Safety Analysis and Minimum-Weight Design of Rigid Frames Based on Reliability Concept

By

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Abstract

This paper deals with the safety analysis and the minimum-weight design of rigid frames based on reliability concepts. A safety analysis of redundant systems has difficult features, which are due to the fact that those systems generally have many various possible failure modes. As a rule, the approximations have been done on the assumptions of statistical independence and complete dependence among failure modes. However, the resulting solutions may be widely different as the number of failure modes increases. Also, usual structures have considerable dependence on their own failure mechanisms.

A simple formula presenting a good upper bound is derived herein by using the correlations between every two modes. The discussion is done with respect to the lower bound. Its improvement is attempted by introducing a simple mathematical model, which is used for estimating the probability that any three events occur simultaneously. This paper also outlines the minimum-weight design with reliability constraints. The proposed method is employed in the safety analysis. The results are discussed and compared with those of the deterministic method. For large systems, an approximate design method is proposed, which decomposes the possible failure modes into basic and non-basic modes.

1. Introduction

There remain many problems to be overcome in the application of reliability concepts to the design of real structures. The reliability analysis still requires some improvements, regardless of whether the classical theory or the extended theory is employed. Large and complex structures may fail in one of many possible mechanisms which are interrelated with each other. In addition to local failure phenomena, common failures may occur due to the initial yielding, the actual collapse and the instability of the structures. Indeterminate trusses can be considered to fail by the initial yielding. There are, however, many routes to reach the system collapse in indeterminate trusses, which while it is considerably

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difficult to trace all of them, some investigations were presented.[1)2)] While the actual collapse criterion is applied without difficulty, several mechanisms must be taken into consideration.

The limit analysis concept is employed here to discuss the safety of rigid frames. Also, only the failure due to the formation of plastic hinges is treated, so the failure through the loss of stability is not accounted for. Since the collapse mechanisms are inter-dependent, this effect is inevitable in the exact evaluation of reliability. Theoretically, the system reliability may be calculated by integrating the joint probability density function. But, actually, it is impossible to estimate this function, and even if it is found, the calculation requires a numerical integration which is prohibitive and time-consuming. Therefore, the approximations have been used on the assumptions of statistically complete dependence and probabilistic independence among modes.[3)] These assumptions give the lower and upper bounds, respectively. However, the difference between the upper and lower bounds may become large as the number of failure modes increases.

In this paper, a simple method presenting a good upper bound is proposed by calling attention to the events that two modes occur simultaneously. A stochastic dependence between modes is introduced by considering the coefficient of the correlation between each modal margin. Then, in order to remove the integration process, the approximate method developed by M. Tichy and M. Vorlicek is used in the actual calculation.[4)] Since the relations used here are purely mathematical, the formula can be applied for the safety analysis of systems possessing different kinds of failure modes.

From the view of safety design, only the value of the upper bound is necessary, but the lower bound is important to check the availability of the approximation. There is a brief discussion with regard to the lower bound. Furthermore, an improvement is attempted based on the fact that the simultaneuos occurrence of three modes is considerably affected and limited by the occurrence of two of these modes. A simple mathematical model is developed to evaluate the effects of those events on the average.

The minimum-weight design of rigid frames is also outlined in the last section. The proposed method is discussed and compared with the method developed by J. Stevenson and F. Moses.[5)6)] Furthermore, we propose an approximate technique for large structures, in which the possible failure modes are partitioned into two groups, that is, basic modes and non-basic modes. This can clarify the contribution of each mode to the total collapse and make the calculation relatively inexpensive.

2. Reliability of Structural System[7)8)9)10)]

Supposing that a structural system has n failure modes, the probability of a system failure, p_f, can be written in the terms of the modal failure events F_i as follows:

$$p_f = P_r(F_1 \cup F_2 \cup \cdots \cup F_n) \tag{1}$$

where the symbol "$\cup$" signifies the union of the events. The event F_i occurs when the modal resistance R_i is less than the modal load effect S_i.

$$F_i = (R_i - S_i < 0) \tag{2}$$

Taking the collapse mechanism as a typical failure mode, the failure event F_i is written by introducing the reserve strength Z_i in the following:

$$F_i = (Z_i < 0) \tag{3}$$

where

$$Z_i = \sum_{k=1}^{n} A_{ik} M_k - \sum_{j=1}^{m} B_{ij} S_j \tag{4}$$

M_k : structural resistance of a structural member at the k-th point in the structure

S_j : effect of the j-th load on the structure

A_{ik} : resistance coefficient determined by the position and condition of the k-th point related to the i-th failure

B_{ij} : load coefficient determined by the position and magnitude of the j-th load on the structure related to the i-th failure mode

The beam failure mode of a single-bent frame shown in Fig. 1 is

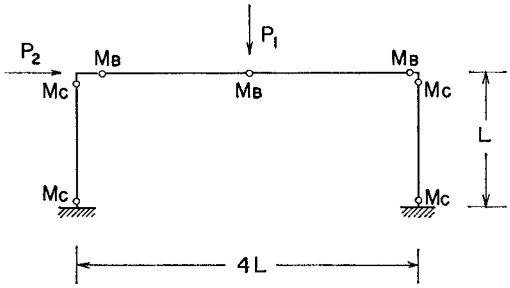
$$Z_1 = M_B + 2M_B + M_B - 4P_1L/2 \tag{5}$$

If the form of Eq. 4 is used,

$$Z_1 = \sum_{k=1}^{7} A_{1k} M_k - \sum_{j=1}^{2} B_{1j} S_j \tag{6}$$

where

$$\begin{aligned} M_1 &= M_2 = M_6 = M_7 = M_c \\ M_3 &= M_4 = M_5 = M_B \\ A_{11} &= A_{12} = A_{16} = A_{17} = 0 \\ A_{13} &= A_{15} = 1 \\ A_{14} &= 2 \\ B_{11} &= 0 \\ B_{12} &= 2L \\ S_1 &= P_2 \\ S_2 &= P_1 \end{aligned} \tag{7}$$



FAILURE MODES

$$Z_1 = M_B + 2M_B + M_B - 4P_1 L/2$$
$$Z_2 = M_C + 2M_B + M_C - 4P_1 L/2$$
$$Z_3 = M_B + 2M_B + M_C - 4P_1 L/2$$
$$Z_4 = M_C + M_C + M_C + M_C - P_2 L$$
$$Z_5 = M_C + M_B + M_B + M_C - P_2 L$$
$$Z_6 = M_C + M_C + M_B + M_C - P_2 L$$
$$Z_7 = M_C + 2M_B + 2M_C + M_C - P_2 L - 4P_1 L/2$$
$$Z_8 = M_C + 2M_B + 2M_B + M_C - P_2 L - 4P_1 L/2$$

Fig. 1. Single Bent Frame.

