&0\\1&-1&2&0&0&0\\0&1&0&2&1&1\end{pmatrix}.$$

Optimal Observer: We obtain

$$\mathrm{Ker}\ C=\mathrm{Im}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.$$

The algorithm (3, 7) gives $\mathscr{S}_0=\boldsymbol{R}^6$,

$$\mathscr{S}_1=\mathrm{Im}\begin{pmatrix}0&0&0&0\\0&1&0&0\\1&2&0&0\\1&0&1&0\\0&2&0&2\\0&0&2&1\end{pmatrix},\qquad \mathscr{S}_2=\mathrm{Im}\begin{pmatrix}0&0\\1&0\\2&0\\1&1\\2&4\\2&4\end{pmatrix},\qquad \mathscr{S}_3=0.$$

so that $p=3$ and the pair (C, A) is reconstructible. It follows that

$$\mathrm{Ker}\ C\cap\mathscr{S}_0=\mathrm{Ker}\ C,\quad \mathrm{Ker}\ C\cap\mathscr{S}_1=\mathrm{Im}\begin{pmatrix}0&0\\0&0\\1&0\\1&1\\0&0\\0&2\end{pmatrix},\quad \mathrm{Ker}\ C\cap\mathscr{S}_2=0.$$

Therefore, by (3, 13) — (3, 14) we have

$$\mathscr{X}_0=\mathscr{X}_1=\mathscr{X}_2=\mathrm{Im}\begin{pmatrix}0&0\\1&0\\2&0\\1&1\\2&4\\2&4\end{pmatrix}.$$

Thus, the optmial observer $V\in \underline{V}^{\circ}$ can be obtained as follows:

$$V'=\begin{pmatrix}0&1&2&0.5&0&0\\0&0&0&0.25&1&1\end{pmatrix}$$

Minimum Time Dynamic Controller: We obtain

$$I-VC=\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&-2&1&0&0&0\\0&-0.5&0&1&-0.25&0\\0&0&0&0&0&0\\0&0&0&0&-1&1\end{pmatrix}.$$

A full rank factorization of $(I-VC)$ yields

$$S=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\qquad T=\begin{pmatrix}1&0&0&0&0&0\\0&-2&1&0&0&0\\0&-0.5&0&1&-0.25&0\\0&0&0&0&-1&1\end{pmatrix}.$$

Consequently, by (4, 1) – (4, 2) the following minimum time dynamic controller can be obtained:

$$u(i)=-\begin{pmatrix}1&0&1&0\\1&2&0&0\\0&0&2&1\end{pmatrix}z(i)-\begin{pmatrix}1.5&0.25\\3&0\\2&2.5-\end{pmatrix}y(i)$$

$$z(i+1)\begin{pmatrix}0&0&0&0\\1&0&0&0\\1&-1&-1&-0.5\\0&-2&2&-1\end{pmatrix}z(i)+\begin{pmatrix}1&0\\-1&0\\-1.5&-0.25\\-2&0.5\end{pmatrix}y(i)+\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{pmatrix}u(i).$$

In this section, we have developed a procedure of designing a dynamic deadbeat controller with an observer under the assumption that the observer is unbiased, so that the resultant controller is of order, at least, n-m. Further reduction of the order may be possible, because the assumption of unbiasedness may not be necessary, but K-unbiasedness[15] for some $K \in \underline{K}^{\circ}$ may be sufficient for controllers to have the required property. This will be shown in section 6.

5. Dynamic Deadbeat Controller with Dual Observer

In the last section we showed that a dynamic controller of order n-m can be designed to achieve a deadbeat performance. On the other hand, it is known[11] that a dynamic controller of order n-r can be constructed such that the 2n-r eigenvalues of the closed-loop system take any preassigned values. Such a controller is called a dual observer. This idea is applied to the design of deadbeat controllers in what follows.

We consider the dual observer, written $D(K, V)$, of the form:

$$u(i) = K\xi(i) \tag{5, 1a}$$

$$\xi(i) = Vy(i) + (A+VC)Sz(i) \tag{5, 1b}$$

$$z(i+1) = T\xi(i) \tag{5, 1c}$$

where $z(\cdot) \in \boldsymbol{R}^p$ and $\xi(\cdot) \in \boldsymbol{R}^n$. The dual observer (5, 1) is said to be unbiased if

$$I - BK = ST.$$

It is assumed that the dual observer is unbiased and that both S and T are of full rank. We shall call, in the sequel, the dynamical system $D_d(K, V)$ a dynamic dual controller rather than a dual observer for consistency with the dynamic controller $D(K, V)$. The symbol t and the notations $x(i\,;K, V)$ and $t(K, V)$ are used for purposes similar to those in section 4.

