A Study on the Mechanism of Consolidation

By

Takeshi TAMURA*

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Abstract

Reducing the linear Biot's equations into a single governing equation, the mechanism of multi-dimensional consolidation is considered by means of a variational principle. Particularly, the investigation is made for the geometrical meaning of the linear relation between the distribution of excess pore water pressure and that of deformation, which is obtained by observing the consolidation process from the final steady state.

As the results of the present study, we can conclude that (1) Biot's equations of consolidation are reduced into a single governing equation with the excess pore water pressure as the only unknown function, if we choose the final steady state as the reference, (2) the consolidation process is interpreted as a series of minimum norm problems in a metric vector space, i.e., the Function Space, and (3) the equilibrium condition of consolidation is equivalent to determining a point in some subset which has the shortest distance from the origin of the Function Space.

1. Introduction

Compared with a one-dimensional case, the behavior of a multi-dimensional consolidation is remarkably complicated. The former is governed merely by a heat conduction type of equation, whereas the latter is considered to be governed strictly by Biot's equations[1] which consist of simultaneous partial differential equations, with both displacement and pore water pressure as the unknown functions.

Several analytical solutions of consolidation can be detected as the boundary value problem, but we have few studies concerning the mechanism of consolidation[2),3)] based upon the mathematical structure of Biot's equations.

The purpose of the present paper is to give a geometrical interpretation of the consolidation mechanism, contrasting one of the simple theorems in the theory of the minimum norm problem. This paper is composed of the following three parts.

Firstly, Biot's equations will be expressed as a single equation only in terms of the excess pore water pressure, by using a unique linear relation between the dis-

* Department of Transportation Engineering

tribution of excess pore water pressure and that of residual volumetric strain. The most important point for this reduction is to observe the consolidation process from its final state, i.e., to choose the final state as the reference.

Secondly, some fundamental variational principles will be briefly reviewed with the concept of the Function Space in order to clarify the geometrical meaning.

Finally, the mechanism of consolidation will be investigated through a single Biot's equation in the Function Space, in which the Lagrange multiplier plays an important role.

We should note that all of the discussions are based upon Biot's equations, assuming the linear elasticity of clay skeleton and the incompressibility of pore water. Therefore, the results in this paper never exceed the range of Biot's theory.

2. Biot's Equation in Terms of Excess Pore Water Pressure[4)]

Biot's equations of multi-dimensional consolidation are generally written as

$$\begin{cases} \sum_j \dfrac{\partial \sigma'_{ij}}{\partial x_j} + \dfrac{\partial p}{\partial x_i} + f_i = 0 \qquad (i=1,2,3) & (1) \\ \dfrac{\partial v}{\partial t} = -\dfrac{1}{\gamma_w} \sum_{i,j} k_{ij} \dfrac{\partial^2}{\partial x_i \partial x_j}(p+\gamma_w \Omega) & (2) \end{cases}$$

in which σ'_{ij}=effective stress tensor, p=pore water pressure, f_i=body force, v= volumetric strain, γ_w=weight of pore water per unit volume, k_{ij}=permeability tensor, Ω=potential of position, x_i=cartesian coordinate and t=time.

Associated with Eqs. (1) and (2), the following boundary conditions (B_u), (B_σ) and (B_D) are set up:

(B_u) boundary condition concerning displacement u_i:

$$u_i = \bar{u}_i \quad (i=1,2,3) \qquad \text{on} \quad S_u$$

(B_σ) boundary condition concerning stress traction T_i:

$$T_i = \bar{T}_i \quad (i=1,2,3) \qquad \text{on} \quad S_\sigma$$

(B_D) boundary condition concerning drainage:

$$p = \bar{p} \quad \text{on} \quad S_D, \quad \sum k_{ij} \frac{\partial}{\partial x_i}(p+\gamma_w \Omega) n_j = 0 \quad \text{on} \quad S_{UD}$$

in which $\bar{u}_i$=prescribed displacement on the displacement boundary S_u, $\bar{T}_i$= prescribed (total) stress traction on the stress boundary S_σ, $\bar{p}$=prescribed pore water pressure on the drained boundary S_D and n_i=unit outward normal vector on the undrained boundary S_{UD}.

As a particular case of consolidation, the problem of the undrained state is often considered. For this case, the equation of equilibrium (Eq. (1)) remains unchanged but that of continuity (Eq. (2)) must be replaced by the following restriction on the volumetric strain:

$$v \equiv 0 \tag{3}$$

Therefore in the analysis of the undrained state, we have no boundary condition directly concerning the pore water pressure.

Meanwhile, at the ultimate state after completion of consolidation, there may exist a steady seepage flow, of which pore water pressure p_f is governed by

$$\sum_{i,j} k_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(p_f + \gamma_w \Omega) = 0 \tag{4}$$

with (B_D). The displacement at this state u_{fi} can be determined by substituting p_f into Eq. (1). Namely, the deformation and the pore water pressure at the final state are solved separately as uncoupled problems.

Choosing this final steady state as the basic reference state, i.e., observing the consolidating process from the final state ($\sigma'_{ij} = \sigma'_{fij}$ $p = p_f$, $v = v_f$, $u_i = u_{fi}$), Biot's equations are rewritten in the same form as Eqs. (1) and (2) in appearance:

$$\begin{cases} \sum_j \frac{\partial \tau'_{ij}}{\partial x_j} + \frac{\partial u}{\partial x_i} = 0 \qquad (i=1, 2, 3) & (5) \\ \frac{\partial \theta}{\partial t} = -\frac{1}{\gamma_w} \sum_{i,j} k_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} & (6) \end{cases}$$

in which

$$\tau'_{ij} = \sigma'_{ij} - \sigma'_{fij} \tag{7}$$

$$u = p - p_f \tag{8}$$

$$\theta = v - v_f \tag{9}$$

$$v_i = u_i - u_{fi} \tag{10}$$

These are the residual values which will disappear with the progress of consolidation. Particularly in this paper, u as defined by Eq. (8), refers to the excess pore water pressure.