Then, the failure probability of the i-th mode, p_{fi}, is given as

$$p_{fi} = P_r(Z_i < 0) \tag{8}$$

By paying attention to the n-th event, Eq. 1 can be expanded to be

$$P_r(F_1 \cup F_2 \cdots \cup F_n) = P_r(F_1 \cup \cdots \cup F_{n-1}) + P_r(F_n) - P_r((F_1 \cup \cdots \cup F_{n-1}) \cap F_n) \tag{9}$$

where the symbol "$\cap$" signifies the intersection of the events. The last term on the right side in Eq. 9 is lowly limited by the probability of the occurrence of two events.

$$P_r((F_1 \cup \cdots \cup F_{n-1}) \cap F_n) \geq P_r(F_k \cap F_n) \tag{10}$$

where F_k is one of the events F_i $(i=1, \cdots, n-1)$
Thus, the upper bound of p_f can be reduced, that is,

$$p_f \leq P_r(F_1 \cup \cdots \cup F_{n-1}) + P_r(F_n) - P_r(F_k \cap F_n) \tag{11}$$

To obtain the closest upper bound, one should select the value which maximizes the last term of inequality 11. When the expansion is continued, a simple formula presenting an upper bound can be obtained.

$$p_f \simeq \sum_{i=1}^{n} P_r(F_i) - \sum_{i=1}^{n-1} \underset{j=1}{\overset{i}{\mathrm{Max}}} \, (P_r(F_j \cap F_{i+1})) \tag{12}$$

Here, assuming a statistically complete dependence between each mode, the failure probability becomes as follows:

$$p_f = P_r(F_{iw}) \tag{13}$$

where F_{iw} is the weakest mode.
On the other hand, assuming a probabilistic independence, it can be expressed as

$$p_f = \sum_{i=1}^{n} P_r(F_i) \tag{14}$$

It is apparent that Eq. 12 givse a better upper bound than Eq. 14, because the value calculated by Eq. 12 is less than that by Eq. 14 by the value of its second term.

When the failure modes do not have a strong dependence, Eq. 13 gives a value very far from the true solution. For such a case, the following equation may give a better result.

$$p_f \geq \sum_{i=1}^{n} P_r(F_i) - \sum_{0<i<j<n} P_r(F_i \cap F_j) \tag{15}$$

3. An Improvement on Evaluation of System Reliability[11]

In the preceding section, the simple method is presented, which requires the probabilities that two events occur simultaneously. These probabilities can be obtained by introducing the inter-dependence between two modes through the coefficients of correlation. Using the approximate method developed by M. Tichy and M. Vorlicek,[4] they can be calculated without integration. When $P_r(F_i)$ is less than $P_r(F_j)$, the probability of the occurrence of two events, F_i and F_j, is expressed as

$$P_r(F_i \cap F_j) = P_r(F_j)\{P_r(F_i) + r_{F_iF_j}^{\varphi_j+2}(1-P_r(F_i))\} \tag{16}$$

where $r_{F_iF_j}$: the coefficient of correlation between F_i and F_j

$$\varphi_j = -\log_{10} P_r(F_j) \tag{17}$$

While the implementation of Eq. 12 needs only the failure probability of each mode and the coefficient of correlation, it may give a too conservative value for a case where the failure modes are considerably dependent. To examine the accuracy of this method, an improvement on the lower bound is considered in this section.
Eq. 1 can be rewritten in the terms of the intersections of the events as follows:

$$P_r(F_1 \cup \cdots \cup F_n) = \sum_{i=1}^{n} P_r(F_i) - \sum_{0<i<j<n} P_r(F_i \cap F_j) + \sum_{0<i<j<k<n} P_r(F_i \cap F_j \cap F_k) - \cdots + P_r(F_1 \cap F_2 \cap \cdots \cap F_n) \tag{18}$$

It is evident that in Eq. 18 any term has a greater value than those of the successive terms. Also, the former is influential on the latter. Generally, higher terms are truncated in the calculation, assuming that those have small values. However, this assumption is not true for some cases, and the case treated here may be one of the exceptions. Therefore, we attempt to take additionally the third term of Eq. 18 into account, in order to improve the approximation. In order to obtain this probability analytically, a simple mathematical model is proposed so as to estimate the probability of those events easily.

Generally, the failure probability of the *i*-th mode is calculated as the negative region shown in Fig. 2. Then, the circle with the same area as this region is transferred to express the failure event F_i and a similar treatment is used for the remainders, F_j and F_k. The probability of the occurrence of two events is given as the intersected region of two circles, as shown in Fig. 3. Also, the event that three modes occur at the same time can be specified as the shading part of Fig. 4. This procedure is summarized as follows:

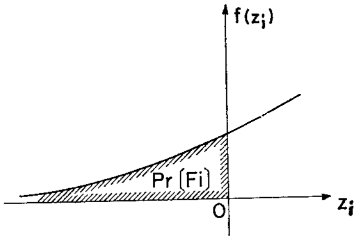


Fig. 2. Failure Probability of the *i*-th Mode.

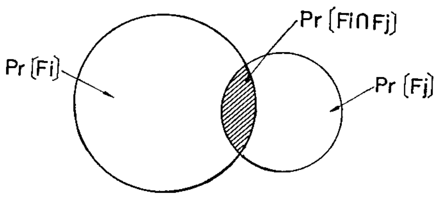
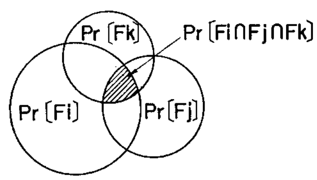


Fig. 3. Probability of Two-Events-Occurrence



Fig, 4. Expression of $P_r(F_i \cap F_j \cap F_k)$.

Step 1 Specify the three circles, where their radii are determined by corresponding their areas to their failure probabilities.

Step 2 The distances between each two circles can be obtained by making the overlapped section areas equivalent to those of the two-events-occurrence probabilities.

Step 3 The three-events-occurrence probability can be calculated as the area of the induced curve-linear triangle.

It should be noted that the actual failure events can never be exactly expressed by that simple figure, which may have a complicated boundary.