DEFINITION 4 : A map $K : \boldsymbol{R}^n \to \boldsymbol{R}^r$ is called an optimal dual controller if

$$\text{Ker } [(I-BK)A]^i (I-BK) = \mathscr{X}_i + \mathscr{B}, \quad \forall i \in \underline{q} \tag{5, 3}$$

where $\mathscr{X}_i$ is already defined by (2, 3).

DEFINITION 5 : A map $V : \boldsymbol{R}^m \to \boldsymbol{R}^n$ is called an optimal dual observer if

$$\text{Im } (A+VC)^i = \mathscr{S}_i, \quad \forall i \in \underline{p} \tag{5, 4}$$

where $\mathscr{S}_i$ is defined by (3, 7).

We shall write $\underline{K}_d^{\circ}$ and $\underline{V}_d^{\circ}$ for the classes of optimal dual controllers and observers, respectively.

DEFINITION 6 : A dynamic controller $D_d(K, V)$ is called a dynamic deadbeat dual controller, if there exists $\rho \in \boldsymbol{i}$ such that

$$x(i\,;K, V) = 0, \quad \forall i \geq \rho \tag{5, 5}$$

for any $x(0)$ and $z(0)$. Furthermore, we call $D_d(K, V)$ a minimum time controller if it minimizes $t(K, V)$.

The problem to be treated in this section is to find the minimum time deadbeat dual controller.

To begin with, we obtain the optimal dual controller and observer.

THEOREM 4 : Let $Y \in M_{r,n}$ be defined by (2, 9) – (2, 10). Then

$$K=(YB)^{-1}Y$$

is an optimal dual controller. Let $Z \in \underline{Z}$ where $\underline{Z}$ is defined by (3, 15). Then

$$V=-AZ(CZ)^{-1}$$

is an optimal dual observer.

The proof of THEOREM 4 is about the same as those for THEOREMs 1 and 2.

The following theorem is the main result of this section.

THEOREM 5 : There exists a dynamic deadbeat dual controller if and only if the pair (A, B) is controllable, and the pair (C, A) is reconstructible. Under these conditions, every $D_d(K, V)$ with $K \ni \underline{K}_d^\circ$ and $V \in \underline{V}_d^\circ$ is a minimum time controller.

The proof of THEOREM 5 is based on the following three lemmas.

LEMMA 8 : Let $i \in \boldsymbol{i}$. Then

$$x(i\,;K,V)=0 \tag{5, 6}$$

for any $x(0)$ and $z(0)$, if and only if

$$[(I-BK)A]^i+\sum_{k=0}^{i-1}[(I-BK)A]^k BK(A+VC)^{i-k}=0 \tag{5, 7}$$

and

$$\sum_{k=0}^{i-1}[(I-BK)A]^k BK(A+VC)^{i-k}S=0. \tag{5, 8}$$

PROOF : We write

$$\varepsilon(i) \triangleq Ax(i)+\xi(i). \tag{5, 9}$$

Then, by (5, 1) – (5, 2) we obtain

$$\varepsilon(i+1)=(A+VC)\varepsilon(i) \tag{5, 10a}$$

$$\varepsilon(0)=(A+VC)[x(0)+Sz(0)] \tag{5, 10b}$$

and

$$x(i+1)=(I-BK)Ax(i)+BK\varepsilon(i). \tag{5, 11}$$

Iteration of (5, 11) and use of (5, 10) yields

$$\begin{aligned} x(i+1) &= [(I-BK)A]^{i+1}x(0)+\sum_{k=0}^{i}[(I-BK)A]^k BK\varepsilon(i-k) \\ &= [(I-BK)A]^{i+1}x(0)+\sum_{k=0}^{i}[(I-BK)A]^k BK(A+VC)^{i-k}\varepsilon(0) \\ &= \{[(I-BK)A]^{i+1}+\sum_{k=0}^{i}[(I-BK)A]^k BK(A+VC)^{i-k+1}\}x(0) \end{aligned}$$

$$+\sum_{k=\cdot}^{i}[(I-BK)A]^{k}BK(A+VC)^{i-k+1}Sz(0).$$

Thus, (5, 7) – (5, 8) follow at once.

