Eigenvalue Problem of Consolidation

By

Takeshi TAMURA*

(Received September 26, 1979)

Abstract

Following the preceding paper[1], the discussion in this paper will be that the governing equation of the multi-dimensional consolidation expressed in terms of the excess pore water pressure alone, can be treated analytically through the eigenvalue problem similar to that of the one-dimensional case. Especially, we will emphasize the importance of the first eigenvalue for the practical application of this theory.

The main conclusions of this study are:

(1) We can find a set of eigenvalues and eigenfunctions of the multi-dimensional consolidation quite similar to those derived from Terzaghi's one-dimensional equation.

(2) The magnitude of the eigenvalue is proportional to the dissipative energy due to the seepage flow.

(3) The degree of consolidation is mostly determined by the first eigenvalue and therefore it can be used to estimate the effectiveness of the sand drain as an application.

1. Introduction

In the preceding paper[1], Biot's equations of consolidation were reduced to a single governing equation, with the excess pore water pressure as the only unknown function under the assumption of linear stress-strain relations of a clay skeleton. This was applied to investigate the mechanism of a multi-dimensional consolination through the concept of the Function Space. Meanwhile in the present paper, the mathematical treatment of this single equation will be explained by using the theory of the eigenvalue problem, just as in the one-dimensional case. Such a method has a remarkable importance as regards both engineering and mathematical aspects.

Indeed, Fig. 1 shows the degree of consolidation - the time factor curves obtained according to Terzaghi's one-dimensional equation under the uniform distribution of the initial excess pore water pressure. In this figure, the solid line corresponds to the rigorous solution while the dotted line corresponds to the approximated solution only

* Department of Transportation Engineering.

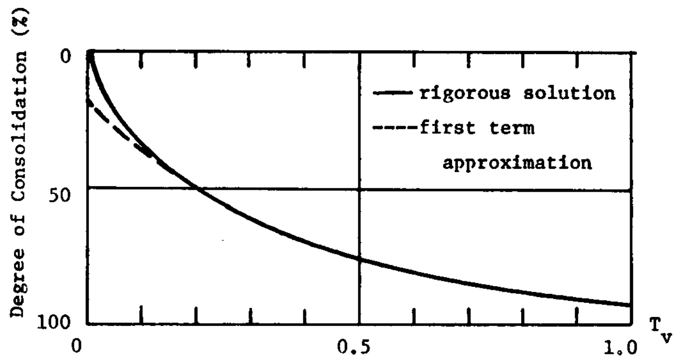


Fig. 1. Degree of consolidation-time factor curve of one-dimensional consolidation.

by the first term of the Fourier series. It can be understood from this figure that the effects of the second and following terms become negligible after the time factor T_v attains about 0. 1. In other words, when we pay attention to the latter part of the consolidation process, it is required not to obtain the complete solution, but to calculate the first eigenvalue and its eigenfunction alone. The main purpose of the present paper is to apply the above idea to a multi-dimensional consolidation which is subject to Biot's equations.

As originated by Biot himself[2], the eigenvalue problems of multi-dimensional consolidation are broaldy investigated and have been frequently applied to construct the closed form solution of Biot's equations. In particular, Mandel's pioneering work[3] is remarkably evaluated. However, the formulation of the eigenvalue problem in these studies remains quite complicated since they treat directly Biot's original equations. However, according to the present method, we are able to not only understand the meaning of eigenvalue and eigenfunction, but also formulate the mathematical theory more simply.

In this paper, firstly, the single governing equation suitable for the eigenvalue problem of consolidation will be introduced, which is equivalent to that reduced in the preceding paper. Secondly, the theory of the eigenvalue problem will be explained. Finally, we will apply it to the problem of an axi-symmetric circular region and to the numerical estimation of the effectiveness of sand drain.

Several analyses of consolidation have been published recently, but there remain several questions concerning the agreement of the calculated values with the observed values in the field. In such circumstances, the statistical method[4] for prediction of ground settlement has made remarkable progress. This is regarded as a method to extract several eigenvalues of velocity of settlement from the observed data. Meanwhile, in the present paper, the eigenvalue of the consolidation process will be evaluated from a mechanical point of view, in order to predict the settlement or the dominant distribution of excess pore water pressure.

2. Basic Equation for Eigenvalue Problem of Consolidation

In this section, the basic equation of multi-dimensional consolidation for the eigenvalue problem will be deduced from Biot's equations. Since all of the assumptions, notations and mathematical treatment of this part are almost the same as those of the preceding paper, we will explain briefly without a detailed commentary.

Choosing the final steady state as the reference state, Biot's equations of consolidation are written as

$$\sum \frac{\partial \tau'_{ij}}{\partial x_j} + \frac{\partial u}{\partial x_i} = 0 \qquad (i=1, 2, 3) \tag{1}$$

$$\frac{\partial \theta}{\partial t} = -\frac{1}{\gamma_w} \sum k_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{2}$$

in which τ'_{ij}=effective stress, u=excess pore water pressure, θ=volumetric strain, k_{ij}= permeability, γ_w=weight of pore water per unit volume and x_i=cartesian coordinate.

Associated with Eqs. (1) and (2), the following homogeneous boundary conditions are set up:

(B_{u0}) boundary condition concerning displacement v_i:

$$v_i = 0 \qquad (i=1, 2, 3) \text{ on } S_u$$

($B_{\sigma 0}$) boundary condition concerning stress traction T_i:

$$T_i = 0 \qquad (i=1, 2, 3) \text{ on } S_\sigma$$

(B_{D0}) boundary condition concerning drainage:

$$\sum k_{ij} \frac{\partial u}{\partial x_j} n_i = 0 \text{ on } S_{UD}, \qquad u=0 \text{ on } S_D$$

in which v_i=displacement, T_i= (total) stress traction, n_i= unit outward normal vector and S_u, S_σ, S_D and S_{UD} denote the displacement boundary, the stress boundary, the drained boundary and the undrained boundary, respectively.