The two-dimensional model employed here may be insufficient to express some special cases. Nevertheless, it can be considered to give the inter-dependence between three modes on the average, though it may include little physical meaning.

The system failure probability, p_f, is calculated by using the first to the third terms of Eq. 18.

$$p_f \leq \sum_{i=1}^{n} P_r(F_i) - \sum_{0<i<j<n} P_r(F_i \cap F_j) + \sum_{0<i<j<k<n} P_r(F_i \cap F_j \cap F_k) \tag{19}$$

While this model is developed to improve the lower bound, the used equation will give an upper bound. It is natural that this approximation presents a conservative value when the model corresponds well to the actual event and the calculation is performed with a good accuracy. The obtained value will approach the true solution more closely, whether it is conservative or not.

4. The Relation between Proposed Method and Ordering Method

F. Moses and D. Kinser[12)13)] proposed the analysis method called the ordering method for estimating the over-all failure probability. By using the relationship of mutually exclusive events, p_f can be wirtten as

$$p_f = P_r(F_1) + P_r(F_2 \cap \bar{F}_1) + P_r(F_3 \cap \bar{F}_1 \cap \bar{F}_2) + \cdots + P_r(F_n \cap \bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_{n-1}) \tag{20}$$

While this equation covers all possible failure events of the entire system, it requires numerous computations. Its implementation can be performed only when the correlations among all modes are found. It is apparent that the following relation exists.

$$P_r(F_n \cap \bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_{n-1}) \leq P_r(F_n \cap \bar{F}_k) \tag{21}$$

where k is one of numbers 1 through $n-1$.

If k is selected as having the minimum value of $P_r(\bar{F}_i \cap F_n)$ $(i=1, \cdots, n-1)$, that is,

$$P_r(\bar{F}_k \cap F_n) = \operatorname*{Min}_{i=1}^{n-1} P_r(\bar{F}_i \cap F_n) \tag{22}$$

the difference between the left hand and the right hand of inequality 21 will be the smallest. Now, the probability, $\operatorname*{Min}_{i=1}^{n-1} P_r(\bar{F}_i \cap F_n)$ can be rewritten by use of the failure event F_i as follows.

$$\operatorname*{Min}_{i=1}^{n-1} P_r(\bar{F}_i \cap F_n) = \operatorname*{Min}_{i=1}^{n-1} \{P_r(F_n) - P_r(F_i \cap F_n)\} \tag{23}$$

Since $P_r(F_n)$ is constant, the minimum value is obtained when $P_r(F_i \cap F_n)$ indicates the maximum value.

$$\operatorname*{Min}_{i=1}^{n-1} P_r(F_i \cap F_n) = P_r(F_n) - \operatorname*{Max}_{i=1}^{n-1} P_r(F_i \cap F_n) \tag{24}$$

Using Eq. 24, Eq. 20 can be approximately expressed as follows:

$$\begin{aligned} p_f &\simeq P_r(F_1) + (P_r(F_2) - P_r(F_2 \cap F_1)) + (P_r(F_3) \\ &\qquad - \operatorname*{Max}_{i=1}^{2} P_r(F_i \cap F_2)) + \cdots + (P_r(F_n) - \operatorname*{Max}_{i=1}^{n-1} P_r(F_i \cap F_n)) \\ &= \sum_{i=1}^{n} P_r(F_i) - \sum_{i=1}^{n-1} \operatorname{Max}\,(P_r(F_1 \cap F_{i+1}), \cdots, P_r(F_i \cap F_{i+1})) \end{aligned} \tag{25}$$

Thus, the relationship between the proposed method and the ordering method is clarified. Namely, they are equivalent, if the ordering method accepts the approximation as shown in Eq. 21.

5. Minimum-Weight Design with Reliability Constraints[14)]

In this section, the minimum-weight design of rigid frames is outlined, based on the reliability concepts. As F. Moses[12)] and others mentioned, the minimization with a probability level moves the design away from having many active individual constraints. However, it is naturally anticipated that the safety analysis process will have more complicated and difficult features. The minimum-weight design procedure is formulated as follows:

$$\begin{aligned} &\text{Minimize} \qquad W = K \sum_{i=1}^{m} A_i L_i \\ &\text{subject to} \qquad p_f \leq p_{fa} \end{aligned} \tag{26}$$

where A_i : the cross sectional area of the i-th member

L_i : the length of the i-th member

K : constant m : number of members

W : total weight

p_{fa}: allowable failure probability

Then, according to Ridha's work[16)], the cross sectional area can be expressed by the full plastic moment.

$$A_i = \frac{1}{3}\left(\frac{36}{S_{yi}}\right)^{2/3} M_i^{2/3} \tag{27}$$

where S_{yi} : the yielding stress of the i-th member

M_i: the full plastic moment of the i-th member

Taking M_i as the design variable, this design problem is reduced to a nonlinear programming problem. The objective function is calculated from Eqs 26 and 27, and the over-all failure probability p_f is obtained from Eq. 12.

There is, however, a point to be careful about in the use of Eq. 12 for the minimum-weight design. While the cross sectional areas of members are prescribed in the analysis, they are variables and are to be specified in the design procedure. Through the optimization, the dominant failure mode may move from one to others. Then, the ordering of possible modes is influential on the accuracy of Eq. 12. Its second terms (two-events-occurrence probabilities) are found to be the maximum value among those probabilities which are related to a failure mode. In this paper, the failure modes are successively ordered from those with the largest failure probability. Since this process is employed in each design stage, it will require some time for computing the data.

6. Approximate Design Method Using Decomposition Technique

As mentioned above, the minimum-weight design of rigid frames is reduced to a mathematical programming problem. Then, using Eq. 12 for the safety analysis, it is easily performed with the aid of an appropriate mathematical programming technique. However, there still remain some problems in its direct application to large structural systems. As the number of the failure modes increases, the implementation of Eq. 12 will consume more time, especially in the calculation of the covariance matrices. This procedure, of course, does not require so many runnings different from the usual deterministic analysis. However, since this procedure appears many times in the design process, the total computation time will become enormous.

In this paper, an approximation based on the decomposition concepts is introduced to remove the problem concerned with computation. It is not likely that all collapse modes will be critical for the deterministic design. While the critical modes are active as constraints, the remainders are not and they don't contribute to the resulting solution. If the active modes whose numbers are the same as those of the design variables are taken as the constraints, the optimum set of values on weight and design variables can be obtained. This can be understood by corresponding the active modes to the "basis" of the coordinates of the design space.