LEMMA 9 : If (5, 6) holds for some $i\in \boldsymbol{i}$ and for any $x(0)$ and $z(0)$, Then

$$\mathscr{X}_{i-j}\supset\mathscr{S}_j,\ \forall j\in \underline{i}. \tag{5, 12}$$

PROOF : We define

$$N_0\triangleq I \tag{5, 13a}$$

$$N_j\triangleq[(I-BK)A]^j+\sum_{k=0}^{j-1}[(I-BK)A]^kBK(A+VC)^{j-k},\quad j=1,2,\cdots,i, \tag{5, 13b}$$

and observe that N_j, $j\in\underline{i}$, satisfies the following recurrence relations :

$$N_j=[(I-BK)A]N_{j-1}+BK(A+VC)^j \tag{5, 14}$$

and

$$N_j=N_{j-1}(A+VC)-(I-BK)[A(I-BK)]^{j-1}VC \tag{5, 15}$$

We show first that (5, 7) implies

$$\text{Ker } N_{i-j}\supset\mathscr{S}_j,\quad \forall j\in\underline{i}. \tag{5, 16}$$

Obviously, $N_i=0$, so that Ker $N_i\supset\mathscr{S}_0=\boldsymbol{R}^n$, and if Ker $N_{i-j}\supset\mathscr{S}_j$, then by (5, 15) we obtain

$$0=N_{i-j}\mathscr{S}_j\supset N_{i-j}(\mathscr{S}_j\cap\text{Ker } C)=N_{i-j-1}A(\mathscr{S}_j\cap\text{Ker } C)$$
$$=N_{i-j-1}\mathscr{S}_{j+1},$$

proving Ker $N_{i-(j+1)}\supset\mathscr{S}_{j+1}$, and hence, induction proves (5, 16). Next, it will be shown that

$$\mathscr{X}_j\supset\text{Ker } N_j,\quad \forall j\in\underline{i}. \tag{5, 17}$$

For this it suffices to verify that

$$\overset{-1}{N}_{j-k}\mathscr{X}_k\subset\overset{-1}{N}_{j-(k+1)}\mathscr{X}_{k+1},\qquad k\in\underline{j-1}.$$

Let $x\in\overset{-1}{N}_{j-k}\mathscr{X}_k$. Then $N_{j-k}x\in\mathscr{X}_k$, and hence, by (5, 14) we get $AN_{j-k-1}x\in\mathscr{X}_k+\mathscr{B}$, or

$$x\in\overset{-1}{N}_{j-k-1}A^{-1}(\mathscr{X}_k+\mathscr{B})=\overset{-1}{N}_{j-k-1}\mathscr{X}_{k+1}$$

as claimed. The lemma now follows from (5, 16) & (5, 17). □

LEMMA 10 : Let $K\in\underline{K}_d{}^\circ$ and $V\in\underline{V}_d{}^\circ$. Suppose that there exists $i\in\boldsymbol{i}$ such that

$$\mathscr{X}_{i-j}\supset\mathscr{S}_j,\qquad \forall j\in\underline{i}. \tag{5, 18}$$

Then $x(i\,;\,K,V)=0$ for any $x(0)$ and $z(0)$.

PROOF : By definition we obtain

$$\text{Im }(A+VC)^j=\mathscr{S}_j,\qquad \forall j\in\underline{i} \tag{5, 19}$$

$$\text{Ker }[(I-BK)A]^{i-j}(I-BK)=\mathscr{X}_{i-j}+\mathscr{B}, \qquad \forall j\in \underline{i} \tag{5, 20}$$

and so

$$\text{Ker }[(I-BK)A]^{i-j}=A^{-1}(X_{i-j-1}+\mathscr{B})=\mathscr{X}_{i-j}, \qquad \forall j\in \underline{i}. \tag{5, 21}$$

It follows from (5, 18) – (5, 21) that

$$[(I-BK)A]^{i-j}(I-BK)(A+VC)^{j}=0, \qquad \forall j\in \underline{i}$$
$$[(I-BK)A]^{i-j}(A+VC)^{j}=0, \qquad \forall j\in \underline{i}.$$

Thus, the lemma follows from LEMMA 8. □

The proof of THEOREM 5 is obvious from LEMMAs 8-10.

REMARK 1 : It is obvious that

$$t^{*}\triangleq\min_{K,V} t(K, V)=\min \{i: \mathscr{X}_{i-j}\supset\mathscr{S}_{j}, \ \forall j\in \underline{i}\}\leq p+q-1,$$

i. e., the minimum of $t(K, V)$ is independent of the choice between $D(K, V)$ and $D_d(K, V)$.

REMARK 2 : Since rank $(I-BK)=n-r$ if $K\in\underline{K}_d^{\circ}$, it follows from (5, 2) that the dynamic dual controller can be designed with order $\mu=n-r$.