Because of the linear stress-strain relations of the clay skeleton and the homogeneous boundary conditions, the relation between the displacement and the excess pore water pressure, which is determined by Eq. (1), is given as

$$v_i(x) = \int_V U_i(x, X) u(X) dV_X \qquad (i=1, 2, 3) \tag{3}$$

in which $U_i(x, X)$ means the displacement generated by $u=\delta(x, X)$ (Dirac's delta function), the subscript X of dV_X means the integral variable and x and X denote arbitrary points in the region.

Taking the divergence operation on both sides of Eq. (3) with respect to the variable x (x_i), the following equation is deduced:

$$\theta(x)=\int_V \Theta(x,\ X)u(X)dV_X \tag{4}$$

in which

$$\Theta(x,\ X)=\sum\frac{\partial U_i(x,\ X)}{\partial x_i} \tag{5}$$

Eq. (4) means the linear relation between the distribution of the volumetric strain and that of the excess pore water pressure, observed from the final steady state of the consolidation process. This equation has a unique inverse relation:

$$u(x)=\int_V \Phi(x,\ X)\theta(X)dV_X \tag{6}$$

which was used in the preceding paper to derive the single governing equation by combining it with the continuity condition (Eq. (2)):

$$\frac{\partial u}{\partial t}(x,\ t)=-\frac{1}{\gamma_w}\int_V \Phi(x,\ X)\sum k_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(X,\ t)dV_X \tag{7}$$

It is also reasonable to deduce the following single governing equation from Eq. (4):

$$\int_V \Theta(x,\ X)\frac{\partial u}{\partial t}(X,\ t)dV_X=-\frac{1}{\gamma_w}\sum k_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(x,\ t) \tag{8}$$

Eqs. (7) and (8) are essentially equivalent to each other. The former is useful for investigating the structure of Biot's equations through a comparison with Terzaghi's equation, since it represents the time rate of the excess pore water pressure explicitly. However, we will adopt the latter one exclusively for the consideration of the eigenvalue problem.

3. Eigenvalue Problem in Consolidation Phenomenon

3. 1 Equation of eigenvalue problem

Decomposing the excess pore water pressure $u(x,\ t)$ into the product of functions of space variable x and time t:

$$u(x,\ t)=u_\alpha(x)e^{-\lambda_\alpha t} \tag{9}$$

and substituting into Eq. (8), we have the following homogeneous equation for $u_\alpha(x)$ with a scalar parameter λ_α:

$$\lambda_\alpha\int_V \Theta(x,\ X)u_\alpha(X)dV_X=\frac{1}{\gamma_w}\sum k_{ij}\frac{\partial^2 u_\alpha(x)}{\partial x_i \partial x_j} \tag{10}$$

λ_α and $u_\alpha(x)$ are, respectively, regarded as the eigenvalue and the eigenfunction, if Eq. (10) has a non-trivial solution under the homogeneous boundary condition (B_{D0}). As is understood from Eq. (9), an eigenfunction $u_\alpha(x)$ means a distribution of excess pore water pressure which never changes its shape but, decays exponentially with time t according to the magnitude of the eigenvalue λ_α. It is obvious that thc eigenfunction associated with a large eigenvalue decays quickly and that the reverse is also true. Hereafter, Eq. (10) will be used as the equation of the eigenvalue problem, which is the basis of the following discussions.

3. 2 Some properties of eigenvalues and eigenfunctions

(1) Positiveness of eigenvalue ($\lambda_\alpha>0$)

Multiplying both sides of Eq. (10) by $u_\alpha(x)$ and integrating over the whole region V, we have

$$\lambda_\alpha\iint_V \Theta(x,\ X)\,u_\alpha(x)\,u_\alpha(x)\,dV_x dV_X=\frac{1}{\gamma_w}\int_V\left(\sum k_{ij}\,\frac{\partial^2 u_\alpha}{\partial x_i\partial x_j}\right)u_\alpha dV_x \tag{11}$$

Firstly, the double integral on the left hand side is transformed as follows:

$$\begin{aligned}
&\iint_V\Theta(x,\ X)\,u_\alpha(x)\,u_\alpha(x)\,dV_x dV_X=\int_V\theta_\alpha(x)\,u_\alpha(x)\,dV\\
&=\int_V\sum\frac{\partial v_{\alpha i}(x)}{\partial x_i}u_\alpha(x)\,dV_x=\int_{S_u+S_\sigma}(\sum_i v_{\alpha i}n_i)\,u_\alpha dS-\int_V\sum v_{\alpha i}\frac{\partial u_\alpha}{\partial x_i}dV_x\\
&=\int_{S_u+S_\sigma}(\sum v_{\alpha i}n_i)\,u_\alpha dS+\int_V\sum v_{\alpha i}\frac{\partial\tau'_{\alpha ij}}{\partial x_j}dV_x\\
&=\int_{S_u+S_\sigma}\sum(\tau'_{\alpha ij}+u_\alpha\delta_{ij})\,n_j v_{\alpha i}dS-\int_V\sum\frac{\partial v_{\alpha i}}{\partial x_j}\tau'_{\alpha ij}dV_x\\
&=-2\int_V\frac{1}{2}(\sum\tau'_{\alpha ij}\eta_{\alpha ij})\,dV_x=-2E_\alpha
\end{aligned}\tag{12}$$

in which we have used the definition of $\Theta(x,\ X)$ (Eq. (4)) for the first equal sign, the definition of $\theta_\alpha(x)$ for the second one, the integration by parts for the third one, the condition of equilbrium (Eq. (1)) for the fourth one, again the integral by parts for the fifth one, the homogeneous boundary conditions (B_{u0}), $(B_{\sigma 0})$ and the definition of strain for the sixth one and the definition of strain energy for the final one, respectively. θ_α, $v_{\alpha i}$, $\tau'_{\alpha ij}$, $\eta_{\alpha ij}$ and E_α denote, respectively, the volumetric strain, the displacement, the effective stress, the strain and the strain energy caused by the excess pore water pressure u_α under homogeneous boundary conditions. Therefore, the above result shows that the value of the double integral of Eq. (11) is always negative. In order to simplify the latter explanation, the following inner product is defined here in the set of distributions of excess pore water pressure:

$$[u_\alpha, u_\beta]=-\frac{1}{2}\iint_V \Theta(x, X)u_\alpha(x)u_\beta(X)dV_x dV_X \tag{13}$$

in which u_α and u_β satisfy the homogeneous boundary condition concerning drainage (B_{D0}). Meanwhile, the right hand side of Eq. (11) is transformed through the integration by parts into:

$$\frac{1}{\gamma_w}\int_V\left(\sum k_{ij}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\right)u_\alpha dV=-\frac{1}{\gamma_w}\int_V\sum k_{ij}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_j}dV \tag{14}$$

which means thc time rate of the dissipative energy due to the seepage flow. It is always negative. Then, the eigenvalue λ_α is proved to be positive from Eq. (11). Rewriting Eq. (11) as

$$2\lambda_\alpha=\frac{\frac{1}{\gamma_w}\int_V\sum k_{ij}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_j}dV}{[u_\alpha, u_\alpha]} \tag{15}$$

it is understood that $2\lambda_\alpha$ signifies the ratio of the dissipative energy to the total strain energy, when the excess pore water pressure u_α exists alone.

It should be noted that the region V is assumed to be bounded in order to guarantee the existence of eigenvalues[5].

(2) Orthogonality of eigenfunctions

u_α and u_β denote the eigenfunctions associated with two eigenvalues λ_α and λ_β, i.e.,

$$\lambda_\alpha\int_V \Theta(x, X)u_\alpha(x)dV_X=\frac{1}{\gamma_w}\sum k_{ij}\frac{\partial^2 u_\alpha(x)}{\partial x_i \partial x_j} \tag{16}$$

$$\lambda_\beta\int_V \Theta(x, X)u_\beta(X)dV_X=\frac{1}{\gamma_w}\sum k_{ij}\frac{\partial^2 u_\beta(x)}{\partial x_i \partial x_j} \tag{17}$$

Multiplying both sides of these equations by $-u_\beta(x)/2$ and $-u_\alpha(x)/2$, respectively, we integrate over the whole region. By subtracting them, the following relation is obtained.

$$(\lambda_\alpha-\lambda_\beta)[u_\alpha, u_\beta]=-\frac{1}{2\gamma_w}\int_{S_D+S_{UD}}\left\{\left(\sum k_{ij}\frac{\partial u_\alpha}{\partial x_i}n_j\right)u_\beta-\left(\sum k_{ij}\frac{\partial u_\beta}{\partial x_i}n_j\right)u_\alpha\right\}dV \tag{18}$$

in which Green's integral formula is used. Because of the homogeneous boundary condition (B_{D0}), it can be easily concluded that if $\lambda_\alpha \neq \lambda_\beta$, then

$$[u_\alpha, u_\beta]=0 \tag{19}$$

which means the orthogonality of the eigenfunctions associated with the different eigenvalues. In other words, the stress state generated by u_α produces no work on the strain field generated by u_β if $\lambda_\alpha \neq \lambda_\beta$. For the sake of simplicity, all eigenfunctions are assumed to be normalized as follows

$$[u_\alpha, u_\alpha]=-\frac{1}{2}\iint_V \Theta(x, X)u_\alpha(x)u_\alpha(x)dV_x dV_X=1 \tag{20}$$

which signifies that the amount of strain energy due to the eigenfunction u_α is unit. If there are more than two eigenfunctions for the same eigenvalue, they are orthonormalized by Schmidt's method. Therefore, it is possible to suppose, without loss of generality, that the set of eigenfunctions constitutes a system of orthonormal basis, i. e.,

$$[u_\alpha,\ u_\beta]=\delta_{\alpha\beta} \tag{21}$$

(3) Expression of general solution in terms of eigenfunctions

By definition, each function $u_\alpha(x)e^{-\lambda_\alpha t}$ $(\alpha=1,\ 2,\cdots)$ is a solution of Eq. (8), which satisfies the homogeneous boundary condition (B_{D0}). In order to also satisfy the initial condition of the excess pore water pressure, it is necessary to superimpose an infinite of such solutions as:

$$u(x,\ t)=\sum_{\alpha=1}^{\infty}a_\alpha u_\alpha(x)e^{-\lambda_\alpha t} \tag{22}$$

whereby the (Fourier) coefficients are determined by the following method. The initial distribution of excess pore water pressure $u_0(x)$ should be obtained under an undrained condition, but it is here supposed to have been already done. Putting $t=0$ in Eq. (22), we have

$$u(x,\ 0)=u_0(x)=\sum_{\alpha=1}^{\infty}a_\alpha u_\alpha(x) \tag{23}$$

Calculating the inner product $[u_0,\ u_\beta]$ and using the condition of orthogonality (Eq. (21)), Eq. (23) yields

$$[u_0,\ u_\beta]=[\sum a_\alpha u_\alpha,\ u_\beta]=\sum a_\alpha\delta_{\alpha\beta}=a_\beta \tag{24}$$

Namely, the Fourier coefficient a_α is determined as

$$a_\alpha=[u_0,\ u_\alpha]=-\frac{1}{2}\iint_V\Theta(x,\ X)u_0(x)u_\alpha(X)dV_x dV_X \tag{25}$$

and the general solution is expressed in the following form:

$$u(x,\ t)=\sum_{\alpha=1}^{\infty}[u_0,\ u_\alpha]u_\alpha(x)e^{-\lambda_\alpha t} \tag{26}$$

Therefore, it is necessary only to obtain the eigenvalues and the eigenfunctions in order to construct the solution of the consolidation problem, if the initial pore water pressure is known.