Here, paying attention to this fact, the possible modes are divided into two groups: a basic group and a non-basic group of modes, which correspond to the groups of active modes and non-active modes. It is, however, to be noted that this explicit classification of modes is hardly allowed in the probabilistic design. This is because all modes contribute to the evaluation of a system failure probability, even if some of them are very small. Furthermore, since the probabilistic design has only one constraint, a sufficient number of modes is not explicitly determined. Therefore, the basic and non-basic modes can be expressed only by the terms of "dominant" and "indominant", respectively.

The basic modes are possibly selected in the following way. If the modes, which have the larger probability and are less correlated, are chosen to be part of the basic modes, the resulting solution may show a good agreement with the exact one. Then, the failure probability of the entire system is calculated by neglecting the effects of the non-basic modes.

$$p_f \simeq P_r(F_b) \tag{28}$$

where F_b denotes the failure event induced by at least one of the basic modes.

It should be noted that some treatment is necessary to use Eq. 28 for the design process, because it underestimates the failure probability. Assuming that the non-basic modes are independent of each other and uncorrelated to the basic modes, p_f can be expressed as

$$p_f = P_r(F_b) + \sum_{i=q+1}^{n} P_r(F_i) \tag{29}$$

where q and n denote the numbers of the basic modes and all modes, respectively. While the use of this equation removes the problem of underestimation, it still involves the problem of the execution time for the case of complex structures having a large number of non-basic modes. For such cases, it is a considerably prohibitive task to calculate the failure probabilities of all the non-basic modes. Then, it is to be desired that safety is guaranteed without using the non-basic modes in the design process.

E. Vanmarcke proposed an iterative design scheme for the design of large systems. At first, the design is performed by solving a relatively simple auxiliary problem in which only a set of basic modes are considered in computing system failure probabilities. Next, by changing the set of basic failure modes, new designs are successively generated. At every step, the upper bounds on the objective function are followed by solving the formulation:

$$\begin{array}{ll} \text{Minimize} & \text{objective function} \quad W \\ \text{subject to} & p_f \leq p_{fa} - p_f' \end{array} \tag{30}$$

where p_f' can be found by subtracting $P_r(F_b)$ from the system failure probability which was obtained by using the design variables of the earlier stage. This method is very useful, because it not only saves the core size and the execution time but also easily gives good solutions which satisfy the design requirement. However, it seems to have problems in convergency and applicability such as how many modes are appropriate for the basic modes, and when or by what the iteration should be terminated. If the structural system to be designed has more dominant modes than the employed basic modes, the resulting upper and lower bounds will not converge, and the modified allowable failure level may become negative for some special cases.

In this paper, an iterative design method is proposed, based on Vanmarcke's method. An improvement of convergency is attempted. The number of basic modes is not fixed and new dominant modes are successively added as the design stage proceeds. The procedure is summarized as follows: (See Fig. 5)

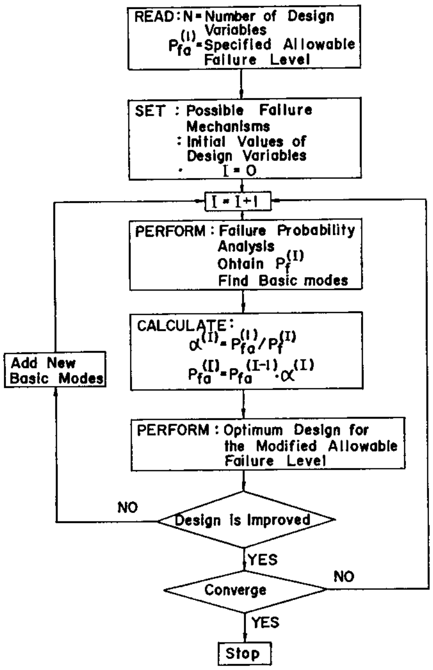


Fig. 5. Flow-Chart: Iterative Design Scheme.

1) Construct the possible failure mechanisms. Specify the number of basic modes as that of the independent design variables. Give the allowable failure probability level. Assume the initial values for the design variables.

2) Perform the failure probability analysis for the current set of design variables. Obtain the system failure probability and find the basic modes.
3) Solve the problem:

$$\text{Minimize } W \text{ (scaled weight)} = \sum_{i=1}^{n} M_i L_i \qquad \text{subject to } P_r(F_b^{(I)}) \leq p_{fa}^{(I)} \tag{31}$$

where $p_{fa}^{(I)}$ is the modified allowable failure level. This value is obtained by using the modification factor $\alpha^{(I)}$.

where
$$p_{fa}^{(I)} = p_{fa}^{(1)} \cdot \alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(I-1)} \tag{32}$$
$$\alpha^{(I)} = p_{fa}^{(1)} / P_r(F_b^{(I)}) \tag{33}$$

Introducing the modification factor $\alpha^{(I)}$, the allowable failure level is forced to converge to an appropriate value.
4) If the improved design is not generated after two cycles, new basic modes are added for the next cycle. That number is found in the subsequent analysis by comparing the basic modes with those of the previous stage. If the design is improved, the basic modes are exchanged, but the number is unchanged.
5) Upon proceeding to the iterative process, the final design is achieved when the basic modes are not changed, or the difference between the predetermined failure level and the calculated failure probability is sufficiently small.

7. Numerical Results

To illustrate the validity or the accuracy of the analysis and design methods proposed in this paper, some test examples are presented. In all examples, the loads and resistances are assumed to be normally distributed, because of the easiness of the treatment and the security of the central limit theorem. At first, the approximation method proposed by M. Tichy and M. Vorlicek[4] is examined for some cases in which the ratio of $P_r(F_i)$ and $P_r(F_j)$ varies. The results are shown in Fig. 6 (a)$\sim$(d), in which the abscissa is the coefficient of correlation. From these figures, this method seems to underestimate the joint probability density function when the probabilities of the i-th and j-th failure events have the same order, or their coefficient of correlation is greater than 0.8. For other cases, this method presents conservative values which are fairly close to the true solutions.

Example 1 Reliability Analysis of Portal Frame

This simple model is used to explain the proposed methods in detail. The load condition and failure modes are illustrated in Fig. 1. The applied loads P_1 and P_2 are considered to be dependent and to have the mean values and the coefficient of variation presented in Table 1.