REMARK 3 : We denote the system (2, 1) & (3, 1) by S, and by S' the dual system to S : i. e.

$$x(i-1)=A'x(i)+C'y(i)$$
$$u(i)=B'x(i).$$

We can formulate for S' the problems corresponding to those which have been treated in sections 2-5 for S. These problems can be easily solved by dualizing the corresponding results. Then the following dualities are readly verfied. Let $K\in M_{r,n}$ and $V\in M_{n,m}$. Then K is an optimal (optimal dual) controller for S if and only if K' is an optimal dual (optimal) observer for S'. V is an optimal (optimal dual) observer for S if and only if V' is an optimal dual (optimal) controller for S'. Furthermore, for S' we define the dynamic controller $D'(V', K')$:

$$z(i-1)=S'A'T'z(i)+S'A'K'u(i)+S'C'y(i)$$
$$y(i)=V'T'z(i)+V'K'u(i),$$

and the dynamic dual controller $D_d'(V', K')$:

$$z(i+1)=T'\xi(i)$$
$$\xi(i)=K'u(i)+(A'+K'B')S'z(i)$$
$$y(i)=V'\xi(i).$$

Then, if $K\in\underline{K}^{\circ}$ and $V\in\underline{V}^{\circ}$, $D(K, V)$ $(D_d'(V', K'))$ is a minimum time dynamic (dynamic dual) controller for S (S'). If $K\in\underline{K}_d^{\circ}$ and $V\in\underline{V}_d^{\circ}$, then $D_d(K, V)$ $(D'$

$(V', K'))$ is a minimum time dynamic dual (dynamic) controller for $S(S')$.

EXAMPLE 2 : Consider the same system as in EXAMPLE 1. We have already obtained

$$Y=\begin{bmatrix}1&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{bmatrix}, \qquad Z'=\begin{bmatrix}0&1&2&1&2&2\\0&0&0&1&4&4\end{bmatrix}$$

Applying THEOREM 4 we get

$$K=\begin{bmatrix}1&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{bmatrix}, \qquad V'=-\begin{bmatrix}1&2&3&0.5&4&2\\0&0&0&0.25&2&2.5\end{bmatrix},$$

so that

$$I-BK=\begin{bmatrix}0&0&0&-1&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\end{bmatrix},$$

A full rank factorization of $(I-BK)$ gives

$$S=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}, \qquad T=\begin{bmatrix}0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{bmatrix}.$$

Consequently, by (5, 1) we obtain the following controller with order $\mu=n-r=3$:

$$u(i)=-\begin{bmatrix}0.5&0&0.25\\4&1&0\\1&-2&1.5\end{bmatrix}z(i)-\begin{bmatrix}1.5&0.25\\3&0\\2&2.5\end{bmatrix}y(i)$$

$$z(i+1)=-\begin{bmatrix}2&0&0\\0.5&0&0.25\\4&0&2\end{bmatrix}z(i)-\begin{bmatrix}2&0\\0.5&0.25\\4&2\end{bmatrix}y(i).$$

6. Rednced Order Dynamic Deadbeat Controllers

In the preceding two sections, it was shown that dynamic deadbeat controllers can be designed with order $n-m$ or $n-r$. It is possible, however, to design deadbeat controllers with order less than either $n-m$ or $n-r$, if the results[15),16)] on the reduced order linear function observers are applied. This will be shown in this section. For simplicity, it is assumed in this section that the pair (A, B) is controllable and the pair (C, A) is reconstructible.

Observer based controller

Let $K \in \underline{K}^\circ$ and write

$$\mathscr{L} \triangleq \text{Ker } K \cap \text{Ker } C. \tag{6, 1}$$

Let $\mathscr{W}$ be a subspace of $\mathscr{L}$ such that Ker $CA \cap \mathscr{W} = 0$ and

$$\text{Ker } C \cap \mathscr{O}_k \subset \mathscr{L} \tag{6, 2}$$

for some $k \in \underline{n}$, where $\mathscr{O}_k$ is computed sequentially by

$$\mathscr{O}_0 = \boldsymbol{R}^n \tag{6, 3a}$$

$$\mathscr{O}_{i+1} = A(\text{Ker } C \cap \mathscr{O}_i) + A\mathscr{W}. \tag{6, 3b}$$

Since $\mathscr{O} \triangleq \{\mathscr{O}_0, \mathscr{O}_1, \cdots\}$ is non-increasing, we can define

$$\rho \triangleq \min\{k : \text{Ker } C \cap \mathscr{O}_k \subset \mathscr{L}\}.$$

