A Method for Solving Mixed-Integer Linear Programming Problems with Angular Structure

By

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(Received June 26, 1979)

Abstract

An algorithm for solving mixed-integer linear programming problems with an angular structure is proposed. The basic idea is to decompose the original problem in the same way as the Dantzig-Wolfe decomposition principle in the linear programs, and to solve a restricted master program and subproblems iteratively. The subproblems are mixed-integer problems of smaller sizes than that of the original one. The termination of this algorithm is checked in two stages. If the optimality test is satisfied, the procedure terminates with the optimal solution. If not, the search for improvement is continued within a restricted extent. If the search terminates with no improved solution, the best solution obtained so far is given as a suboptimal solution. The numerical results making comparisons between the present method and the branch and bound method are shown.

1. Introduction

Although a number of combinatorial optimization problems may be formulated as mixed-integer programs, solving them is often hampered by size[1)]. In dealing with large-scale systems in the real world, large problems almost always have a special structure, such as the so called angular structure. In such a case, it may be efficient to devise the computational procedure in due consideration of that structure.

The purpose of this paper is to develop a method for solving mixed-integer linear programming problems with an angular structure. The integer variables are contained separately in their respective block constraints, but have interrelations among them through the coupling constraint for the continuous variables. An example of such problems is a dynamical planning problem over several periods of time, in which the planning for single time period is given as a mixed-integer problem.

The Dantzig-Wolfe decomposition principle has been already proposed for

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solving linear programming problems with an angular structure. The method proposed in this paper can be viewed as an extension of the Dantzig-Wolfe decomposition principle to mixed-integer problems. The solution is obtained by solving a restricted master program and subproblems iteratively. One is a linear programming problem, while the others are mixed-integer problems of smaller sizes than that of the original problem.

The termination of the present algorithm is checked in two stages. If the optimality test is satisfied, the procedure terminates and the optimal solution is obtained. If not, the search for improvement is continued within a restricted extent of the search region. The procedure is finite. If the procedure terminates with no improved solution, the best solution obtained so far is provided as a suboptimal solution. Accordingly, this algorithm can not necessarily give any guarantee for obtaining the optimal solution, but, if not, it may be expected to obtain a good suboptimal solution quickly.

Benders[3] has presented a partitioning approach for solving mixed-integer problems. In this approach, the original problem is partitioned into a relaxed pure-integer problem and a linear programming problem. Then the two problems are solved iteratively. The pure-integer problem has the same size as the original problem. Consequently, this approach seems to be effective only for a problem with a relatively small number of integer variables. On the other hand, the size of the mixed-integer problem to be solved in the proposed algorithm is smaller than that of the original, as the result of decomposition of the original problem based upon the structure of the constraints. In view of this, the proposed algorithm seems to be promising for the problem mentioned above.

With an application to the optimal planning of blending raw coal in the iron industry, the present algorithm compares favourably with the conventional branch and bound method.

2. Problem Statement

Consider the following mixed-integer linear programming problem:

(P1)
$$\min_{x(t),y(t)} z=\sum_{t=1}^{T} c(t)'y(t) \qquad \text{subject to}$$

$$\sum_{t=1}^{T} A(t)y(t)=b \tag{1}$$

$$\left.\begin{array}{l} B(t)y(t)+D(t)x(t)\geqq d(t) \\ y(t)\geqq 0, \qquad x(t)\in X_t \end{array}\right\} \tag{2}$$

$$t=1,2,\cdots,T$$

where,

$$X_t \triangleq \{x(t) \mid x_i(t) = 0 \text{ or } 1, i = 1, 2, \cdots, m_t\} \tag{3}$$

In (P1), $y(t)$ is an $n_t \times 1$ continuous vector, while $x(t)$ is an $m_t \times 1$ integer vector. The vectors $c(t)$, $d(t)$ and b are of the dimension $n_t \times 1$, $l_t \times 1$ and $l_0 \times 1$, respectively. $A(t)$, $B(t)$ and $D(t)$ are the matrices of appropriate dimensions. A prime denotes transposition of a vector or a matrix.