Table 1. Reliability Analysis of Single Bent Frame

a) Condition of Input Data

Mean Value of Moment Resistance (K-FT)	Mean Value of Load		Coefficient of Variation	
	P_1 (K)	P_2 (K)	Resist.	Load
40	0.5	1.0	0.2	0.2

b) Calculated Probabilities of Failure

Dependent	Lower Bound (Eq 19)	Simulation	Stevenson's M.	Upper Bound (Eq 12)	Independent
$\times 10^{-2}$ 0.808	$\times 10^{-2}$ 0.920	$\times 10^{-2}$ 0.958	$\times 10^{-2}$ 0.886	$\times 10^{-1}$ 0.100	$\times 10^{-1}$ 0.150
	0.8 sec.	5×10^4 trials	1.9 sec.	0.8 sec.	

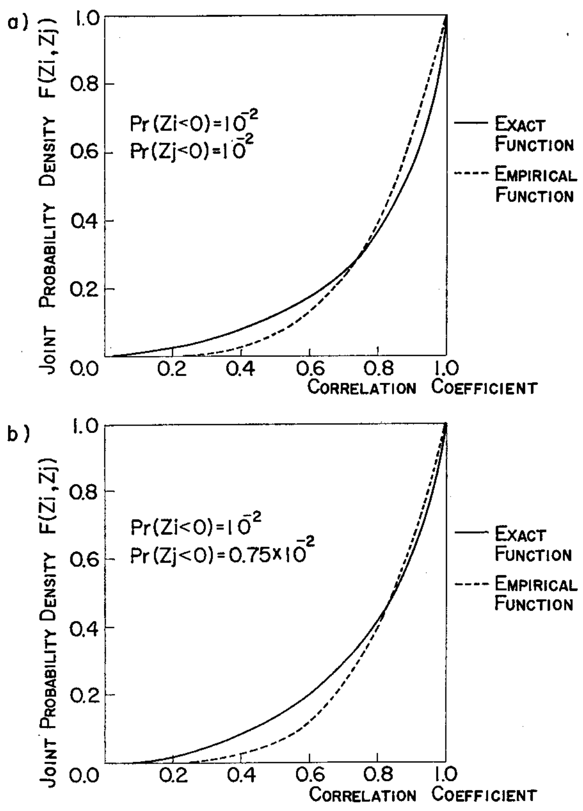


Fig. 6

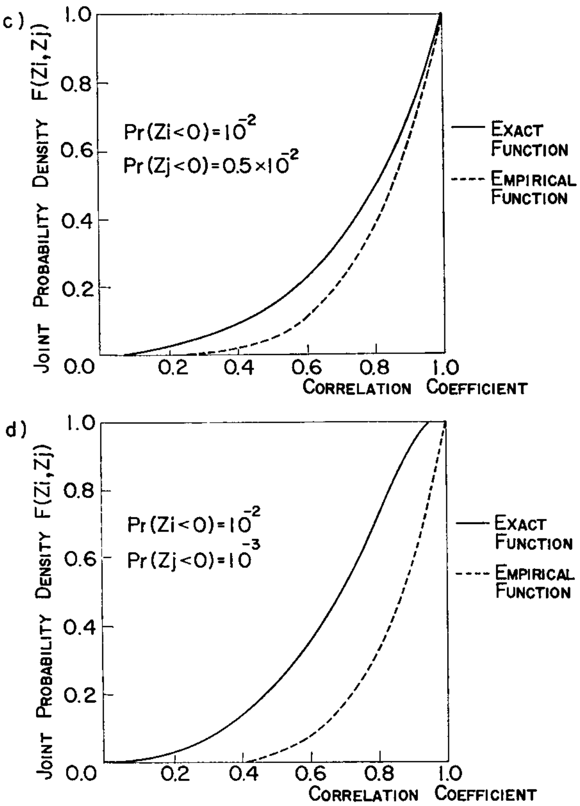


Fig. 6. Comparison of Empirical and Exact Function. (Tichy's Method and Exact Solutions)

Assuming the statistical independence and the complete dependence, the resulting band is considerably wide, and the proposed method (Eq. 12) gives an upper bound which is closer to the true solution than Eq. 14. By using a circle model described in section 3, Eq. 19 presents a good lower bound. While the method by J. Stevenson shows a good agreement, it may lead to unconservative solutions, and also it requires twice the execution time of Eq. 12 in computing. (See Table 1)

Example 2 Reliability Analysis of One-Bay Two-Story Frame

This model, whose geometry and applied load are shown in Fig. 7, is considered to have many dominant failure modes which are considerably inter-dependent on each other. Then, the assumption of statistical independence will give a too conservative value. By taking the correlation of two failure events into account, the proposed method improves the upper bound effectively. (See Table 2 and Table 3)

Table 2. Reliability Analysis of Two-Story Single-Bay Frame (I)

a) Condition of Input Data

Case No.	Mean Value of Moment Resistance	Mean Value of Load		Coefficient of Variation		P_1, P_2
		P_1	P_2	Moment Resistance	Load	
1	40	1.0	0.5	0.2	0.2	Dependent
2	40	0.8	0.4	0.2	0.2	Dependent
3	40	0.6	0.3	0.2	0.2	Dependent

b) Calculated Probabilities of Failure

Case No.	Dependent	Independent	Simulation		J. Stevenson	Proposed Method
			Limit Analysis	Elastic Analysis		
1	0.941×10^{-1}	0.462	0.118	0.722	0.113	0.157* 0.132
			10^4 trials	10^4 trials	7.8 sec	2.5 sec
2	0.841×10^{-2}	0.369×10^{-1}	0.131×10^{-1}	0.414	0.112×10^{-1}	0.160×10^{-1} 0.147×10^{-1}
			10^4 trials	10^4 trials	7.2 sec	2.5 sec
3	0.271×10^{-3}	0.831×10^{-3}	0.640×10^{-3}	0.109	0.587×10^{-3}	0.710×10^{-3} 0.649×10^{-3}
			5×10^5 trials	10^4 trials	5.7 sec	2.5 sec

Proposed Method
* Upper Value, by Tichys' Method
Lower Value, by Numerical Integration

Table 3. Reliability Analysis of Two-Story Single-Bay Frame (II)

a) Condition of Input Data

No.	Mean Value of Moment Resistance	Mean Value of Load		Coefficient of Variation		P_1, P_2
		P_1 (K)	P_2 (K)	Moment Resistance	Load	
1	40(K-FT)	1.0	0.5	0.1	0.1	Dependent
2	40	0.8	0.4	0.1	0.1	Dependent

b) Calculated Probabilities of Failure

No.	Dependent	Independent	Simulation		J. Stevenson	Proposed Method
			Limit Analysis	Elastic Analysis		
1	0.425×10^{-2}	0.969×10^{-2}	0.504×10^{-2}	0.776	0.480×10^{-6}	0.515×10^{-6}
			5×10^4 trials	10^4 trials	4.8 sec	2.5 sec
2	0.864×10^{-6}	0.113×10^{-5}		0.199	0.938×10^{-6}	0.961×10^{-6}
				10 trials^4	3.2 sec	2.5 sec