Compute $\mathscr{M}_n$ according

$$\mathscr{M}_0 = \mathscr{L}$$

$$\mathscr{M}_{i+1} = \mathscr{L} \cap A^{-1}\mathscr{M}_i \oplus \mathscr{W}$$

and put

$$\mathscr{M} \triangleq \mathscr{M}_n. \tag{6, 4}$$

Then we can choose the 3-tuple (V, S, T) so that[15),16)]

$$(I - VC)A\mathscr{M} \subset \mathscr{M} \subset \mathscr{L} \subset \text{Ker } K \tag{6, 5}$$

$$K(I - VC)[A(I - VC)]^\rho = 0 \tag{6, 6}$$

$$I - VC - ST = 0 \pmod{\mathscr{M}} \tag{6, 7}$$

and we obtain the observer of order $l = n - m - \dim \mathscr{M}$, which reconstructs the control law $Kx(i)$ in at most ρ-steps. It is now enough to note that (6, 5) and (6, 7) implies

$$BK[(I - VC)A]^i(I - VC)ST = 0, \quad i \in \boldsymbol{i}$$

and hence (4, 10) yet holds, in order to verify that the observer having the property (6, 5)-(6, 7) actually achieves deadbeat control. A procedure is given by Akashi and Imai[15),17)] for obtaining such an observer. This is summarized below.

We note first that there exists a sequence $\underline{\underline{\mathscr{X}}} \triangleq \{\mathscr{X}_0, \mathscr{X}_1, \cdots, \mathscr{X}_\rho\}$ such that

$$\mathscr{O}_i = \text{Ker } C \cap \mathscr{O}_i \oplus \mathscr{X}_i \oplus A\mathscr{W}$$

$$\mathscr{X}_\rho \subset \mathscr{X}_{\rho-1} \subset \cdots \subset \mathscr{X}_0.$$

We can choose $Z \in M_{n,m}$ so that

$$\text{Im } Z \triangleq \mathscr{X}_0 \oplus A\mathscr{W}. \tag{6, 8}$$

It is readily verfied that $CZ \in M_{m,m}$ is invertible, and so we can define

$$V \triangleq (I - P_{\mathscr{M}}) Z (CZ)^{-1} \tag{6, 9}$$

and choose S and T of full rank so that

$$ST \triangleq (I - P_{\mathscr{M}})(I - Z(CZ)^{-1}C), \tag{6, 10}$$

where P_M is the orthogonal projection on $\mathscr{M}$. The 3-tuple (V, S, T) actually has the property (6, 5)-(6, 7) as shown in 15)-17).

Summarizing the above argument we have the following result.

THEOREM 6 : Let $K \in \underline{K}^\circ$ and let (V, S, T) be given by (6, 9)-(6, 10) with S and T of full rank. Then the 4-tuple (K, V, S, T) provides a deadbeat controller of order

$$l = n - m - \dim \mathscr{M}.$$

REMARK 1 : The controller obtained in THEOREM 6 drives the system state to the origin in at most $(q+\rho)$-steps.

REMARK 2 : Nothing has been mentioned about how to choose $\mathscr{W}$ so that (6, 2) is satisfied. It is shown in 16) that when $\dim L < m$, an arbitrarily chosen $\mathscr{W}$ generically has the property (6, 2). On the other hand, when $\dim L \geq m$, an arbitrarily chosen $\mathscr{W}$ satisfies (6, 2) generically only if $\dim \mathscr{W} \leq m-1$.

EXAMPLE 3 : Consider the system treated in EXAMPLE 1. We have already obtained a $K \in \underline{K}^\circ$ in EXAMPLE 1. Thus it is easy to check that

$$\mathscr{L} = \text{Im}\,[-2 \quad 0 \quad 1 \quad 2 \quad 0 \quad -4]', \text{ Ker } CA \cap \mathscr{L} = 0.$$

Therefore, we can choose $\mathscr{W} = \mathscr{L}$. Then by (6, 3) we get

$$\mathscr{O}_1 = \text{Im} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathscr{O}_2 = \text{Im} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 6 \\ 2 & 4 & 3 \end{pmatrix}, \quad \mathscr{O}_3 = Im \begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 0 & -4 \\ 0 & 1 \\ -6 & 8 \\ 0 & 8 \end{pmatrix}.$$

so that $\mathscr{O}_3 \cap \text{Ker } C = \mathscr{O} \subset \mathscr{L}$. Therefore, by (6, 9) and (6, 10) we obtain

$$V' = \begin{bmatrix} 4.08 & 1. & 3.96 & -5.58 & 0. & -3.84 \\ 0.68 & 0. & 0.66 & -0.93 & 1. & -0.64 \end{bmatrix}, \quad S = \begin{pmatrix} 0.5 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 0.08 & -3.96 & 0.96 & -0.08 & -0.66 & 0.16 \\ 0.16 & 5.58 & -0.08 & 0.84 & 0.93 & 0.32 \\ -0.32 & 3.84 & 0.16 & 0.32 & 0.64 & 0.36 \end{pmatrix},$$