The constraints of this problem have a special structure, called angular. The integer vector $x(t)$ for each t is contained in the t-th block constraint (2) alone, but has interrelations among $x(t)$ for all other t through the coupling constraint (1) imposed on the continuous vector $y(t)$.

In this paper, we propose an efficient algorithm for solving (P1) with special regard to the angular structure of the problem. In the way similar to the Dantzig-Wolfe decomposition technique for the linear programming problems, the solution is obtained by solving a restricted master program and subproblems iteratively. One is a linear programming problem, while the others are mixed-integer problems of small sizes.

3. Construction of the Restricted Master Program

The t-th block constraint (2) gives the feasible region for the continuous vector $y(t)$, if the integer vector $x(t)$ is assigned at an appropriate value in X_t. Then, the feasible region for $y(t)$ consists of the convex polyhedrons in R^{n_t} defined by

$$S_t^k \triangleq \{y(t) \mid B(t)y(t) \geqq d(t) - D(t)x^k(t),\ y(t) \geqq 0,\ x^k(t) \in X_t\}$$
$$k = 1, 2, \cdots, k_t \tag{4}$$

where k_t is the number of feasible points of $x(t)$ in X_t. We also define the union of the polyhedrons as

$$Y_t \triangleq \bigcup_{k=1}^{k_t} S_t^k \tag{5}$$

Then the problem is to choose $x^k(t)$, or equivalently S_t^k, for each t so that the resultant combination of S_t^k can attain the satisfaction of the coupling constraint (1) and the minimization of the objective function z.

Let S_t^k be bounded for simplicity, and let $\{{}^jy_t^k\}$ be the set of its extreme points. The convex combination of all the extreme points of the t-th block forms the convex hull of Y_t, i.e.

$$\bar{Y}_t \triangleq \{y(t) \mid y(t) = \sum_{j,k} {}^j\mu_t^k\, {}^jy_t^k,\ \sum_{j,k} {}^j\mu_t^k = 1,\ {}^j\mu_t^k \geqq 0\} \tag{6}$$

which contains the feasible region for $y(t)$.

In the same way as the conventional Dantzig-Wolfe decomposition principle, we construct the master program in terms of all extreme points of all blocks. Substituting $y(t)\in \bar{Y}_t$ into the objective function z and the coupling constraint (1) gives the following linear programming problem:

(P2)
$$\min_{\lambda} z = f'\lambda \qquad \text{subject to}$$
$$F\lambda = b \tag{7}$$
$$E\lambda = e \tag{8}$$
$$\lambda \geqq 0 \tag{9}$$

where λ and f are the $\nu\times 1$ vectors whose components are ${}^j\mu_t^k$ and $c(t)'{}^jy_t^k$ respectively, ν being the number of all the combinations among the superscripts j, k and the subscript t. F is the $l_0\times\nu$ matrix with the columns $A(t){}^jy_t^k$, and E denotes the $T\times\nu$ matrix given by

$$E \triangleq \text{diag}\ (e_1', e_2', \cdots, e_T') \tag{10}$$

In (8) and (10), $e_t(t=1,2,\cdots,T)$ and e denote the vectors all the components of which are equal to unity.

Problem (P2) is called the master program. In the case of linear programming problems, the master program is completely equivalent to the original one[2]. In the present case, however, (P2) is not equivalent to (P1) because the set $\bar{Y}_t$ gives a larger region than the feasible region for the t-th block. In order to hold the equivalence between (P1) and (P2), we impose the following condition on the basic feasible solution of (P2):

C1. The extreme points of the t-th block, which form the basic feasible solution of (P2), must belong to the same polyhedron S_t^k.

Since ν is large, solving (P2) directly may be impossible. Therefore, we deal with the restricted master program (RMP, for short) whose variables are restricted to candidates for the basic variables in (P2).