It is, however, considerably noteworthy that Eqs. (5) and (6) are accompanied with the following homogeneous boundary conditions (B_{uo}), $(B_{\sigma o})$ and (B_{Do}), which signify $\bar{u}_i = 0$, $\bar{T}_i = 0$, $\bar{p} = 0$ in (B_u), (B_σ) and (B_D), respectively. The boundary conditions (B_{uo}) and $(B_{\sigma o})$ indicate that the displacement v_i determined by Eq. (5) is exclusively due to the existence of excess pore water pressure u alone. In other words, assuming u as known, the displacement v_i can be obtained through Eq. (5)

under the boundary condition (B_{uo}) and ($B_{\sigma o}$). Because of the linearity of the stress—strain relation of clay skeleton, there must be a unique linear relation between the distribution of such a displacement and that of excess pore water pressure. We can obtain this linear relation by the following consideration.

Firstly, let $U_i(x, X)$ be the Green function of Eq. (5), i.e., the displacement determined by

$$\sum_j \frac{\partial \tau'_{ij}}{\partial x_j} + \frac{\partial \delta(x, X)}{\partial x_i} = 0 \qquad (i = 1, 2, 3) \tag{11}$$

with (B_{uo}) and ($B_{\sigma o}$) in which $\delta(x, X)$ denotes Dirac's delta function. Eq. (11) means the equilibrium condition when a unit amount of concentrated pore water pressure is applied at a point X. Using the principle of superposition, it is possible to write down the displacement v_i in an integral form:

$$v_i(x) = \int_V U_i(x, X)u(X)dV_X \qquad (i=1, 2, 3) \tag{12}$$

in which the subscript X of dV_X means the integral variable. In the above and following descrpition, there is not noticed any distinction between the points x or X and their cartesian coordinates x_i or X_i. Eq. (12) is equivalent to the following ones:

$$\eta_{ij}(x) = \int_V H_{ij}(x, X)u(X)dV_X \qquad (i=1, 2, 3) \tag{13}$$

$$\theta(x) = \int_V \Theta(x, X)u(X)dV_X \tag{14}$$

in which

$$H_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \qquad (i,j=1, 2, 3) \tag{15}$$

$$\eta_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (i,j=1, 2, 3) \tag{16}$$

$$\Theta = \sum_i H_{ii} \tag{17}$$

$$\theta = \sum_i \eta_{ii} \tag{18}$$

We will consider, then, the inverse relation of Eq. (14):

$$u(x) = \int_V \Phi(x, X)\theta(X)dV_X \tag{19}$$

Of course, there is little possibility to determine $\Phi(x, X)$ directly from Eq. (14), but it is easy to understand the meaning of the integral kernel $\Phi(x, X)$. Namely, $\Phi(x, X)$ signifies the distribution of excess pore water pressure, which is caused in a case where a unit amount of concentrated volumetric strain is imposed at a point

X, with no volume change at any other points under (B_{uo}) and ($B_{\sigma o}$). In other words, $\Phi(x, X)$ is the excess pore water pressure u determined by

$$\begin{cases} \sum_j \frac{\partial \tau'_{ij}}{\partial x_j} + \frac{\partial u}{\partial x_i} = 0 \qquad (i=1, 2, 3) & (20) \\ \theta = \delta(x, X) & (21) \end{cases}$$

with (B_{uo}) and ($B_{\sigma o}$).

Taking the time derivative of Eq. (19) and substituting it into Eq. (6), the single governing equation of a multi-dimensional consolidation is finally obtained, with the excess pore water pressure as the only unknown function:

$$\frac{\partial u}{\partial t} = -\frac{1}{\gamma_w} \sum_{i,j} k_{ij} \int_V \Phi(x, X) \frac{\partial^2 u}{\partial x_i \partial x_j}(X, t) dV_X \tag{22}$$

which assumes the form of the equation of evolution just like the equation of heat conduction. With the boundary condition (B_{Do}), Eq. (22) can be solved for the excess pore water pressure u independently of the displacement v_i, if the initial condition of u is given.

3. Variational Principles and Concept of Function Space

Using the concept of the Function Space proposed by Prager and Synge[5], we will explain the way to intrepret several variational principles of linear elasticity by means of simple geometry. Before going to the subject, let us define a few vector notations to simplify the description.

displacement	$u_i, v_i, \bar{u}_i$	$\rightarrow \boldsymbol{u}, \boldsymbol{v}, \bar{\boldsymbol{u}}$
strain	$\varepsilon_{ij}, \eta_{ij}$	$\rightarrow \boldsymbol{\varepsilon}, \boldsymbol{\eta}$
stress	σ'_{ij}, τ'_{ij}	$\rightarrow \boldsymbol{\sigma}', \boldsymbol{\tau}'$
body force	f_i	$\rightarrow \boldsymbol{f}$
surface traction	$T_i, \bar{T}_i$	$\rightarrow \boldsymbol{T}, \bar{\boldsymbol{T}}$
stress-strain	$\sigma'_{ij} = \sum D_{ijkl}\varepsilon_{kl}$	$\rightarrow \boldsymbol{\sigma}' = \boldsymbol{D\varepsilon}$
relation	$\varepsilon_{ij} = \sum F_{ijkl}\sigma'_{kl}$	$\rightarrow \boldsymbol{\varepsilon} = \boldsymbol{F\sigma}'$

Although it is not necessary for the time being to distinguish the effective stress from the total stress, the prime (′) is attached to the stress to avoid confusion.