Cosequently, we obtain the following third order controller:

$$u(i)=\begin{bmatrix}-0.5 & -2 & 2\\ -2.5 & -1 & 2\\ -0 & -2 & -1\end{bmatrix} z(i)+\begin{bmatrix}0.5 & 0.25\\ -11.0 & -2.0\\ 14.0 & 1.5\end{bmatrix} y(i)$$

$$z(i+1)=\begin{bmatrix}-2.92 & 1.12 & -2.92\\ 7.66 & 2.24 & 0.66\\ 5.68 & 1.52 & 0.68\end{bmatrix} z(i)+\begin{bmatrix}-7.32 & -0.94\\ 15.86 & 2.87\\ 11.28 & 2.26\end{bmatrix} y(i)+$$
$$+\begin{bmatrix}0.08 & 0.96 & 0.16\\ 0.16 & -0.08 & 0.32\\ -0.32 & 0.16 & 0.36\end{bmatrix} u(i).$$

We observe that $t(K, V)=5$ for this reduced order controller while $t^*=4$ for the minimum time controller. Although the order of controller can be reduced by one, the number of control steps increases by one.

Dual observer based controller

The above result can be readily dualized to yield a dynamic dual controller with an order less than $n-r$. To show this we need the following preliminary result, which will later be dualized.

LEMMA 11 : Let $K\in \underline{K}_0$ and $\mathscr{M}$ denote the subspace defined by (6, 4). Suppose that there exists $\pi\in \boldsymbol{i}$ such that

$$\mathcal{O}_\pi\cap \mathrm{Ker}\ C\subset\mathscr{M}\subset\mathscr{L} \tag{6, 11}$$

where $\mathcal{O}_\pi\in \underline{\underline{\mathcal{O}}}$ is defined by (6, 3). with Z given by (6, 8) let

$$V=(I-P_{\mathscr{M}})Z(CZ)^{-1} \tag{6, 12}$$
$$ST=(I-P_{\mathscr{M}})[I-Z(CZ)^{-1}C]. \tag{6, 13}$$

Then

$$K(I-VC-ST)=0 \tag{6, 14}$$
$$TA(I-VC-ST)=0, \tag{6, 15}$$

and for some $i\in \boldsymbol{i}$

$$[A(I-VC)]^i+\sum_{k=0}^{i-1}(A+BK)^{i-k}VC[A(I-VC)]^k=0 \tag{6, 16}$$
$$T\{\sum_{k=0}^{i-1}(A+BK)^{i-k}VC[(I-VC)]^k\}=0 \tag{6, 17}$$

PROOF : We obtain

$$I-VC-ST=P_{\mathscr{M}}. \tag{6, 18}$$

Thus, (6, 14) follows. Since V, as given by (6, 12), satisfies $(I-VC)A\mathscr{M}\subset\mathscr{M}$, we obtain

$$STAM=(I-P_{\mathscr{M}})(I-VC)AM=0, \tag{6, 19}$$

and since S is of full rank, (6, 15) follows from (6, 18)-(6, 19). As to (6, 16), we first obtain

$$(A+BK)^{i}=0, \qquad \forall i \geq q. \tag{6, 20}$$

Also we see by (6, 11) that

$$\text{Im } (I-VC)[A(I-VC)]^{k} \subset \mathscr{M} \subset \mathscr{L}, \qquad \forall k \geq \pi. \tag{6, 21}$$

Hence,

$$H_{k}=(A+BK)^{k}BK(I-VC)[A(I-VC)]^{i-1-k}=0, \qquad k \in \underline{i-1} \tag{6, 22}$$

if $i \geq q+\pi$. From (6, 20) (6, 22) we claim that if $i \geq q+\pi$

$$(A+BK)^{i}-\sum_{k=0}^{i-1}(A+BK)^{k}BK(I-VC)[A(I-VC)]^{i-1-k}=0. \tag{6, 23}$$