4. Optimality Test

Assume that an initial basic feasible solution for the master program (P2) is available by choosing l_0+T points out of ν extreme points of all polyhedrons $S_t^k(k=1,2,\cdots,k_t;\ t=1,2,\cdots,T)$. Let π be the simplex multipliers associated with this basis. Partition π as

$$\pi = (\pi_0', \pi_1, \pi_2, \cdots, \pi_T)' \tag{11}$$

where the $l_0\times 1$ vector π_0 corresponds to the constraint (7) and the $T\times 1$ vector $(\pi_1,\pi_2,\cdots,\pi_T)'$ to (8). Then, the relative cost factor for the nonbasic variable ${}^j\mu_t^k$ is given by

$$ {}^j\bar{f}_t^k = [c(t)'-\pi_0'A(t)]^jy_t^k-\pi_t \tag{12}$$

If ${}^j\bar{f}_t^k\geqq 0$ for all nonbasic variables, no change in the nonbasic variables can cause the objective function z to decrease. Consequently, the current basic feasible solution is optimal. However, since $\bar{Y}_t$ is larger than the feasible region for the t-th block, the optimality test for the current solution need not be checked for all nonbasic variables corresponding to the extreme points in $\bar{Y}_t$. Thus, it follows that the condition mentioned above is sufficient for the optimality, but not necessary.

The procedure for checking the optimality test is as follows. Solve the following subproblem for each block:

(P3) $$\min_{x(t),y(t)} z_t = [c(t)'-\pi_0'A(t)]y(t) \qquad \text{subject to (2)}$$

Let the minimum objective value for the above be z_t^*. If

$$\bar{f}(t)\triangleq z_t^*-\pi_t\geqq 0 \qquad \text{for all } t \tag{13}$$

then the current basic feasible solution is optimal.

Note that (P3) is a mixed-integer problem of a smaller size than that of the original (P1).

5. The Procedure for the Case of Negative Relative Cost Factors

In this section we consider the case where, as the result of solving (P3), the relative cost factor $\bar{f}(t)$ is negative for some t. Let the solution of (P3) be $(x^{k*}(t), y^*(t))$, where $y^*(t)\in S_t^{k*}$. Assume that the current basic feasible solution of RMP belongs to S_t^k for the t-th block.

First, if $S_t^{k*}=S_t^k$, the solution $y^*(t)$ can enter the basis of RMP because the condition C1 is satisfied by doing so. Then, the procedure due to the Dantzig-Wolfe decomposition principle is permissible and consequently, the solution of RMP is necessarily improved.

Second, if $S_t^{k*}\neq S_t^k$, the solution $y^*(t)$ can not enter the basis of RMP keeping the current basic solution for the t-th block as the basis. Therefore, by replacing S_t^k by S_t^{k*} according to C1, a new RMP is constructed with variables corresponding to the extreme points obtained by the new combination of polyhedrons. In this case, the RMP does not necessarily have any improved solution, because the current basic solution is replaced.

Therefore, we must find out a new combination of polyhedrons, if it exists, which assures the improved solution of RMP. The search procedure for this purpose, however, may be tedious, if not impossible. We confine the search for improvement to the following restricted extent:

C2. If $\bar{f}(t)<0$ and $S_t^{k*} \neq S_t^k$ for some t, the exchange of S_t^k for S_t^{k*} is made for one block at a time.

The search procedure is continued in the following way. Assume that the polyhedron for the t^*-th block is replaced by $S_{t^*}^{k*}$. Then, solve the following linear programming problem:

(P4)
$$\min_{y(t)} \ z = \sum_{t=1}^{T} c(t)'y(t) \qquad \text{subject to (1) and}$$
$$\left.\begin{aligned} B(t)y(t) &\geqq d(t) - D(t)\hat{x}(t) \\ y(t) &\geqq 0 \end{aligned}\right\} \tag{14}$$
$$t = 1, 2, \cdots, T$$

In this problem the integer vectors of (P1) are fixed as follows:

$$\hat{x}(t) = \begin{cases} x^{k*}(t^*) & \text{for } t = t^* \\ \bar{x}(t) & \text{for } t \neq t^* \end{cases} \tag{15}$$

where $\bar{x}(t)$ is the integer solution associated with the current basic feasible solution of RMP.