Table 4. Reliability Analysis of Two-Story Single-Bay Frame (III)

Mean Value of Moment Resistance	Dependent	Lower Bound by Eq. (19)	Proposed M. (Lower Bound)	Simulation	Proposed M. (Upper Bound)	Independent
60	0.355×10^{-2}	0.947×10^{-2}	0.112×10^{-1}	0.121×10^{-1} (4.61×0^{4} tri)	0.138×10^{-1}	0.170×10^{-1}
70	0.123×10^{-2}	0.375×10^{-2}	0.375×10^{-2}	0.380×10^{-2} (4.6×10^{4} tri)	0.400×10^{-2}	0.407×10^{-2}

Mean Value of Load: $P_1=1.0$(K) $P=0.5$(K)

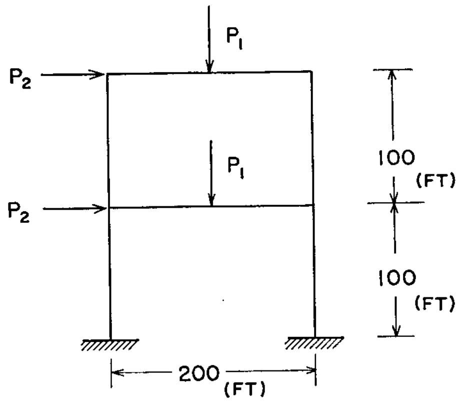


Fig. 7. Two-Story Single-Bay Rigid Frame.

The failure probability decreases and approaches the result obtained by assuming the statistical independence, as the mean value of the load decreases. This phenomena can be explained by the fact that the coefficient of correlation between each mode becomes smaller according to the decrease of the load. However, this tendency accompanied with the decrease of failure probability is not observed when the coefficients of variation of the load and resistance decrease. The approximate solution by the method of M. Tichy and M. Vorlicek is compared with the exact solution for this model to show that this method always presents conservative values. Its application can be considered useful for ordinal structures, because it can save the execution time by the elimination of numerical integration. Next, the failure probabilities are calculated for those cases in which the moment resistances have 60 and 70 (K-FT) as mean values. Then, the numerical results are summarized in Table 4, including the results of the improving method proposed in section 3.

This method is considered to improve the lower bound. In the calculation process, the curve-linear triangles, which correspond to three-events occurrence probabilities, are approximated by the linear triangles, in order to save the execution time and make the programming simple. In spite of this approximation, this method gives lower bounds very close to the solutions obtained by the Monte Carlo simulation. However, this method seems to have the possibility of presenting greater values for cases where the difference between the curve-linear triangle and the linear triangle is small, or the estimating method for the two-events-occurrence probability presents values that are too conservative. Nevertheless, these results show good agreement with the true solutions.

Example 3 Reliability Analysis of Two-Bay One-Story Frame

In this model, the beam mechanism can be considered to be the most dominant mode, and to have less correlation among the modes. (See Fig. 8) Table 5

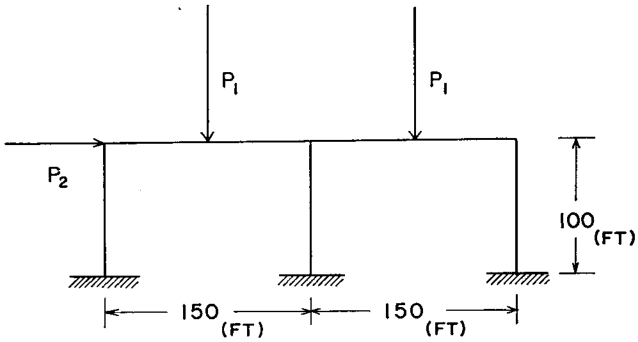


Fig. 8. One-Story Two-Bay Rigid Frame.

Table 5. Reliability Analysis of One-Story Two-Bay Frame

a) Condition of Input Data

No.	Mean Value of Moment Resistance	Mean Value of Load		Coefficient of Variation		P_1, P_2
		P_1 (K)	P_2 (K)	Moment Resistance	Load	
1	40(K-FT)	1.0	0.5	0.2	0.2	Dependent
2	40	1.0	0.5	0.2	0.2	Independent

b) Calculated Probabilities of Failure

No.	Dependent	Independent	Simulation		J. Stevenson	Proposed Method
			Limit Analysis	Elastic Analysis		
1	0.808×10^{-1}	0.200×10^{-1}	0.158×10^{-1}	0.736×10^{-1}	0.162×10^{-1}	0.169×10^{-1}
			5×10^4 trials	5×10^4 trials	4.2 sec.	1.8 sec.
2	0.808×10^{-2}	0.199×10^{-1}	0.158×10^{-1}	0.163×10^{-1}	0.163×10^{-1}	0.169×10^{-1}
			0.610×10^{-1}	5×10^4 trials	3.6 sec.	1.8 sec.

shows that all solutions are almost equal, excepting those obtained by assuming complete dependence or independence. Considering the matter of less calculation time, the proposed method is also useful for this case, though Stevenson's method presents the closest value to the true solution.

Example 4 Minimum-Weight Design of Portal Frame

By using the proposed analysis method, a simple portal frame, shown in Fig. 1, is designed to minimize its total weight or its cost with some specified allowable failure probability levels. The obtained results are shown in Tables 6 and 7, and compared with the results obtained by J. Stevenson as well as the results based on the assumption of the independence of the failure modes. The probability of

Table 6. Input Data for Single Bent Frame Optimum Design

L=10 FT

No.	Start (K-FT)		Load (K)		Coefficient of Variation				Prob. of Failure
	M_C	M_B	P_1	P_2	M_C	M_B	P_1	P_2	to be allowed
1	220	260	30	20	0.10	0.10	0.10	0.10	0.08
2	320	360	30	20	0.10	0.10	0.10	0.10	0.008
3	320	360	30	20	0.10	0.10	0.10	0.10	0.0008
4	380	420	30	20	0.10	0.10	0.10	0.10	0.00008
5	320	360	30	20	0.10	0.10	0.05	0.05	0.0008
6	320	360	30	20	0.10	0.10	0.15	0.15	0.0008
7	320	360	30	20	0.10	0.10	0.20	0.20	0.0008

each failure mode is affected by its mean value and variance. For example, the failure modes, Z_1 and Z_2, have nearly equal central safety factors, but their failure probabilities have a relatively large difference. This is due to the difference of their variances. Then, one will reach the conclusion that the number of independent design variables, which are included in the failure modes, is very important in the probabilistic design. Therefore, the failure modes which are considered to be dominant in the deterministic design do not always become dominant from the probabilistic point of view.