a) Principle of minimum potential energy

Among the whole set of geometrically admissible displacement fields (satisfying (B_u)), the unique solution of the given boundary value problem makes the following total potential energy minimum:

$$\pi[\boldsymbol{u}] = \int_V \left\{\frac{1}{2}(\boldsymbol{D\varepsilon}, \boldsymbol{\varepsilon}) - (\boldsymbol{f}, \boldsymbol{u})\right\} dV - \int_{S\sigma} (\bar{\boldsymbol{T}}, \boldsymbol{u}) dS \tag{23}$$

in which $\boldsymbol{\varepsilon}$=the strain derived from $\boldsymbol{u}$ and (,) denotes the usual inner product:

$$(\boldsymbol{D\varepsilon}, \boldsymbol{\varepsilon}) = \sum_{i,j,k,l} D_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{24}$$

$$(\boldsymbol{f}, \boldsymbol{u}) = \sum_i f_i u_i \tag{25}$$

b) Principle of maximum complementary energy

Among the whole set of mechanically admisisble stress fields (satisfying Eq. (1) and (B_σ)), the unique solution of the given boundary value problem makes the following complementary energy maximum:

$$\pi^*[\boldsymbol{\sigma}'] = -\int_V \frac{1}{2}(\boldsymbol{F\sigma}', \boldsymbol{\sigma}') dV + \int_{Su} (\bar{\boldsymbol{T}}, \boldsymbol{u}) dS \tag{26}$$

c) Concept of Function Space[5)]

Prager and Synge introduced the concept of the Function Space in order to interpret geometrically the above principles, and to give a method of error estimation for the Rayleigh-Ritz approximation (Hypercircle Method). We will define the following set of strain fields $\tilde{E}$ and its subsets E, E*, E_0 and E_0^* for the explanation of this idea.

$\tilde{E}$: set of all strain fields which are distributed over the domain

$$E: \left\{\boldsymbol{\varepsilon} \mid \boldsymbol{\varepsilon} \in \tilde{E},\ \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\quad u_i = \bar{u}_i \text{ on } S_u\right\}$$

$$E^*: \left\{\boldsymbol{\varepsilon}^*(=\boldsymbol{F\sigma}^{*\prime}) \mid \varepsilon^* \in \tilde{E},\ \sum \frac{\partial \sigma_{ij}^{*\prime}}{\partial x_j} + f_i = 0,\quad T_i^* = T_i \text{ on } S_\sigma\right\}$$

$$E_0: \left\{\boldsymbol{\varepsilon}_0 \mid \boldsymbol{\varepsilon}_0 \in \tilde{E},\ \varepsilon_{0ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\quad u_i = 0 \text{ on } S_u\right\}$$

$$E_0^*: \left\{\boldsymbol{\varepsilon}_0^*(=\boldsymbol{F\sigma}_0^{*\prime}) \mid \boldsymbol{\varepsilon}_0^* \in \tilde{E},\ \sum \frac{\partial \sigma_{0ij}^{*\prime}}{\partial x_j} = 0,\quad T_i^* = 0 \text{ on } S_\sigma\right\}$$

Fig. 1 shows the conceptual view of the correlation between $\tilde{E}$, E, E_0, E* and E_0^*. O means the origin of space $\tilde{E}$, of which the strain distribution is equal to zero identically. Set E is equivalent to the admissible set in the principle of minimum potential energy, while Set E* is equivalent to the admissible set in the principle of maximum complementary energy. Any element in set E_0 is defined as the difference of some two elements in E($=\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in$ E), while any element in set E_0^* is defined as the difference of some two elements in E*($=\boldsymbol{\varepsilon}_2^* - \boldsymbol{\varepsilon}_2^*$, $\boldsymbol{\varepsilon}_1^*$, $\boldsymbol{\varepsilon}_2^* \in$ E*).

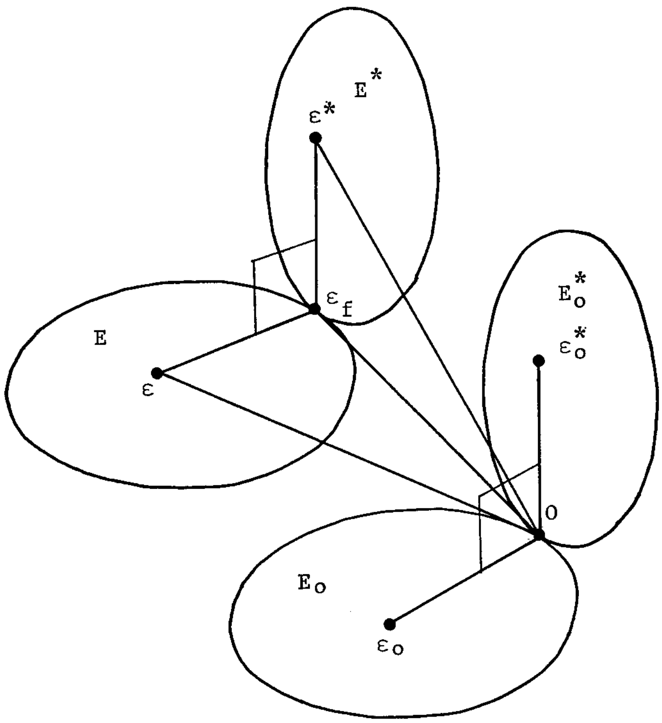

Fig. 1. Set of strain distributions $\tilde{E}$ and its subsets E, E*, E_0 and E_0^*

The following inner product is introduced in the linear space $\tilde{E}$:

$$\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \rangle = \frac{1}{2}\int_V \sum D_{ijkl}\varepsilon_{1ij}\varepsilon_{2kl}dV \tag{27}$$

Using this inner product, the norm of strain:

$$||\boldsymbol{\varepsilon}|| = \sqrt{\langle \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \rangle} \tag{28}$$

and the measure of two elements in E:

$$d(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) = ||\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|| \tag{29}$$

are defined. It is easy to prove the following geometric relation among sets $\tilde{E}$, E, E*, E_0 and E_0^* (refer to Fig. 1).