(4) Meaning of Fourier coefficient a_α

Calculating the inner product $[u,\ u]$ of the solution (Eq. (22)), we have so-called Parceval's identity:

$$[u,\ u]=[\sum a_\alpha u_\alpha e^{-\lambda_\alpha t},\ \sum a_\alpha u_\alpha e^{-\lambda_\alpha t}]=\sum a_\alpha^2 e^{-2\lambda_\alpha t} \tag{27}$$

which shows that the residual strain energy $[u,\ u]$ at t is expressed in the form of a summation of the strain energy carried by eigenfunctions, and that their velocities of decay are $e^{-2\lambda_\alpha t}$.

The degree of consolidation of the one-dimensional case is usually defined as the ratio of the algebraic sum of the residual excess pore water pressure to that of the initial one. There is no well-established definition for the degree of multi-dimensional consolidation. One reasonable definition is the method which uses the concept of strain energy as follows:

$$U(t)=1-\frac{E}{E_0}=1-\sqrt{\frac{\sum[u_0,\ u_\alpha]^2 e^{-2\lambda_\alpha t}}{[u_0,\ u_0]}} \tag{28}$$

in which E and E_0 are, respectively, the residual and the initial strain energy. For a large t, it becomes approximately

$$U(t)\doteqdot 1-\frac{|[u_0,\ u_1]|}{\sqrt{E_0}}e^{-\lambda_1 t} \tag{29}$$

which shows that the degree of consolidation at the latter stage depends upon the first eigenvalue and the strain energy carried by the first eigenfunction.

4. Applications

4.1 Consolidation of axi-symmetric circular region

We will apply the theory explained above to the consolidation of the axi-symmetric circular region as a simple example. This problem has been solved by Omaki[6], using the Laplace transformation method, but the solution will be presented along the eigenvalue problem. The clay material is assumed to be isotropic for simplicity.

After a little calculation, it is easy to obtain $\Theta(x, X)$ (defined by Eq. (5)) for the circular region of radius R as

$$\Theta(r,\ s)=-\frac{1}{\lambda+2\mu}\left\{\delta(r,\ s)+\frac{2\mu}{\lambda+\mu}\ \frac{s}{R^2}\right\} \tag{30}$$

in which r and s denote arbitrary points in the radial coordinate and λ, μ denote Lame's constants. Eq. (30) means the distribution of the volumetric strain which is generated by the excess pore water pressure on a circle $r=s$, i. e., $u=\delta(r, s)$. Substituting Eq. (30) into Eq. (8), we have the following equation for the axi-symmytric circular region:

$$\frac{\partial u}{\partial t}+2(1-2\nu)\frac{1}{R^2}\int_0^1 s\frac{\partial u}{\partial t}(s)dS=\frac{2(1-\nu)\mu}{1-2\nu}\ \frac{k}{\gamma_w}\Delta^2 u \tag{31}$$

in which $\nu=$Poisson's ratio and $k=$isotropic permeability.

Before the theory of the eigenvalue problem is applied to Eq. (31), several values are modified into the non-dimensional form:

$$\frac{2(1-\nu)\mu}{1-2\nu}\frac{k}{\gamma_w}\frac{t}{R^2}\to t\,(\text{time factor})$$

$$2(1-2\nu)\to c,\quad \frac{r}{R},\ \frac{s}{R}\to r,\ s$$

Then Eq. (31) is simplified as

$$\frac{\partial u}{\partial t}+c\int_0^1 s\frac{\partial u}{\partial t}ds=\varDelta^2 u \tag{32}$$

Writing $\lambda_\alpha=\eta_\alpha^2$ and substituting

$$u(r,\ t)=u_\alpha(r)e^{-\eta_\alpha^2 t} \tag{33}$$

into Eq. (32), we have the equation of the eigenvalue problem:

$$\frac{d^2u_\alpha}{dr^2}+\frac{1}{r}\frac{du_\alpha}{dr}+\eta_\alpha^2[u_\alpha+cI_\alpha]=0 \tag{34}$$

in which

$$I_\alpha=\int_0^1 su_\alpha(s)ds \tag{35}$$

The boundary condition for Eq. (34) is

$$(B_{D0})\qquad u(r=0)<+\infty,\quad u(r=1)=0$$

η_α is determined so that $u_\alpha(x)$ exists satisfying Eq. (34) and the above boundary condition (B_{D0}). Noting cI_α is a constant, it is easy to understand that the general solution of Eq. (34) is constructed by the following two independent solutions:

$$J_0(\eta_\alpha r)-cI_\alpha,\qquad Y_0(\eta_\alpha r)-cI_\alpha$$

in which J_0 and Y_0 denote the first and second kinds of Bessel functions of the zeroth order.

Howerer, the latter is inappropriate because it becomes infinitive if $r\to 0$. There fore, the solution of Eq. (34) is written as follows because the magnitude of the coefficient is not essential:

$$u_\alpha(r)=J_0(\eta_\alpha r)-cI_\alpha \tag{36}$$

in which η_α and I_α are unknown, and they are determined by satisfying the boundary condition ($u_\alpha(r=1)=0$) and Eq. (35). From the former, we have

$$cI_\alpha=J_0(\eta_\alpha) \tag{37}$$

which is substituted into Eq. (36) to write the eigenfunction:

$$u_\alpha(r)=J_0(\eta_\alpha r)-J_0(\eta_\alpha) \tag{38}$$

Further substituting Eq. (38) into Eq. (35) and calculating the integral, the equation

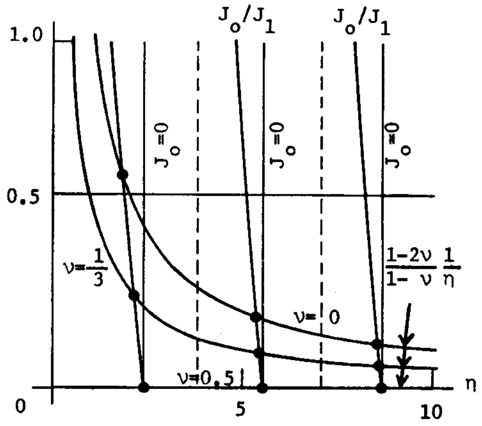


Fig. 2. Roots of eigenvalue equation in the consolidation of circular region.

to determine the eigenvalue $\lambda_\alpha(=\eta_\alpha^2)$ is finally obtained as follows:

$$\frac{J_0(\eta_\alpha)}{J_1(\eta_\alpha)}=\frac{1-2\nu}{1-\nu}\ \frac{1}{\eta_\alpha} \tag{39}$$

in which J_1 denotes the Bessel function of the first order.