Table 8 indicates the failure probabilities calculated by the proposed method and Stevenson's method. The proposed method gives a greater value than Stevenson's method. The reasons are that Stevenson's method does not have the security of safety, and the assumption adopted in his work is not adequate, except for the case where the effect of variance is sufficiently small compared with the difference of the mean values.

Table 7. Numerical Results of Single Bent Frame Example

a) Optimum Solutions

Case No.	Final (K-FT)		Object Function	Prob. of Failure	Prob. of Failure (Independent)
	M_G	M_B			
1	171.12	188.01	197.41	0.888×10^{-1}	0.163
2	178.95	219.14	213.61	0.800×10^{-2}	0.160×10^{-1}
3	195.06	241.68	227.66	0.798×10^{-3}	0.149×10^{-2}
4	190.51	247.72	240.07	0.798×10^{-5}	0.124×10^{-3}
5	168.98	229.61	215.99	0.800×10^{-3}	0.142×10^{-2}
6	208.63	266.66	241.36	0.800×10^{-3}	0.151×10^{-2}
7	238.48	285.19	256.05	0.798×10^{-3}	0.157×10^{-2}

b) Failure Probabilities of Individual Modes

Case No.	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
1	0.573×10^{-2}	0.572×10^{-1}	0.543×10^{-1}	0.0	0.0	0.0	0.218×10^{-2}	0.183×10^{-2}
2	0.463×10^{-2}	0.653×10^{-2}	0.472×10^{-2}	0.0	0.0	0.0	0.115×10^{-3}	0.479×10^{-4}
3	0.635×10^{-3}	0.416×10^{-3}	0.434×10^{-3}	0.0	0.0	0.0	0.207×10^{-5}	0.137×10^{-5}
4	0.339×10^{-4}	0.577×10^{-4}	0.319×10^{-4}	0.0	0.0	0.0	0.358×10^{-6}	0.745×10^{-7}
5	0.499×10^{-3}	0.497×10^{-3}	0.413×10^{-3}	0.0	0.0	0.0	0.720×10^{-5}	0.256×10^{-5}
6	0.413×10^{-3}	0.669×10^{-4}	0.425×10^{-3}	0.0	0.0	0.0	0.459×10^{-5}	0.145×10^{-5}
7	0.547×10^{-3}	0.551×10^{-4}	0.469×10^{-3}	0.0	0.0	0.0	0.225×10^{-5}	0.124×10^{-5}

c) Safety Factors for Mean Load and Moment Resistance

Case No.	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
1	1.25	1.20	1.22	3.42	3.59	3.51	1.33	1.37
2	1.46	1.32	1.39	3.57	3.97	3.77	1.44	1.54
3	1.61	1.46	1.53	3.90	4.37	4.13	1.58	1.70
4	1.83	1.55	1.69	3.81	4.65	4.23	1.64	1.85
5	1.53	1.33	1.43	3.38	3.98	3.68	1.42	1.57
6	1.78	1.58	1.68	4.15	4.74	4.45	1.70	1.85
7	1.90	1.75	1.82	4.77	5.24	5.00	1.91	2.02

Table 8. Optimum Solutions Calculated by Stevenson's Method

Case No.	Final (K-FT)		Object Function	Prob.of Failure	
	M_G	M_B		J. Stevenson	Proposed Method
2	196.38	196.89	207.68	0.765×10^{-1}	0.325×10^{-1}
3	215.45	213.07	219.57	0.762×10^{-3}	0.831×10^{-2}

Example 5 Optimum Design of Two-Story Single -Bay Frame

A model similar to Example 2 is employed to demonstrate the approximate design method based on the decomposition concept. The optimization is performed by the SUMT incorporating Powell's direct search technique, which does not require the derivatives of the functions. The possible modes are divided into basic and non-basic modes. This model has more than fifty collapse mechanisms. It is difficult to specify the appropriate basic modes at the start of the design process, because the dominant modes may change at any stage of the optimization. Here, their selection is automatically carried out at each design stage.

At first, the example frame model is designed by using Eq. 29 in computing the system failure probability. Then, sixteen modes are used for the basic ones. While this method needs less calculation time than the entire mode design, it gives a heavier design. (See Table 10)

Next, according to Vanmarcke's method, some designs are generated, where 8 and 16 are employed as the number of basic modes so as to investigate its influence. As shown in Table 9, Table 10 and Fig. 9, infeasible designs are generated

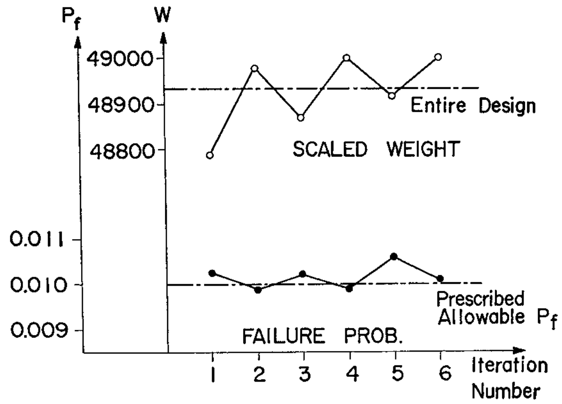


Fig. 9. Convergency of Decomposition Method.