(1) The theorem of existence and uniqueness of solution in the linear elasticity implies that sets E and E* possess a unique common point $\boldsymbol{\varepsilon}_f$(=the point of solution).

(2) Similarly to (1), sets E_0 and E_0^* contain the origin O as a unique common point.

(3) Because of the homogeneity of the boundary conditions, any element ε_0 of E_0 is perpendicular to all the elements ε_0^* of E_0^* in the sense of the inner product defined by Eq. (27), i.e.,

$$\langle \varepsilon_0, \varepsilon_0^* \rangle = -\int_V \sum \frac{\partial \sigma_{ij}^{*\prime}}{\partial x_j} u_i dV + \int_{S\sigma+Su} \sum T_i^* u_i dS = 0 \qquad (30)$$

(4) Similarly to (3), the difference of any two elements in E is perpendicular to the differences of all the pairs of elements in E*, i.e.,

$$\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1^* - \varepsilon_2^* \rangle = 0 \qquad (\varepsilon_1, \varepsilon_2 \in E, \ \varepsilon_1^*, \varepsilon_2^* \in E^*) \qquad (31)$$

As a special case, $(\varepsilon_1 - \varepsilon_f)$ and $(\varepsilon_1^* - \varepsilon_f)$ are perpendicular to each other.

If the variational principles are considered in space $\tilde{E}$, we can say that the principle of minimum potential energy is to seek after a point ε in E which has the smallest distance from an arbitrary point ε^* fixed in E*. Furthermore, the principle of maximum complementary energy is to seek after a point ε^* in E* which has the smallest distance from an arbitrary point ε fixed in E. (Refer to Figs. 2, 3) It is also obvious that these minimum distances are given by the point ε_f in both cases.

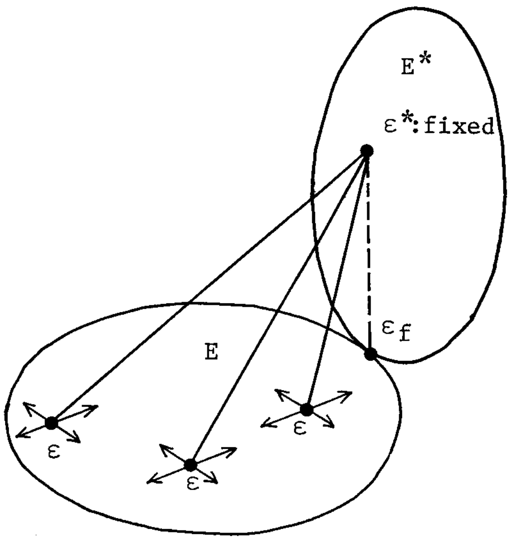

Fig. 2. Principle of minimum potential energy.

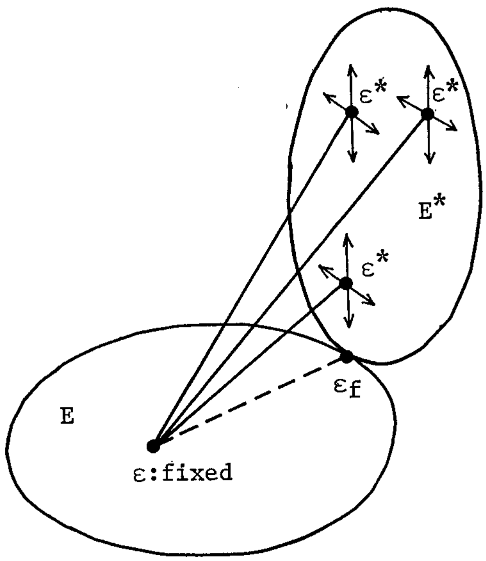

Fig. 3. Principle of maximum complementary energy

d) Sakai's principle[6)]

It has been said that the two fundamental variational pricinples are reciprocally related to each other, but that they are independent in themselves. Sakai, however, proved recently that these principles are deduced as the sationary conditions of the following single functional:

$$||\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*||^2=\frac{1}{2}\int_V(\boldsymbol{D}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*),\ (\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*))dV \tag{32}$$

by showing

$$||\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*||^2=\pi[\boldsymbol{\varepsilon}]-\pi^*[\boldsymbol{\varepsilon}^*] \tag{33}$$

in which $\pi[\boldsymbol{\varepsilon}]$ and $\pi^*[\boldsymbol{\varepsilon}^*]$ are the functionals defined by Eqs. (23) and (26), respectively. In other words, Sakai's principle verifies the fact that the two variational principles are equivalent to the minimization of the distance from $\boldsymbol{\varepsilon}$ to $\boldsymbol{\varepsilon}^*$ by moving $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^*$ at the same time. (Refer to Fig. 4)

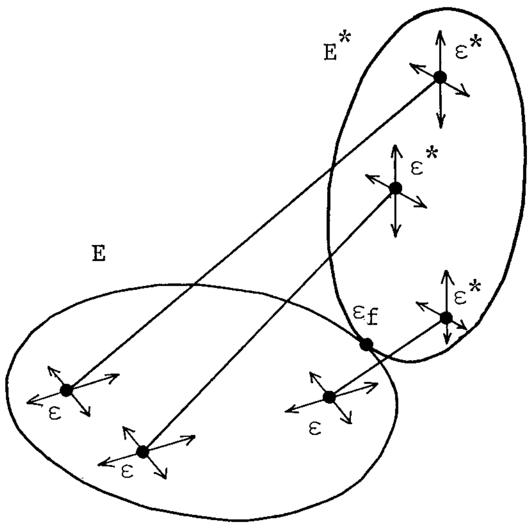

Fig. 4. Sakai's principle.