For all η_α saisfying the above equation, the function $u(r, t)$ defined by Eqs. (33) and (38) is a solution of Eq. (32) under the homogeneous boundary condition (B_{D0}). It should be noted that Eq. (39) is identical with that obtained by Omaki in the other method. We now investigate the values of η_α practically for several Poisson's ratios ($\nu=0$, 0.33, 0.5). Fig. 2 shows the curves expessed by both sides of Eq. (39). As the simplest case, $\lambda_\alpha(=\eta_\alpha^2)$ is determined by

$$J_0(\eta_\alpha)=0 \tag{40}$$

when $\nu=0.5$, namely, when Biot's equations are reduced to Tezaghi's single equation. For each α, the eigenvalue becomes larger as $\nu\rightarrow 0.5$, but regardless of Poisson's ratio, the eigenvalue (therefore also the eigenfunction) approaches asymptotically that of Terzaghi's type as $\alpha\rightarrow\infty$. Table 1 shows the eigenvalues λ_α for $\alpha=1, 2,\cdots, 5$. The

Table 1. Eigenvalue λ_α in the consolidation of circular region.

α	$\nu=0$	$\nu=0.33$	$\nu=0.5$
1	3.390	4.691	5.783
2	28.424	29.457	30.472
3	72.868	73.881	74.887
4	137.030	138.037	139.039
5	220.927	221.930	222.932

whole set of u_α (Eq. (38)) associated with the individual eigenvalues constitute a system of eigenfunctions, and saitsfy the relation of orthogonality (Eq. (21)). Taking into consideration Eq. (30), the inner product defined by Eq. (13) is written in this case as follows:

$$[u_\alpha, u_\beta] = \frac{1}{2}\left\{\int_0^1 r u_\alpha(r) u_\beta(r) dr + c I_\alpha I_\beta\right\} \tag{41}$$

It is also necessary to normalize the eigenfunction into $u_\alpha/\sqrt{[u_\alpha, u_\alpha]}$ so as to satisfy Eq. (20). Representing the normalized eigenfunction by $u_\alpha(r)$ again, we then have

$$u_\alpha(r) = \frac{J_0(\eta_\alpha r) - J_0(\eta_\alpha)}{\sqrt{\frac{J_0^2(\eta_\alpha)}{4} + \frac{J_1^2(\eta_\alpha)}{4} - \frac{1}{2\eta_\alpha} J_0(\eta_\alpha) J_1(\eta_\alpha)}} \tag{42}$$

Fig. 3 shows a few normalized eigenfunctions for $\alpha = 1$, 2. Fig. 4 shows the degree of the consolidation-time factor curves calculated from the solutions (Eq. (26)), assuming the uniform initial excess pore water pressure. In this figure, the solid lines correspond to the usual degree of consolidation while the dotted lines correspond to that defined by using the ratio of strain energy (Eq. (28)). The latter curves show slightly small values, but they have a tendency quite similar to the usual ones. Therefore $U(t)$,

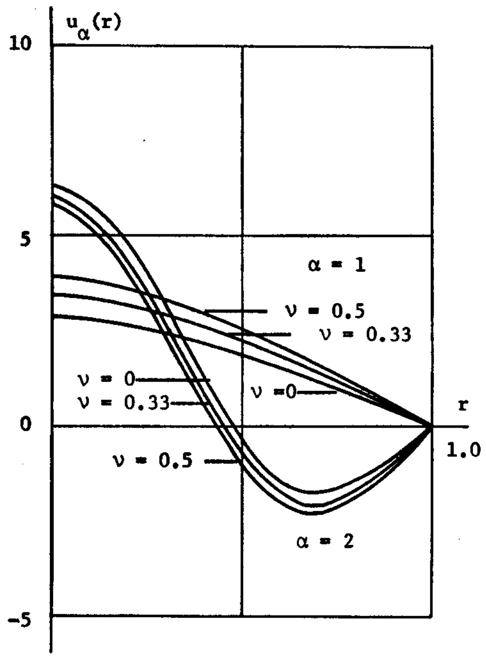


Fig. 3. Normalized eigenfunctions in the consolidation of circular region.

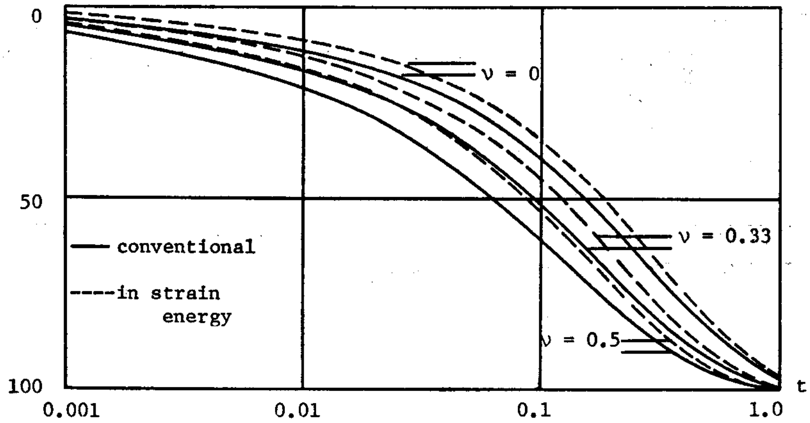


Fig. 4. Degree of consolidation-time factor curves in the consolidation of circular region.

defined by Eq. (28), can be used as an index which represents the degree of multi-dimensional consolidation.