Table 9. Approximate Design with 8 Basic Modes (Vanmarcke's method)

Iteration Number	X_1	X_2	X_3	X_4	Scaled Weight	Failure Prob.
1	69.83	39.12	62.73	62.53	46,842	0.015916
2	75.95	43.26	71.00	71.78	52,398	0.005117
3	69.54	41.14	65.23	61.28	47,438	0.013771
4	74.12	41.34	67.55	67.64	50,132	0.007790
5	69.54	41.14	65.23	61.28	47,438	0.013771
6	74.12	41.34	67.55	67.65	50,132	0.007790

$X_1 \sim X_4$: Design variables which mean moment resistances (K-FT)

Table 10. Numerical Results of Two-Story Single-Bay Frame Design

a) Decomposition Method (Vanmarcke's method)

Iter. No.	X_1	X_2	X_3	X_4	Object Func.	Failure Prob.
1	73.26	41.63	65.57	63.48	48,788	0.0102509
2	72.07	40.42	66.91	65.49	48,978	0.0099098
3	74.72	40.61	64.60	64.41	48,868	0.0102136
4	73.89	38.96	68.08	64.69	49,002	0.0099248
5	74.71	40.70	64.61	64.41	48,886	0.0101669
6	75.31	40.58	66.21	62.14	49,020	0.0099342

Cal. Time 20 sec./iteration

b) Entire Mode Design

Entire D.	71.69	40.79	67.25	64.93	48,932	0.0099925

Cal. Time 150 sec.

c) Approximate Method (Based on Eq. 29)

Appr. D.	79.52	38.98	66.30	65.53	49,066	0.0099558

Cal. Time 120 sec.

d) Proposed Iterative Method

Iteration	X_1	X_3	X_3	X_4	P_{fa}	P_f	Scaled Weight	No. of Basic Mode
1	63.02	33.54	61.38	62.20	0.010000	0.037163	44,628	4
2	55.63	67.75	58.40	49.97	0.026910	0.038872	46,350	4
3	91.11	0.	91.03	91.91	0.000692	1.282900	54,808	4
4	69.83	39.12	62.73	62.53	0.010000	0.015916	46,842	8
5	71.28	42.85	68.30	67.72	0.006283	0.007913	50,030	8
6	74.39	39.21	65.73	65.41	0.007991	0.010147	48,948	8
7	72.09	39.36	68.08	65.31	0.007826	0.010189	48,968	8
8	71.17	42.56	64.80	66.39	0.007861	0.009705	48,984	8
9	70.32	42.82	64.31	62.35	0.010000	0.012112	47,960	12
10	77.52	40.22	65.60	65.51	0.008256	0.008656	49,770	12
11	71.08	41.81	66.43	63.07	0.009539	0.010073	48,878	12
12	73.57	40.90	64.24	65.89	0.009466	0.010008	48,920	12
13	73.57	40.89	64.33	65.93	0.009465	0.009953	48,944	12

X_i: Mean Plastic Moment of Member (K-FT) Cal. Time 118 sec.

$i=1, 3$: Beams of Upper and Lower Stories

$i=2, 4$: Columns of Upper and Lower Stories

at the first design cycle. These are induced by the lack of effects of the non-basic failure modes. Next, upper and lower bounds on weight are produced in turn, and they are conservative and unconservative, respectively. In both cases, safety is examined by calculating the system failure probability with an analysis of all the modes.

For the case of 16 modes, the following items are observed from Table 11. The approximate method requires only 20 seconds of computation for one iteration, while the complete mode design requires 150 seconds. Also, the results show acceptable values on weight, though the obtained values are slightly different in design variables from the exact values. Among the feasible designs, the second design shows the least weight, which is only 0.1 per cent heavier than the entire mode design. Other designs have larger failure probabilities than this design in spite of heavier weights. This fact may imply that the optimization for this problem is very sensitive to the change of the allowable failure level. It may also imply that the induced constraint surface does not have the distinct vertices which are observed in linear programming problems. The design space obtained for a two-variable problem indicates this tendency. (See Fig. 10)

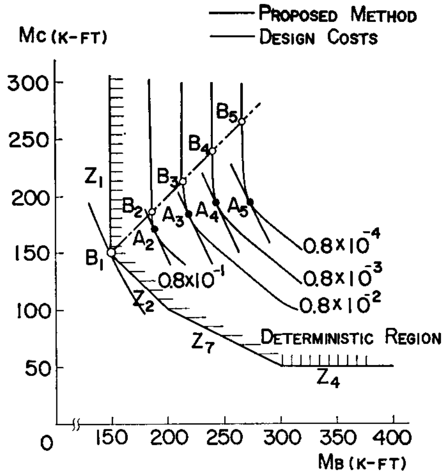


Fig. 10. Design Space of Single Bent Frame.

Taking 8 as the number of basic modes, this method does not give good solutions, for which the iterative procedure is terminated after four cycles, because the solutions hitherto begin to diverge as shown in Table 10. Then, there are considerably large gaps on the weights of the upper and lower bounds. The accuracy of the decomposition method is explicitly dependent on how to select the basic modes.

Table 10 (d) presents the results obtained by the proposed iterative method. The design starts with the same number of basic modes as that of the independent variables. (i.e. 4) After three iterations, four modes are newly added to the basic modes. After five more cycles, the number is changed to 12, and an acceptable design is generated at the 13th cycle. This method requires 118 seconds in computation, but the calculation time can be reduced by starting from more basic modes. The first three steps contribute nothing to the final design, and they should be eliminated. Also, this method gives a good design with only 8 basic modes. The obtained design is only 0.03 per cent heavier than the entire mode design. The convergency may be improved by introducing the modification factor.

8. Conclusions

This paper treats the safety analysis and the minimum-weight design of rigid frames based on reliability concepts. The improvements on the upper and lower bounds of system failure probability are attempted by considering the correlations between two or three failure events. Through this investigation, one may reach the following conclusions:

1) A simple mathematical model proposed herein proves to be acceptable for the rigid framed structures treated in this paper. A very close failure probability can be obtained by considering the effect of three-events-occurrence on the average. However, this method seems to be more suitable for examining the availability of the proposed method presenting an upper bound, because it consists of relatively complex procedures. For that reason, it requires more execution time than the latter.

2) By taking the failure probability as the index of safety, the minimum-weight design can be reduced to a mathematical programming problem with only one constraint, or to an unconstrained optimization problem. Many designs, whose weights are very close to the optimum one, are obtained through the optimization. This can be seen in the deterministic bridge design with the deflection constraints. If these designs are accepted as approximate solutions, the minimum-weight design will become simple and practical by utilizing the advantage that many various constraints can be reduced to one constraint.

4) The use of the decomposition technique makes the proposed method effective in the redesign procedure. The classification of failure modes can clarify their contributions to the system failure. Conservative approximate designs can be obtained by using parts of possible modes. If a sufficient number of modes is taken as basic modes, the iterative scheme proposed by E. Vanmarcke is considered to be suitable. It can save the core size and make the calculation inexpensive. The iterative method proposed herein has similar merits. This method is superior to Vanmarcke's method with

regard to convergency and applicability. There, the modified allowable failure level is forced to converge and the appropriate number of basic modes is sought for in the program.

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