From the above explanation, it is understood that several variational principles can be interpreted by means of a simple geometrical concept in space $\tilde{\mathrm{E}}$.

4. Consolidation Mechanism and Minimum Norm Problem

In this section, the mechanism of consolidation will be investigated through the variational principle, particularly by using the concept of the Function Space. As a practical procedure of disscussion, we will firstly explain that the meaning of Eq. (13), (14) or (19) can be understood by a simple theorem of the minimum norm problem. Also, the (excess) pore water pressure can be defined as the Lagrange multiplier which is concerned with the restriction of the minimum norm problem. Secondly, the meaning of the condition of continuity (Eq. (2) or (6)) will be explain in detail.

4.1 Variational principle with restriction on volumetric strain

As was discussed in 3, the equilibrium condition is equivalent to minimizing the total potential energy in the usual boundary value problem of elasticity. Then, let us investigate the problem accompanied with the restriction that the volumetric strain $v(x)$ must be equal to a given function $\bar{v}(x)$. The deformation state, even in a consolidation process, is analysed by regarding it as this type of problem since it is quasi-static. Therefore such a problem as is accompanied with some restriction, is essentially fundamental in the present investigation.

When several restrictions exist among the independent variables, the variational principle is generally formulated by introducing the so-called Lagrange multiplier $\lambda(x)$. Namely, the total potential energy $\pi[\boldsymbol{u}]$ must be modified into the following form:

$$\pi[\boldsymbol{u}, \lambda] = \pi[\boldsymbol{u}] + \int_V \lambda(x)(v(x) - \bar{v}(x))dV \tag{34}$$

Indeed, the first variation of the functional $\pi[\boldsymbol{u}, \lambda]$ yields the following Euler's equations (stationary conditions):

$$\begin{cases} \sum_j \dfrac{\partial \sigma'_{ij}}{\partial x_j} + \dfrac{\partial \lambda}{\partial x_i} + f_i = 0 \qquad (\mathrm{i}=1, 2, 3) & (35) \\ v - \bar{v} = 0 & (36) \end{cases}$$

with the natural boundary condition:

$$\sum_j (\sigma'_{ij} + \lambda\delta_{ij})n_j - \bar{T}_i = 0 \qquad (i=1, 2, 3) \quad \text{on} \quad S_\sigma \tag{37}$$

If the Lagrange multiplier λ is regarded as the pore water pressure p, Eqs. (35) and (37) become identical with the equilibrium condition of Biot's equations (Eq. (1)) and the stress boundary condition (B_σ), respectively. Hence, it is concluded that the equilibrium condition with the restriction on the volumetric strain is equivalent to that of consolidation.

Then, we will consider what is discussed above in set $\tilde{E}$. In subset E ($\subset \tilde{E}$), a new subset $E_{\bar{v}}$ is further defined, such that all elements of $E_{\bar{v}}$ satisfy the constrained condition $v = \bar{v}$ (refer to Fig. 5):

$$E_{\bar{v}} = \{\boldsymbol{\varepsilon} \mid \boldsymbol{\varepsilon} \in E, \ v(= \sum \varepsilon_{ii}) = \bar{v}\}$$

Then, the following minimum norm problem is set up:

"Determine an element $\boldsymbol{\varepsilon}$ in $E_{\bar{v}}$ such that the norm $||\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^*||$ becomes minimum when $\boldsymbol{\varepsilon}^*$ is fixed in E^*."

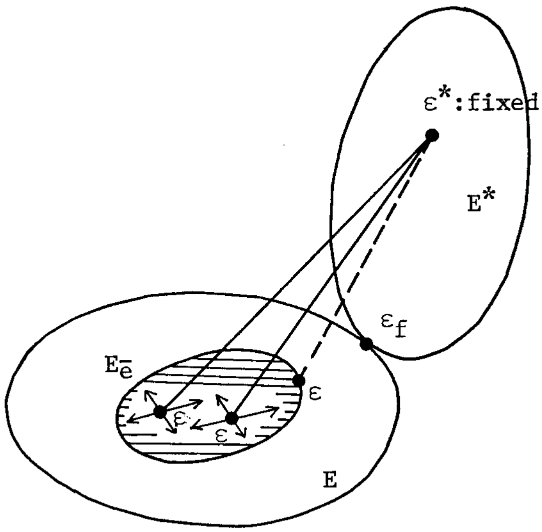

Fig. 5. Subset $E_{\bar{\varepsilon}}$ and principle of minimum potential energy with constrained condition.

This problem is formulated by defining a functional $\pi'[\boldsymbol{\varepsilon}, \lambda]$:

$$\pi'[\boldsymbol{\varepsilon}, \lambda] = \langle \boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*, \boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^* \rangle + \int_V \lambda(x)(v(x)-\bar{v}(x))dV \tag{38}$$

in which $\langle \boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*, \boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^* \rangle$ is the functional to be minimized, and the second term is an additional one to take into consideration the restriction; $\boldsymbol{\varepsilon} \in E_{\bar{\varepsilon}}$. Using the method of Prager and Synge or Sakai, it can be easily proved that the stationary conditions of $\pi'[\boldsymbol{\varepsilon}, \lambda]$ are the same as those of $\pi[\boldsymbol{u}, \lambda]$. Indeed, $\pi'[\boldsymbol{\varepsilon}, \lambda]$ is developed as follows:

$$\pi'[\boldsymbol{\varepsilon}, \boldsymbol{\lambda}] = \pi[\boldsymbol{u}, \lambda] - \pi^*[\boldsymbol{\varepsilon}^*] \tag{39}$$

As is understood from Eq. (39), the stationary conditions of $\pi'[\boldsymbol{\varepsilon}, \lambda]$ and $\pi[\boldsymbol{u}, \lambda]$ are coincident if $\boldsymbol{\varepsilon}^*$ is fixed. Therefore, the problem with the constrained condition concerning the volumetric strain can be regarded as a problem of determining an element $\boldsymbol{\varepsilon}$ in $E_{\bar{\varepsilon}}$ which makes $||\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*||$ minimum. (Refer to Fig. 5)