When the time factor t exceeds 0.2, the solution can be approximated only by the following first term:

$$u(r,\ t) \fallingdotseq 1.96\{J_0(2.17r)-0.129\}e^{-4.69t} \quad [\nu=0.33] \tag{43}$$

4. 2 Numerical analysis of eigenvalue

It is so difficult to obtain the eigenvalues and the eigenfunctions analytically for the general region that numerical techniques are unavoidable. Reducing what was discussed in **2.** and **3.** into the numerical analysis of the eigenvalue by means of the finite element method, we will explain the results of an example concerning the consolidation with sand drain.

4. 2. 1 Calculation of eigenvalue by means of finifte element method

Biot's equations, (Eqs. (1) and (2)), cad be converted into the forllowing matrix form by the method of Christian and Boehmer[7]:

$$\boldsymbol{Kv}+\boldsymbol{Lw}=\boldsymbol{0} \tag{44}$$

$$\frac{d}{dt}\boldsymbol{V}=-\boldsymbol{Yw} \tag{45}$$

in which

$\boldsymbol{K}$: stiffness matrix

$\boldsymbol{v}$: vector of whole nodal displacements

$\boldsymbol{w}$: vector of excess pore water pressures of whole elements

$\boldsymbol{V}$: vector of volume changes of whole elements

$\boldsymbol{L}$: transformation matrix from $\boldsymbol{w}$ to equivalent nodal forces

$\boldsymbol{Y}$: transformation matrix from $\boldsymbol{w}$ to $\boldsymbol{V}$ according to Darcy's law

These values are calculated when the finite element mesh, the boundary conditions and the material constants are given. Solving $\boldsymbol{V}$ from Eq. (44) and considering

$$\boldsymbol{V}=\boldsymbol{L}^T\boldsymbol{v} \tag{46}$$

$\boldsymbol{V}$ is expressed in terms of $\boldsymbol{w}$ as follows:

$$\boldsymbol{V}=-(\boldsymbol{L}^T\boldsymbol{K}^{-1}\boldsymbol{L})\boldsymbol{w} \tag{47}$$

And substituting it into Eq. (45), the equation for $\boldsymbol{w}$ is obtained:

$$\boldsymbol{A}\frac{d\boldsymbol{w}}{dt}=-\boldsymbol{Y}\boldsymbol{w} \tag{48}$$

in which T means the transpose of matrix, and the matrix A:

$$\boldsymbol{A}=-\boldsymbol{L}^T\boldsymbol{K}^{-1}\boldsymbol{L} \tag{49}$$

has such column vectors which represent the volume changes of elements caused by a unit intensity of the excess pore water pressure at an element. It is easy to see that Eqs. (47) and (48) correspond to Eqs (4) and (8), respectively. When we substitute

$$\boldsymbol{w}(t)=\boldsymbol{w}_\alpha e^{-\lambda_\alpha t} \tag{50}$$

into Eq. (48), the equation of the eigenvalue is written as follows:

$$\lambda_\alpha \boldsymbol{A}\boldsymbol{w}_\alpha=\boldsymbol{Y}\boldsymbol{w}_\alpha \qquad (\alpha=1,\ 2,\ \cdots,\ m) \tag{51}$$

in which m means the total number of all elements in the region. If Eq. (51) has a non-trival solution, then λ_α and $\boldsymbol{w}_\alpha$ are called the eigenvalue and the eigenvector of consolidation, respectively. The matrices $\boldsymbol{A}$ and $\boldsymbol{Y}$ are proved to be not only symmetric but also negative definite. Therefore, the eigenvalue λ_α is always positive and the eigenvectors constitute a set of orthonormal basis, i. e.,

$$[\boldsymbol{w}_\alpha,\ \boldsymbol{w}_\beta]=\delta_{\alpha\beta} \tag{52}$$

in which

$$[\boldsymbol{w}_\alpha,\ \boldsymbol{w}_\beta]=-\frac{1}{2}\boldsymbol{w}_\alpha^T\boldsymbol{A}\boldsymbol{w}_\beta \tag{53}$$

For an arbitrary initial excess pore water pressure $\boldsymbol{w}_0$, the solution can be expressed as follows:

$$\boldsymbol{w}(t)=\sum_{\alpha=1}^{m}[\boldsymbol{w}_0,\ \boldsymbol{w}_\alpha]\,\boldsymbol{w}_\alpha e^{-\lambda_\alpha t} \tag{54}$$

The meaning of the Fourier coefficient $[\boldsymbol{w}_0,\ \boldsymbol{w}_\alpha]$ is similar to what was explained in **3.**

4. 2. 2 Estimation of effectiveness of sand drain

The sand drain method is one of the most popular ways to improve soft clay ground. Its major purpose is to obtain the strength of clay during a comparatively short time by facilitating the consolidation progress. The effectiveness of sand drain is usually estimated by Barron's method which is based upon the multi-dimensional Terzaghi's equation, and this method is evaluated to some extent by the model test. However, there are few investigations concerning the effects of perneability or stiffness of a sand column. As was mentioned proviously, the consolidation process is ruled by the first eigenvalue and its function. Therefore, it is reasonable to estimate the effectiveness of the sand drain by the magnitude of the first (the smallest) eigenvalue

In this section, the first eigenvalue is calculated for the axisymmetric region divided into the finite element mesh as shown in Fig. 5. The pore water is assumed to be drained through the upper surface, and the other boundary conditions are shown in the same figure.