It is not difficult to verify that (6, 23) is equivalent to (6, 16). Finally, we have from (6, 19) and (6, 21) that

$$STA(I-VC)[A(I-VC)]^{i-1}=0$$

for some $i \in \boldsymbol{i}$, and since S is of full rank,

$$T[A(I-VC)]^{i}=0. \tag{6, 24}$$

Therefore, (6, 17) follows immediately from (6, 16) and (6, 24). □

Then we obtain

THEOREM 7 : Let $V \in \underline{V}_{d}^{\circ}$ and define

$$\underline{M}' \triangleq \{M' : \text{Ker } B' \cap A'\mathscr{M}' \subset \mathscr{M}' \subset \text{Ker } B' \cap \text{Ker } V', \quad \exists k \in n, \ \mathscr{O}_{k}' \cap \text{Ker } B' \subset \mathscr{M}'\} \tag{6, 25}$$

where $\mathscr{O}_{k}'$ is computed sequentially according to

$$\mathscr{O}_{0}'=(\boldsymbol{R}^{n})' \tag{6, 26a}$$

$$\mathscr{O}_{i}'=A'(\text{Ker } B' \cap \mathscr{O}_{i-1}')+A'\mathscr{W}' \tag{6, 26b}$$

where $\mathscr{W}'$ is such that

$$\mathscr{M}'=\text{Ker } B'A' \supset \mathscr{M}' \oplus \mathscr{W}'.$$

Let $\underline{\underline{\mathscr{X}}}'$ be the sequence such that

$$\mathscr{O}_{i}'=\text{Ker } B' \cap \mathscr{O}_{i}' \oplus \mathscr{X}_{i}' \oplus A'\mathscr{W}' \tag{6, 27}$$

$$\mathscr{X}_{0}' \supset \mathscr{X}_{i}' \supset \cdots \supset \mathscr{X}_{n}'. \tag{6, 28}$$

with $Y^{\circ} \in M_{r,m}$ such that

$$\text{Im } Y^{\circ\prime}=\mathscr{X}_{0}'+A'\mathscr{W}', \tag{6, 29}$$

define

$$K' \triangleq (I-P_{\mathscr{M}'})Y^{\circ\prime}(B'Y^{\circ\prime})^{-1} \tag{6, 30}$$

$$T'S' \triangleq (I-P_{\mathscr{M}'})[I-Y^{\circ\prime}(B'Y')^{-1}B']. \tag{6, 31}$$

Then, the quadruple (K, V, S, T) supplies a deadbeat dual controller with order

$$\mu=n-r-\dim \mathscr{M}'. \tag{6, 32}$$

PROOF : By dualizing (6, 14)-(6, 17), i. e., by replacing (A, B, C, K, V, S, T) by $(A', C', B', V', K', T', S')$, we get

$$V'(I-K'B'-T'S')=0$$
$$S'A'(I-K'B'-T'S')=0$$
$$[A'(I-K'B')]^i+\sum_{k=0}^{i-1}(A'+C'V')^{i-k}K'B'[A'(I-K'B')]^k=0$$
$$S'\sum_{k=0}^{i-1}(A'+C'V')^{i-k}K'B'[A'(I-K'B')]^{i-k}=0,$$

or

$$(I-BK-ST)V=0 \tag{6, 33}$$
$$(I-BK-ST)AS=0 \tag{6, 34}$$
$$[(I-BF)A]^i+\sum_{k=0}^{i-1}[(I-BK)A]^kBK(A+VC)^{i-k}=0 \tag{6, 35}$$
$$\sum_{k=0}^{i-1}[(I-BK)A]^kBK(A+VC)^{i-k}S=0. \tag{6, 36}$$

It follows from (5, 1b) and (6, 33)-(6, 34) that

$$(I-BK-ST)\xi(i)=0,$$

and hence

$$\varepsilon(i+1)=Ax(i+1)+\xi(i+1)=(A+VC)[x(i+1)+Sz(i+1)]$$
$$=(A+VC)[Ax(i)+(BK+ST)\xi(i)]=(A+VC)\varepsilon(i).$$

Thus, LEMMA 8 can be applied to this case as well. Therefore, we see by (6, 35) -(6, 36) that the dual controller (5, 1) achieves deadbeat performance if $V\in \underline{V}_d^\circ$ and if (K, S, T) is given by (6, 30)-(6, 31). The proof of (6, 32) follows about the same lines as for THEOREM 4 in Akashi and Imai[18].

EXAMPLE 4 : Consider the same system as in EXAMPLEs 1-3. We have already obtained $V\in \underline{V}_d^\circ$ in EXAMPLE 2. It is readily verified that

$$\mathscr{M}'=\mathrm{Im}[0\quad 0\quad 0\quad -2\quad 0.25\quad 0]'=\mathscr{W}'$$

has all the required conditions of THEOREM 7, i. e., $\mathscr{M}'\in\mathscr{M}'$. Then by (6, 26)-(6, 28) we obtain

$$\mathscr{X}_0'=\mathrm{Im}\begin{bmatrix}1&0&0&1&0&0\\0&0&1&0&0&0\end{bmatrix}'.$$