It should be noted here that the vector $(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*)$ does not generally satisfy the compatibility condition. Namely, an element $(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*)$ is not necessarily derived from some displacement field. For such a condition, it is required to choose the solution of the usual problem $\boldsymbol{\varepsilon}_f$ as $\boldsymbol{\varepsilon}^*$. In relation to the consolidation problem, $\boldsymbol{\varepsilon}_f$ means the final state of the process. Hence $\boldsymbol{\varepsilon}_f$ is fixed as $\boldsymbol{\varepsilon}^*$ and is regarded as the reference point of set E. In order to simplify the description, the vector $(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_f)$ will be denoted by $\boldsymbol{\eta}$. Consequently, $E_{\bar{\varepsilon}}$ is replaced by $E_{\bar{\eta}}$

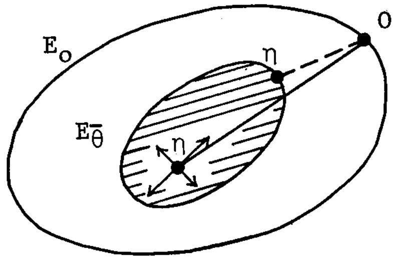

Fig. 6. Subset $E_{\bar\theta}$ and element η.

in which $\bar{\theta}=v-v_f$ and all consideration is confined in E_0. (Refer to Fig. 6) This alternation corresponds to choosing the final state as the reference and making the boundary conditions homogeneous in 2.

From the above explanation, if we

(1) select the final state of consolidation $\boldsymbol{\varepsilon}_f$ as the reference point of set E, and
(2) replace the Lagrange multiplier $\lambda(x)$ by the excess pore water pressure $u(x)$ $(=p(x)-p_f(x))$.

then , the functional $\pi'[\boldsymbol{\varepsilon}, \lambda]$ defined by Eq. (38) is modified into

$$\Pi\,[\boldsymbol{\eta}, u] = \langle\boldsymbol{\eta}, \boldsymbol{\eta}\rangle+\int_V u(x)(\theta(x)-\bar{\theta}(x)dV \qquad (40)$$

4.2 A simple theorem of minimum norm problem[7]

The stationary conditions of the functional $\Pi\,[\boldsymbol{\eta}, u]$ can be interpreted from a different point of view. Firstly, one of the simple theorems in the minimum norm problem (optimal control theory) in the n-dimensional Euclidian space R^n (or more generally in a Hibert space) is explained. S denotes a subspace of R^n defined by m-linear equations:

$$\sum_{j=1}^{n} a_{ij}z_j = c_i \quad \text{or} \quad (\boldsymbol{a}_i, \boldsymbol{z}) = c_i \qquad (i=1, 2, \cdots, m) \qquad (41)$$

in which $\boldsymbol{z}$ means an n-dimensional vector expressed by the components of the orthogonal coordinates $(z_1, z_2, \cdots, z_n)$, $\boldsymbol{a}_i(=(a_{i1}, a_{i2}, \cdots, a_{in}))$ and c_i denote the vector of coefficients and the constant of the i-th equation, respectively. Without a loss of generality, $\boldsymbol{a}_i$ is assumed to be a unit vector:

$$||\boldsymbol{a}_i|| = \sqrt{\sum_{j=1}^{n} a_{ij}^2} = 1 \qquad (i=1, 2, \cdots, m) \qquad (42)$$

The subsapce S is the $(n-m)$-dimensional intersection of the m-hyperplanes expressed by Eq. (41). $\boldsymbol{a}_i$ and $|c_i|$ mean the unit normal and the distance from the origin O to each hyperplane.

The following minimum norm problem is, then, considered:

"Determine a point P in subspace S which has the shortest distance from the origin O."

Fig. 7 shows a schematic view of the above circumstance in the case of $n=3$, $m=2$.

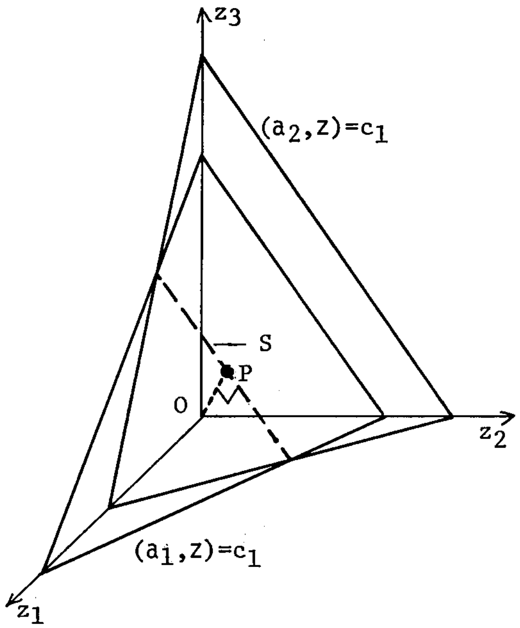

Fig. 7. Subset S and solution point P ($n=3$, $m=2$).