The solution of (P4) is obtained by the Dantzig-Wolfe decomposition technique. If the solution of (P4) improves the current solution of RMP, proceed to the optimality test and check the solution of (P4) thus obtained. If not, the search procedure, i.e. solving (P4) is continued by replacing the polyhedron for another t with $\bar{f}(t)<0$.

We consider the strategy for choosing the number t^* for which block the polyhedron is replaced. The block numbers t_i with $\bar{f}(t_i)<0$ are listed, by arranging them in the increasing order of the relative cost factor, as

$$H \triangleq \{t_i \mid t_i \in \{1, 2, \cdots, T\}, \bar{f}(t_1) \leqq \bar{f}(t_2) \leqq \cdots < 0\} \tag{16}$$

Then the block number t^* should be chosen according to the order listed in H.

If the search procedure does not succeed even for the last block listed in H, return to the first block in H and solve (P3) again, excluding the values of $x(t)$ obtained so far by solving (P3). That is to say, solve the following mixed-integer problem:

$$\text{(P5)} \qquad \min_{x(t),y(t)} z_t = [c(t)' - \pi_0' A(t)] y(t) \qquad \text{subject to (2) and}$$
$$x(t) \neq x^k(t) \qquad k = k_1, k_2, \cdots, k_\alpha \tag{17}$$

where $x^k(t)$ ($k = k_1, k_2, \cdots, k_\alpha$) denote the integer solutions of (P3) which has failed so far to improve the solution of RMP.

If $\bar{f}(t^*) < 0$ holds for the solution of (P5), proceed to solving (P4) by substituting the integer solution thus obtained into the $x^{k*}(t^*)$ in (15). On the other hand, if $\bar{f}(t^*) \geqq 0$ holds for the solution of (P5), remove the number t^* from the list H.

The procedure mentioned above is terminated when the list H becomes empty. Since the number of the solutions of (P3) with $\bar{f}(t) < 0$ is finite, the procedure is completed in a finite number of iterations.

In the case of exchanging the polyhedrons based upon condition C2, the polyhedrons, or equivalently the basic feasible variables, for the block with $\bar{f}(t) \geqq 0$

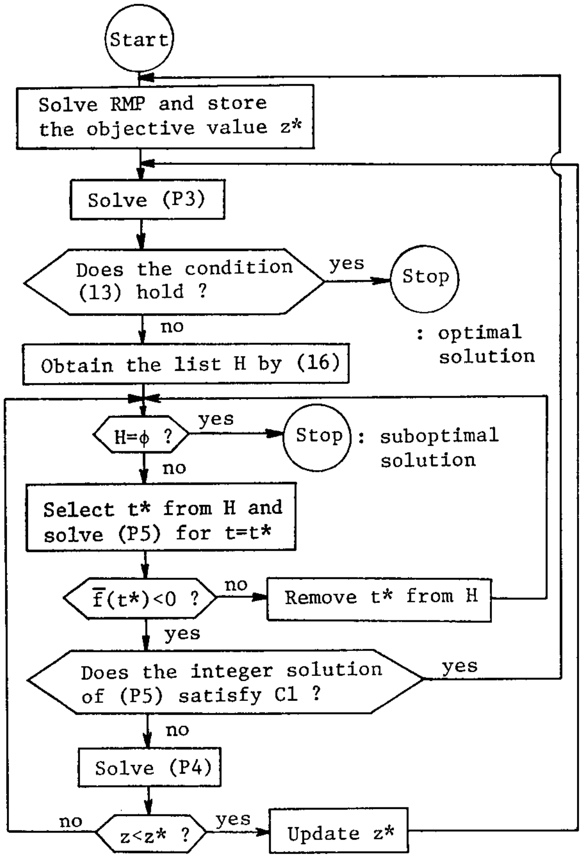

Fig. 1. Flowchart of the proposed algorithm

need not be replaced. Accordingly, the above algorithm gives a procedure examining all possibilities for improving the current basic solution of RMP by replacing a single polyhedron one by one. However, the simultaneous replacement of multiple polyhedrons is not considered in this procedure. Therefore, if the procedure terminates with $H=\phi$, the best solution obtained so far is called a suboptimal solution.