Young's modulus and permeability of the clay region are denoted by E_c and k_c, and those of the sand column by E_s and k_s, respectively. Poisson's ratio ν is 0. 33 in the whole region. Denoting the depth of the region by H, the outer radius by R_0 and the radius the of sand column by R_i, the following parameters are defined:

$$n=\frac{R_0}{R_i} \tag{55}$$

$$h=\frac{R_0}{H} \tag{56}$$

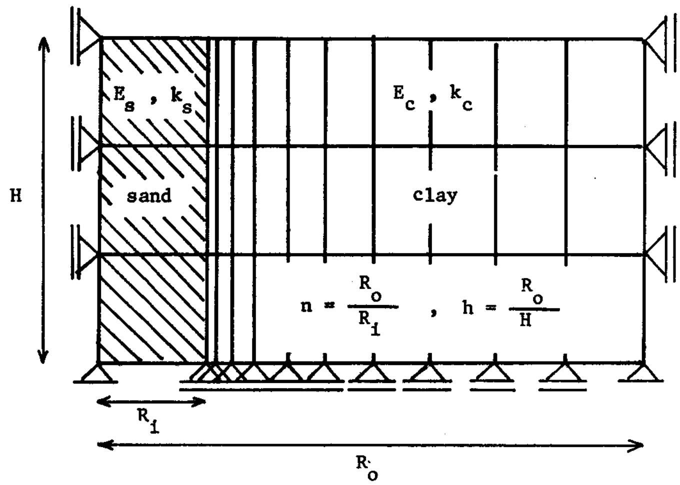
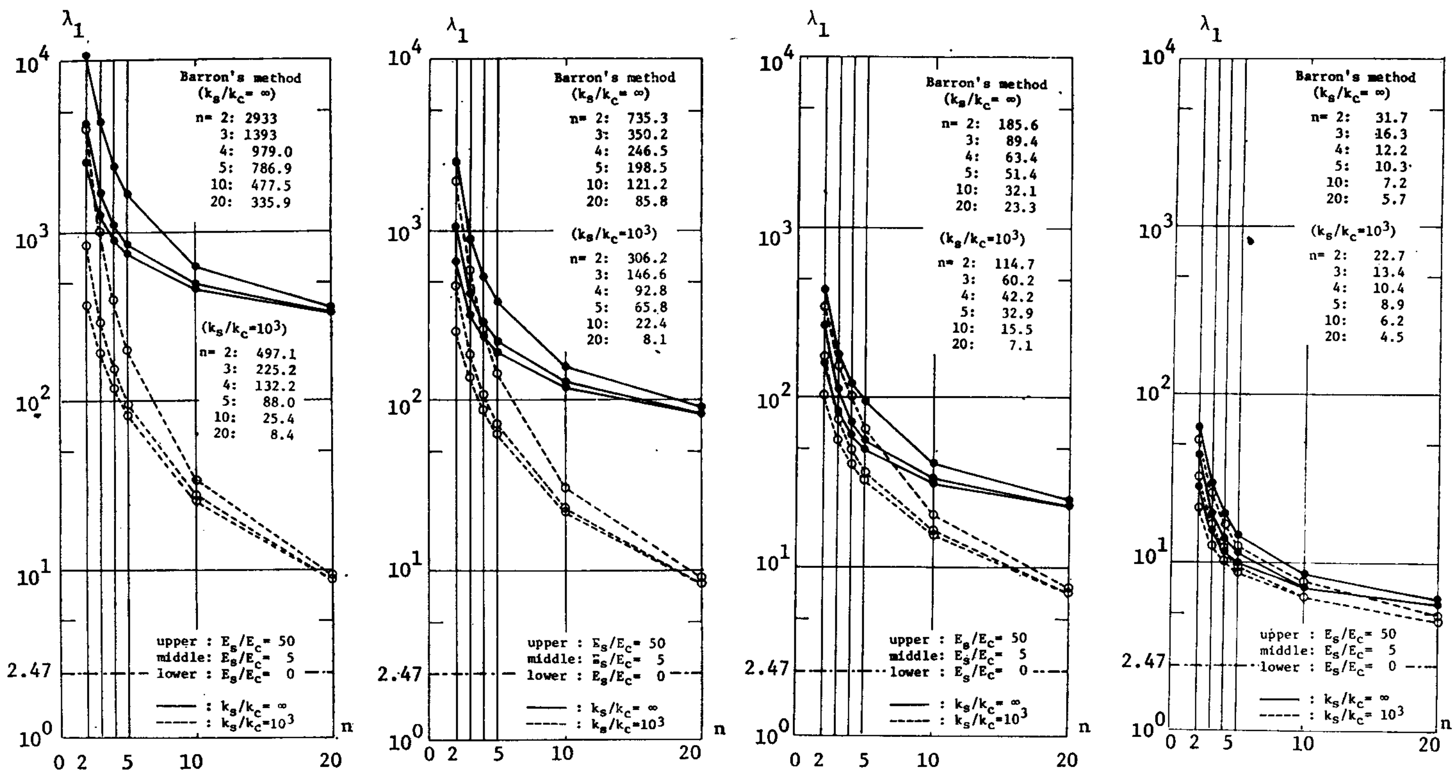


Fig. 5. Finite element mesh and boundary conditions for the sand drain model.