This problem can be easily solved as follows. In order to minimize the distance $||\boldsymbol{z}||=\sqrt{(\boldsymbol{z}, \boldsymbol{z})}$ from the origin under the constrained equation (41), it is required to obtain the stationary conditions of the following function $f(\boldsymbol{z}, \boldsymbol{\lambda})$:

$$f(\boldsymbol{z}, \boldsymbol{\lambda}) = (\boldsymbol{z}, \boldsymbol{z})-\sum_{i=1}^{m} 2\lambda_i\{(\boldsymbol{a}_i, \boldsymbol{z})-c_i\} \tag{43}$$

in which $\boldsymbol{\lambda}=(\lambda_1, \lambda_2, \cdots, \lambda_m)$ are the Lagrange multipliers. Therefore differentiating $f(\boldsymbol{z}, \boldsymbol{\lambda})$ with respect to $\boldsymbol{z}$ and $\boldsymbol{\lambda}$, we have

$$\boldsymbol{z} = \sum_{i=1}^{m} \lambda_i \boldsymbol{a}_i \tag{44}$$

$$(\boldsymbol{a}_i, \boldsymbol{z}) = c_i \qquad (c_i = 1, 2, \cdots, m) \tag{45}$$

Eq. (45) is simply identical with the constrained equation (41), but the form of Eq. (44) is quite remarkable. This equation indicates that the position vector of solution $\boldsymbol{z}$ is expressed as a linear combination of m vectors $\boldsymbol{a}_i (i=1, 2, \cdots, m)$ which define subspace S itself, and that their coefficients are coincident with the Lagrange multipliers. Such a fact results from the particularity of the minimum norm

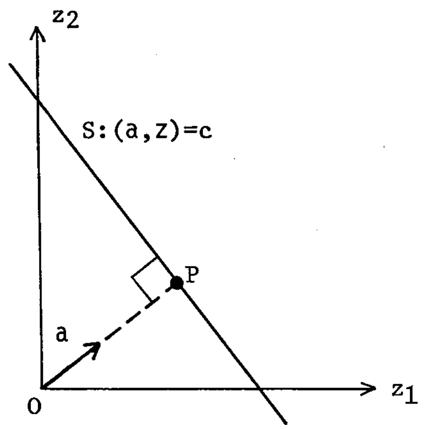

Fig. 8. Vector $\boldsymbol{a}$ and solution point P (n=2, m=1).

problem. Fig. 8 shows what is explained above in the simplest case (n=2, m=1). Substituting Eq. (44) into Eq. (45), the simultaneous linear equations for λ_i are obtained:

$$\sum_{j=1}^{m} \lambda_j(\boldsymbol{a}_j, \boldsymbol{a}_i) = c_i \qquad (i=1, 2, \cdots, m) \tag{46}$$

Determining λ_i by Eq. (46), the position vector $\boldsymbol{z}$ of point P is settled.

4.3 Mechanism of consolidation

Let us apply the above simple theory of the minimum norm problem to the interpretation of the consolidation mechanism. It should be noted that the functional $\Pi\,[\boldsymbol{\eta}, u]$ (Eq. (40)) and the function $f(\boldsymbol{z}, \boldsymbol{\lambda})$ (Eq. (43)) are written in similar forms. Both were introduced to minimize the distance from the origin O under some restrictions. A great difference found between them is the form of their second terms which are related to the constrained conditions. Namely, the second term of $f(\boldsymbol{z}, \boldsymbol{\lambda})$ is expressed in the form of an inner product of $\boldsymbol{z}$, while that of $\Pi\,[\boldsymbol{\eta}, u]$ is not in such a form. Yet, it is possible to rewrite it in the inner product form of u as follows. Let A denote a function which maps a strain distribution $\boldsymbol{\eta}(\in \mathrm{E}_0)$ into the volumetric strain at a point X i.e., $\theta_\eta(x=X)$:

$$A:\ \eta(x) \rightarrow \theta_\eta(x = X)$$

It should be noted that this function A can be expressed in the following inner product form if a particular element $\boldsymbol{E}_X$ is chosen in E_0:

$$\theta_\eta(x = X) = -2\langle \boldsymbol{\eta}, \boldsymbol{E}_X \rangle . \tag{47}$$

The above equation becomes valid when $\boldsymbol{E}_X$ is equal to the strain distribution

$\boldsymbol{H}(x, X)$, which can occur by a unit amount of concentrated excess prer water pressure at X under the homogeneous boundary conditions (Eq. (15)). Induced we deduce

$$
\begin{aligned}
-2\langle \boldsymbol{H}, \boldsymbol{\eta}\rangle &= -\int_V \sum_{i,j} \tau'_{ij}\eta_{ij}dV = \int_V \sum_{i,j} \frac{\partial \tau'_{ij}}{\partial x_j} v_i dV - \int_S \sum_{i,j} \tau'_{ij} n_j v_i dS \\
&= \int_V \delta(x, X) \sum_i \frac{\partial v_i}{\partial x_i} dV - \int_S \sum_{i,j} (\tau'_{ij} + \delta(x, X)\delta_{ij}) n_j v_i dS \\
&= \theta(x, X) \,.
\end{aligned}
\tag{48}
$$

Using the inner product expression obtained above, $\Pi\,[\boldsymbol{\eta}, u]$ (Eq. (40)) is rewritten in the same form as $f(\boldsymbol{z}, \boldsymbol{\lambda})$ (Eq. (43)):

$$
\Pi\,[\boldsymbol{\eta}, u] = \langle \boldsymbol{\eta}, \boldsymbol{\eta}\rangle - 2\int_V u(x)\left\{\langle \boldsymbol{\eta}(X), \boldsymbol{H}(x, X)\rangle + \frac{\bar{\theta}}{2}\right\} dV_X \tag{49}
$$

in which the integral variable in the second term is X. Of course, Eq. (5) and the condition $\theta - \bar{\theta} = 0$ are naturally obtained as the stationary conditions by taking the first variation of $\Pi\,[\boldsymbol{\eta}, u]$ in the usual sense. However, concerning the stationary condition with respect to $\boldsymbol{\eta}$, $\Pi\,[\boldsymbol{\eta}, u]$ can be treated as if it were a quadratic form. $\Pi\,[\boldsymbol{\eta}, u]$ is differentiated under the topology of $\tilde{\mathrm{E}}$ defined by the inner product $\langle\, ,\, \rangle$ (Fréchet or Gâteaux differentiation)[8], and the following condition is deduced:

$$
\boldsymbol{\eta}(x) = \int_V \boldsymbol{H}(x, X) u(X) dV_X \quad \text{or} \quad \eta_{ij}(x) = \int_V H_{ij}(x, X) u(X) dV_X \tag{50}
$$

which is the same as Eq. (13). This procedure is quite similar to the method used to obtain Eq. (44) from $f(\boldsymbol{z}, \boldsymbol{\lambda})$. It is of particular interest to note that we can directly obtain Eq. (50), which is usually obtained from a part of the stationary conditions Eq. (5) and the homogeneous boundary conditions (B_{uo}), ($B_{\sigma o}$). In other words, even the form of solution can be determined as a part of the stationary conditions in some sense. Substituting Eq. (50) into the other stationary condition:

$$
\langle \boldsymbol{\eta}, \boldsymbol{H}\rangle + \frac{\bar{\theta}}{2} = 0 \quad \text{or} \quad \theta - \bar{\theta} = 0 \tag{51}
$$

we have the following relation between $u(x)$ and $\bar{\theta}(x)$:

$$
\bar{\theta}(x) = \int_V \Theta(x, X) u(X) dV_X \tag{52}
$$

which is identical with Eq. (14) and has a meaning similar to Eq. (46).

As is understood from the above explanation, it is possible to interpret the geometrical meaning of the functional $\Pi\,[\boldsymbol{\eta}, u]$ in E_0 through the function $f(\boldsymbol{z}, \boldsymbol{\lambda})$

in R^n. Then, let us consider the mechanism of the consolidation process using the geometry of E_0. At the beginning of the consolidation process, the distribution of the volumetric strain $v(x)$ is restricted to be identical to zero, i.e., $v(x)=0$. But since the final state of consolidation is chosen as the reference in the present study, $v(x)=0$ means $\theta(x)=\bar{\theta}(x)$ $(=-v_f(x)\neq 0)$, which is equivalent to the following equation expressed in the inner product form:

$$\langle \boldsymbol{\eta}(x), \boldsymbol{H}(x\ \ X)\rangle = -\frac{1}{2}\bar{\theta}(X) \tag{53}$$

This equation defines an infinite of hyperplanes in E_0 for all X. $\boldsymbol{H}(x, X)$ and $|-\theta(x)/2|$ represent its normal vector and the distance from the origin in an abstract sense, respectively. The common intersection of hyperplanes forms $E_{\bar{\theta}}$. (Refer to Figs. 6 and 9.) The position of the equilibrium point P_1 in this state is

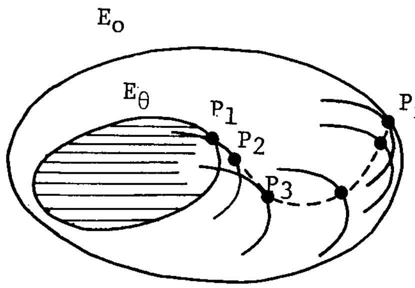

Fig. 9. Mechanical model of consolidation in subset E_0.

expressed in the linear combination of the vectors $\boldsymbol{H}(x, X)$ (Eq. (50)), of which the coefficients (the excess pore water pressure) are determined by Eq. (52). In other words, point P_1 becomes the nearest point in $E_{\bar{\theta}}$ to the origin O (the final state of consolidation). As the consolidation process progresses, the value of the volumetric strain $\bar{\theta}$ varies at all points, namely, set $E_{\bar{\theta}}$ begins to move in E_0. It is the condition of continuity (Eq. (6)) that determines the direction and velocity of such a movement. Since all normal vectors $\boldsymbol{H}(x, X)$ are kept unchanged, all hyperplanes and also $E_{\bar{\theta}}$ itself are subject to parallel translation. Therefore, after a small amount of time Δt, $\bar{\theta}$ changes into

$$\bar{\theta}' = \bar{\theta} + \left(-\frac{1}{\gamma_w}\sum_{i,j} k_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\Delta t \tag{54}$$

and according to Eq. (54), $E_{\bar{\theta}}$ changes into $E_{\bar{\theta}}'$. The condition of equilibrium at this moment is satisfied at P_2, which is the nearest to the origin O among all points of $E_{\bar{\theta}}'$, and also the excess pore water pressure is obtained as the Lagrange multiplier at the same time. Repeating the above procedure, P_3, P_4 ··· are determined successively until set $E_{\bar{\theta}}$ moves so as to contain the origin O (the final state of consolidation process). (Refer to Fig. 9.)

Eq. (52) can be inverted to write the excess pore water pressure explicitly as shown in Eq. (19). Eq. (22), which is obtained by substituting the condition of continuity (Eq. (6)) into Eq. (19), represents the mechanism of consolidation just mentioned above in a single equation.

5. Conclusions

Using the concept of the Function Space, the mechanism of consolidation was studied from the viewpoint of a variational method. The following conclusions were made.

(1) Biot's equations are reduced into a single equation with the excess pore water pressure as the unknown function.

(2) The excess pore water pressure can be regarded as the Lagrange multiplier accompanied with the restriction on the volumetric strain, and it means the potentiality of volume change.

(3) The condition of equilibrium at each stage of consolidation process is satisfied at a point P, which becomes located nearest the final state in the strain set $E_{\bar{\theta}}$.

(4) The position of P is expressed as a linear combination of vectors $\boldsymbol{H}(x, X)$ which define set $E_{\bar{\theta}}$. Each coefficient (Lagrange multiplier) coincides with the excess pore water pressure.

(5) The condition of continuity plays a role in determining the direction and velocity of the translation of $E_{\bar{\theta}}$ according to the distribution of the excess pore water pressure.

(6) The process of consolidation can be interpreted as the trace of P in set E_0, determined by the repetitive procedures of (3) and (5).

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