Thus,

$$Y^\circ = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and hence, by (6, 30)-(6, 31) we get

$$S = \begin{pmatrix} -1/65 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1/65 & 0 \\ 8/65 & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & 1 & 8 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 1 & 0 & 0 & 1/65 & 8/65 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, we obtain the following dual deadbeat controller with order $\mu = n - r - \dim \mathscr{M}' = 1$.

$$u(i) = -\begin{pmatrix} 1.5 & 0.25 \\ 3 & 0 \\ 2 & 2.5 \end{pmatrix} y(i) - \begin{pmatrix} 2/65 & 0.5 \\ 1.65 & 4 \\ 2/13 & 1 \end{pmatrix} z(i)$$

$$z(i+1) = -\begin{pmatrix} 32.5 & 16.25 \\ 2 & 0 \end{pmatrix} y(i) - \begin{pmatrix} 2 & 32.5 \\ 0 & 2 \end{pmatrix} z(i).$$

7. Conclusion

This paper has considered the design problem of dynamic deadbeat controllers using an observer and a dual observer in discrete time linear multivariable systems. First, we defined and obtained the optimal controller and observer. Then we formulated and solved the problem of designing the minimum time dynamic deadbeat controller, and proved the separation theorem which states that the minimum time dynamic deadbeat controller can be obtained by combining the optimal controller and observer. All these results were dualized to derive the corresponding results on the design of the dynamic dual controller. Finally, an algorithm of designing a linear function observer was applied to the design of reduced order controllers. It was shown in an example that the reduction of the controller order was indeed attained, but that the number of control steps may increase. The problem of designing the minimum order minimum time deadbeat controller is unsolved, but it is an interesting problem for the future.

References

1) R. E. Kalman; Proc. 1st IFAC Congress, Moscow, Butterworths, London, pp. 2020-2030 (1960).
2) J. B. Farison and F. C. Fu; IEEE Trans. Automat. Contr., Vol. **AC-15**, pp. 390-391 (1970).
3) V. Kucera; IEEE Trans. Automat. Contr., Vol. **AC-16**, pp. 375-377 (1971).

4) C. T. Mullis; IEEE Trans. Automat. Contr., Vol. AC-**17**, pp. 265-266 (1972).
5) B. Leden; Automatica, Vol. **13**, pp. 185-188 (1977).
6) B. Leden; IEEE Trans. Automat. Contr., Vol. AC-**21**, pp. 791-792 (1976).
7) H. Seraji; Int. J. Control, Vol. **21**, pp. 213-223 (1975).
8) B. Porter and A. Bradshaw; Int. J. Control, Vol. **22**, pp. 149-155 (1975).
9) B. Porter and A. Bradshaw; Trans. ASME, Vol. **98**, Ser. G, pp. 152-155 (1976).
10) H. Akashi and M. Adachi; Systems and Control, Vol. **21**, pp. 671-677 (1977) (in Japanese).
11) D. G. Luenberger; IEEE Trans. Automat. Contr., Vol. AC-**16**, pp. 596-602 (1971).
12) P. Dorato and A. H. Levis; IEEE Trans. Automat. Contr., Vol. AC-**16**, pp. 613-620 (1971).
13) T. Yoshikawa; Systems and Control, Vol. **18**, pp. 125-126 (1974) (in Japanese).
14) H. Akashi and H. Imai; Systems and Control, Vol. **21**, pp. 636-644, (1977) (in Japanese).
15) H. Akashi, H. Imai, and M. Adachi; Systems and Control, Vol. **22**, pp. 104-110, (1978) (in Japanese).
16) H. Imai; "Design of observers and Controllers for Discrete Time Linear Multivariable Systems," Dissertation, Kyoto University (1978).
17) H. Akashi and H. Imai; Systems and Control, Vol. **22**, pp. 753-760 (1978) (in Japanese).
18) H. Akashi and H. Imai; Proc. 7th IFAC Congress, Helsinki, pp. 1717-1724 (1978).

Appendix

LEMMA A: Let $\mathscr{S}$, $\mathscr{R}$, $\mathscr{V}$ be subspaces such that $\mathscr{R}\oplus\mathscr{V}=\boldsymbol{R}^n$ and $P_{\mathscr{R},\mathscr{V}}$ denote the projection on $\mathscr{R}$ along $\mathscr{V}$. Then

$$P_{\mathscr{R},\mathscr{V}}\mathscr{S}=\mathscr{R}\cap(\mathscr{S}+\mathscr{V}) \quad \text{(A, 1)}$$

$$(P_{\mathscr{R},\mathscr{V}})^{-1}\mathscr{S}=\mathscr{R}\cap\mathscr{S}\oplus\mathscr{V}. \quad \text{(A, 2)}$$

The proof can be found in Akashi and Imai[18].