The entire procedure of the algorithm mentioned above is shown by the flowchart in Fig. 1. Since the subproblem (P3) corresponds to (P5) without the constraint (17), both subproblems are solved by the same procedure due to the branch and bound method.

6. Application to Optimal Planning of Blending Materials

As an example of problems formulated in (P1), we consider a problem of scheduling the use of raw materials in order to manufacture a certain product. The quantities of the supply and the requirement of the materials are given in the time period $[1, T]$. Assume that there are N kinds of materials with various qualities and different costs, to be substituted for one another. Then some of N materials are chosen and blended so that the resultant qualities may be acceptably close to the specific levels. There is a physical constraint on the number of equipments that can be handled at one period to blend the materials. Then, the goal of our planning is to determine the quantity of each material so as to produce the specific product with a minimum cost, under the constraints mentioned above.

The above problem may be written as follows:

(P6)
$$\min_{x_n(t),\, y_n(t)} z = \sum_n \sum_t c_n(t) y_n(t) \qquad \text{subject to}$$

$$u_n(t+1) = u_n(t) + \alpha_n(t) - y_n(t) \tag{18}$$

$$\sum_n y_n(t) = W(t) \tag{19}$$

$$q_i^1 W(t) \leqq \sum_n Q_{in} y_n(t) \leqq q_i^2 W(t) \quad i = 1, 2, \cdots, I \tag{20}$$

$$r_n^1 W(t) x_n(t) \leqq y_n(t) \leqq r_n^2 W(t) x_n(t) \tag{21}$$

$$\sum_n x_n(t) \leqq L \tag{22}$$

$$y_n(t) \geqq 0, \quad x_n(t) = 0 \text{ or } 1 \tag{23}$$

$$u_n(t) \geqq 0 \tag{24}$$

$$t = 1, 2, \cdots, T;\ n = 1, 2, \cdots, N$$

where

$u_n(t)$ = quantity of the n-th material in stock at the beginning of period t

$y_n(t)$ = quantity of the n-th material used in period t
$\alpha_n(t)$ = quantity of the n-th material supplied in period t
$c_n(t)$ = cost coefficient of the n-th material in period t
$W(t)$ = total amount of material required in period t
I = number of qualities of material
Q_{in} = level of the i-th quality for the n-th material
q_i^1 = lowest permissible level of the i-th quality
q_i^2 = highest permissible level of the i-th quality
r_n^1 = minimum permissible rate of use, imposed when the n-th material is used
r_n^2 = maximum permissible rate of use for the n-th material
L = maximum number of kinds of materials used at one period

Equations (19)–(23) give the constraint for each period, which is independent of those for other periods. On the other hand, we obtain from (18) and (24)

$$\sum_{\tau=1}^{t} y_n(\tau) \leqq u_n(1) + \sum_{\tau=1}^{t} \alpha_n(\tau) \triangleq h_n(t) \qquad (25)$$
$$t = 1, 2, \cdots, T;\ n = 1, 2, \cdots, N$$

Since $h_n(t)$ is a known quantity, the relation (25) gives a constraint interconnected with the whole period. By introducing the $N \times 1$ vectors as

$$\left.\begin{aligned} x(t) &\triangleq (x_1(t), x_2(t), \cdots, x_N(t))' \\ y(t) &\triangleq (y_1(t), y_2(t), \cdots, y_N(t))' \end{aligned}\right\} \qquad (26)$$

the problem mentioned above is rewritten in the form of (P1).

There have been some studies on the blending problems of raw materials[4,5]. Of these, Tabata[4] deals with an oil-blending problem with the constraint on the number of available pumps, and proposes an algorithm based upon the branch and bound method. However, his approach is confined to the planning for single time period. In that case, it may be difficult to treat the planning over a long time period.

In this paper, we investigate the optimal problem of blending raw coal in order to produce coke in an iron industry, with special regard to the dynamic planning over several periods of time.