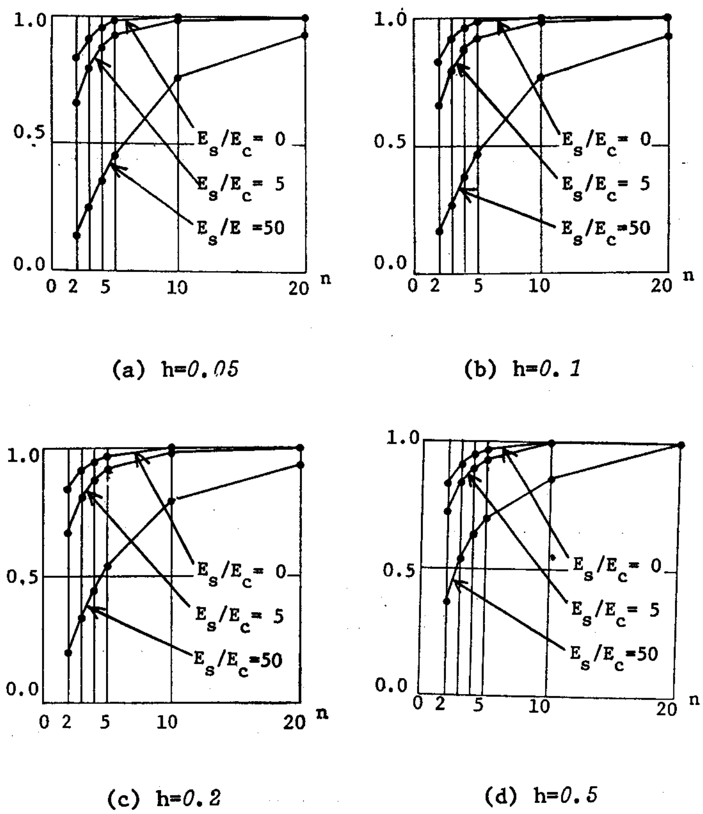
(a) h=0.05 (b) h=0.1 (c) h=0.2 (d) h=0.5

Fig. 6. First eigenvalue λ_1 in sand drain model.

The time t in the following figures is replaced by the usual time factor of a one-dimenesional consolidation of the clay region:

$$T_v = \frac{k_c}{\gamma_w} \frac{(1-\nu)E_c}{(1-2\nu)(1+\nu)} \frac{t}{H^2} \tag{57}$$

Comparing the first eigenvalues calculated by the present method with Barron's rigorous solutions, the accuracy of the calculation was checked. As a result, its error is proved to be practically negligilbe. Fig. 6 shows the variation of the first eigenvalue due to the change of n and E_s/E_c when $h=0.05$, 0.1, 0.2 and 0.5, respectively. In this figure, the solid lines correspond to the case of $k_s/k_c=\infty$, and the dotted lines to $k_s/k_c=10^3$ The broken lines denote the first eigenvalue of the original clay ground ($\lambda_1=2.47$). The numerical values in the same figure are calculated values based upon Terzaghi's equation, which correspond to those of Barron's method. In order to estimate the increase of strength, Fig. 7 shows the mean values of the volumetric strain of the clay region generated by the uniform upper load for each case of Fig. 6,



(a) h=0.05 (b) h=0.1

(c) h=0.2 (d) h=0.5

Fig. 7. Normalized mean volumetric strain in sand drain model.

in which the values are normalized by those of the original ground. Form these figures, the following conclusions are remarked.

(1) The first eigenvalue of Barron's method locates between those of the cases $E_s/E_c=0$ and 5.

(2) For the case of $n>10$, the stiffness of the sand column does not cause serious effects upon the first eigenvalue or the volumetric strain.

(3) When $n<5$, the degree of consolidation becomes remarkably large as E_s/E_c increases, but on the other hand, the value of compression becomes small. Namely, the effect of the sand compaction pile appears considerably for large E_s/E_c.

(4) There can be seen a serious influence of the permeability of the sand column upon the rate of consolidation for a small h.

(5) When h increases to about 0.5, the lateral drainage distance of the pore water becomes comparatively large. Therefore, neither the stiffness nor the permeability of the sand column causes a large effect upon the consolidation, since the time necessary for such lateral flow of pore water is predominant.

5. Conclusions

Using the governing equation expressed in terms of the excess pore water pressure, the following conclusions are noted concerning the eigenvalue problems of consolidation and their applications.

(1) There exist a set of eigenvalues λ_α and eigenfunctions $u_\alpha(\alpha=1, 2, \cdots)$ in multi-dimensional consolidation quite similar to those in the one-dimensional case. Namely, each eigenvalue λ_α is a positive real number, and any two eigenfunctions satisfy the orthogonal relation in the sense of the inner product defined by Eq. (13).

(2) The magnitude of the eigenvalue λ_α is proportional to the dissipative energy due to the seepage flow caused by the excess pore water pressure u_α.

(3) In most stages of the consolidation process, except for the early part, the same figure of distribution of excess pore water pressure to the first eigenfunction u_1 is predominant.

(4) The degree of consolidation is mostly determined by the first eigenvalue.

(5) As an index of consolidation progress, we can define the degree of consolidation by using using the concept of residual strain energy.

(6) In the consolidation of spherical and circular regions, the total mean stress is kept constant in the mean sense over these domalns.

(7) The eigenvalue of the composite ground, such as the sand drain model, can be easily calculated by meas of the finite element method.

Acknowledgement

The author expresses his heartfelt thanks to Professor Koichi Akai of Kyoto University for his supervision of this study.

References

1) Tamura, T.: A Study on the Mechanism on Consolidation, Mem. Fac. Eng., Kyoto Univ., Vol. XII, No. 3 pp. 518-535, 1979.
2) Biot, M. A.: General Solutions of the Equations of Elasticity and Consolidation, Journ. Appl. Mech., Vol. 78, pp. 91-96, 1956.
3) Mandel, J.,: Consolidation des Sols (Étude Mathématique), Géotechnique, Vol. 3, pp. 287-299, 1953.
4) Asaoka, A.: Observational Procedure of Settlement Prediction, Soils and Foundations, Vol. 18, No. 4, pp. 87-101, 1978.
5) Mizohata, S.: Theory of Partial Differential Equation, Modern Mathematics, No. 9, Iwanami-shoten, 1965 (in Japanese).
6) Omaki, S.,: On the Three Dimensional Consolidation Theory of Clay, D. P. R. I. Annuals, Kyoto Univ., No. 14B, pp. 585-601, 1971 (in Japanese).
7) Christian, J. T. and J. W. Boehmer: Plane Strain Consolidation by Finite Elements, Proc. ASCE, Vol. 96, SM 4, pp. 1435-1457, July, 1970.