7. Numerical Results

The numerical results are shown in order to make comparisons between the present algorithm and the conventional branch and bound method (BBM, for short). For simplicity, we treat the problem with $N=5$, though some forty kinds of raw coal are used practically. Three cases are considered for the time period; namely,

$T=2$, 3 and 4. In this case, the size of the problem is, corresponding to (P1), as follows:

$$m_t = n_t = 5, \ l_t = 24 \qquad \text{for all } t$$
$$l_0 = 5T \,(= 10, 15 \text{ and } 20)$$

Table 1 compares the size necessary for the present algorithm with that for BBM for various values of T. Table 2 summarizes the comparison of the computational results for fifteen problems prepared by giving the various values of T, L and $h_n(t)$.

Table 1. Comparison of the present method with BBM

(a) Number of constraints

T	The present method RMP	Subproblem	BBM
2	12	24	58
3	18	24	87
4	24	24	116

(b) Number of variables

T	The present method RMP	Subproblem	BBM
2	13	10	20
3	19	10	30
4	25	10	40

(c) Storage requirements (byte)

T	The present method	BBM
2	67584	30776
3	87308	68044
4	103144	119904

Referring to Table 2, both algorithms require nearly the same computing time for $T=2$. However, as T increases, the present algorithm tends to have less computing time than BBM. Further, the solution obtained as the suboptimal solution in the present algorithm seems to be optimal in practice, as compared with the result obtained by BBM.

8. Conclusion

A new algorithm has been developed for solving mixed-integer linear programming problems with an angular structure. The original decomposition tech-

Table 2. Comparison of the computational results

(a) $T=2$

Problem No.	1	2	3	4	5
The present method					
(i)	2.5900*	2.7973	2.5987	2.7000*	2.5724
(ii)	76	155	163	71	97
(iii)	6.32	13.07	22.44	6.35	17.78
BBM					
(i)	2.5900	2.7972	2.5987	2.7000	2.5724
(ii)	3	3	17	13	25
(iii)	3.26	3.13	16.68	12.66	24.58

(b) $T=3$

Problem No.	6	7	8	9	10
The present method					
(i)	4.1997*	4.2972	3.8192*	4.2000*	3.9631
(ii)	65	201	82	121	139
(iii)	5.48	19.76	20.84	9.30	13.94
BBM					
(i)	4.1999	4.2972	3.8192	4.1999	3.9630
(ii)	11	3	25	25	25
(iii)	41.54	10.01	66.38	80.09	80.62

(c) $T=4$

Problem No.	11	12	13	14	15
The present method					
(i)	5.3558*	5.1165	5.1974	5.3999*	5.4000*
(ii)	170	460	566	140	160
(iii)	19.04	43.03	18.54	23.11	38.81
BBM					
(i)	5.3557	5.1164	5.1974	5.3998	5.3998
(ii)	5	9	35	21	33
(iii)	56.25	118.32	443.26	276.04	447.85

Notes: (i) Objective function. The value with an asterisk indicates that the optimality condition (13) holds.
(ii) Number of linear programming problems actually solved.
(iii) Computing time in seconds.

nique due to Dantzig and Wolfe is confined to solving linear programs with the continuous variables alone. However, by modifying this technique, the present method can be applied to the mixed-integer problems given by (P1). In this paper, the admissible values of the integer variables are restricted to 0 and 1, as shown in (3). But the present method is also applied, without any difficulty, to the problem with bounded integer variables.

By examining illustrative examples, the present algorithm is expected to be efficient from the standpoint of computing time. This feature is demonstrated especially for the planning problem over many periods of time, or the problem with many integer variables. The strict procedure for the case of failing in the optimality test (13) is being investigated with applications to other case studies.

Acknowledgment

The author wishes to express his appreciation to Professor Nishikawa of Kyoto University for his helpful discussion, and to Mr. Inazu of Toyota Motor Corp. and Mr. Tsukabe of Kyoto University for their excellent cooperation. The author is likewise grateful to Mr. Tokuyama of Sumitomo Metal Industry for offering the data of the numerical examples.

All the numerical computations were made by FACOM M-190 at the Data Processing Center of Kyoto University.

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