A Simple Theory for Heat and Mass Transfer in a Compressible Tubulent Boundary Layer on a Flat Plate with Foreign Gas Transpiration

By

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Summary

As a preparatory study of ablation cooling or transpiration of combustible gases into a boundary layer, a simple theory has been developed for the calculation of skin-friction, heat transfer and mass transfer coefficients of a flat plate turbulent boundary layer transpired with foreign gas. For this purpose, the theory previously developed by the author has been extended. The extension was made in two points, namely that the assumption on the Prandtl and Schmidt number with their values being unity was removed, and that a compressibility of fluid was successfully taken into account. The extended theory is also similar to those by Rubesin and Pappas, and by Denison.

However, an improvement has been made in that any number of chemical species is allowed to compose the fluid. This is very neccessary in order to apply the theory to the problems mentioned above. The results abtained are in fairly good agreement with the available experimental results. Also, the computational time is believed to be much shorter than that needed in similar numerical colculations by Landis and Mills.

1. Introduction

Combustible gas transpiration into a boundary layer, or ablation cooling, is a familiar technique for the protection of a re-entry space vehicle from kinetic heating. This type of boundary layer may be found in other fields of technology. For a preparatory study of the boundary layer, a flat plate, non-isothermal turbulent boundary layer with foreign gas transpiration, is dealt with in this article.

Landis and Mills [1] carried out a numerical computation of a compressible boundary layer with foreign gas transpiration, and obtained results for skin-friction and the heat and mass transfer coefficients. A numerical approach is believed to be more useful in the future. In the present situation, however, an analytical approach still has usefulness. It is free from any numerical instability and can save the computation time and money.

In recent years, Rubesin and Pappas [2] proposed an analytical theory to calculate surface skin friction, and the heat and mass transfer coefficients in the turbulent boundary layer with foreign gas transpiration. In their theory, the fluid was assumed to be composed of only two chemical species. Therefore, it cannot be applied to a case with combustion, or even with mixed gas transpiration. Denison [3] has modified their theory, and treated the case of carbon ablation in a high speed flow. In his theory, however, an empirical formula was introduced for the fluid density profile, so that it cannot be applied to other problems. Economos [4] developed an integral method with transformed coordinates, which is effective to reduce the governing equations formally into those for an incompressible flow. The method was applied by the present author to a case with combustion, but the results obtained were in poor agreement with the experiments [5].

In the authors previous report [6], a theory going along with those of Rubesin and Pappas and of Denison was developed allowing any number of chemical species in the fluid. However, the Prandtl and Schmidt numbers were assumed to be unity, and the heat and mass transfer could not be treated. The present analysis is an extension of the previous study, and the above assumption for the Prandtl and Schmidt numbers is removed. Additionally, an extension is made in the present study so as to deal with a supersonic flow situation, which could not be treated in the previous work either.

2. Outlines of the method

2.1 Basic equations and thin film theory

The continuity equation and the governing equations for momentum, heat and the mass fraction of chemical species i are given by the following equations:

$$\frac{d}{dx}(\overline{\rho u})+\frac{d}{dy}(\overline{\rho v})=0, \tag{1}$$

$$\overline{\rho u}\frac{d\bar{u}}{dx}+\overline{\rho v}\frac{d\bar{u}}{dy}=\frac{d\bar{\tau}}{dy} \tag{2}$$

$$\overline{\rho u}\frac{d\bar{h}}{dx}+\overline{\rho v}\frac{d\bar{h}}{dy}=\frac{d\bar{q}}{dy} \tag{3}$$

$$\overline{\rho u}\frac{d\omega_i}{dx}+\overline{\rho v}\frac{d\bar{\omega}_i}{dy}=\frac{d\bar{J}_i}{dy} \tag{4}$$

where u and v are the streamwise and normal velocity components respctively, x and y the streamwise and normal coordinates, h the sensible total enthalpy, ω_i the mass fraction of species i, ρ the fluid density, τ the shear stress, q the heat flux, J_i the mass flux of the species i, and the superscript $(\overline{\ \ })$ denotes the averaged value. In the following, only the averaged quantities are considered so that the superscript $(\overline{\ \ })$ is dropped out for simplicity.

The thin film theory ignores the first term on the left side of each governing equation, and approximates $(\overline{\rho v})$ on the left by $(\rho v)_w$, the mass blowing rate at the wall. Making use of the theory, Eqs. (2) through (4) turn out to be simple forms. They are:

$$\tau=\tau_w(1+B\phi), \tag{5}$$

$$-q=\rho_e U_e(\hat{h}_{wad}-\hat{h}_w)St(1+B\hat{H}), \tag{6}$$

$$-J_i=\rho_e U_e(\omega_{ie}-\omega_{iw})St_m(1+BW_i), \tag{7}$$

where

$$B=\frac{(\rho v)_w}{\rho_e U_e}\frac{2}{Cf}, \tag{8}$$

$$\hat{H}=\frac{\hat{h}-\hat{h}_w}{\hat{h}_{wad}-\hat{h}}\frac{Cf}{2St}, \tag{9}$$

$$W_i=\frac{\omega_i-\omega_{iw}}{\omega-{}_{ie}\omega_{iw}}\frac{Cf}{2St_m}. \tag{10}$$

ϕ in the above equation is the normalized veocity (u/U_e), the subscript e denotes the main stream condition, δ the boundary layer thickness, Cf the skin friction coefficient, St the Stanton number for heat transfer, St_m the same for mass transfer. The subscripts w and ad specify the wall and adiabatic conditions, and the superscript means the sensible total enthalpy, given by the total enthalpy relative to the reference state enthalpy of the mixture.

2.2 The expressions for τ, q and Ji

In the viscous sublayer, the expressions for τ, q and J_i can be given as follows:

$$\tau=\frac{\mu U_e}{\delta}\frac{d\phi}{d\eta}, \tag{11}$$

$$q=-\frac{\mu}{Pr}\left[\frac{\hat{h}_{wad}-\hat{h}_w}{\delta}\frac{d\hat{H}}{d\eta}+(Pr-1)\frac{U_e^2}{2}\frac{d\phi^2}{d\eta}\right], \tag{12}$$

$$J_i=-\frac{\mu}{Sc_i}\frac{\omega_{ie}-\omega_{iw}}{\delta}\frac{dw_i}{d\eta}. \tag{13}$$

where ω_{iw} has been assumed to be small compared to unity, and the Soret and Dufour effects have been neglected. η is the normalized distance from the wall (y/δ), μ the viscosity of the fluid, and Pr and Sc_i the Prandtl and Schmidt numbers.

Accepting Prandl's mixing length hypothesis and assuming that the Prandtl and Schmidt turbulent numbers are respectively unity, the counterparts of the above expressions in the fully turbulent region of the boundary layer can be written as follows:

$$\tau=\rho\kappa^2 U_e^2\eta^2\left|\frac{d\phi}{d\eta}\right|\frac{d\phi}{d\eta}, \tag{14}$$

$$q=-\rho\kappa^2 U_e(\hat{h}_{wad}-\hat{h}_w)\eta^2\left|\frac{d\phi}{d\eta}\right|\frac{d\hat{H}}{d\eta} \tag{15}$$

$$J_i=-\rho\kappa^2 U_e(\omega_{ie}-\omega_{iw})\eta^2\left|\frac{d\phi}{d\eta}\right|\frac{dW_i}{d\eta} \tag{16}$$

where κ is the Kármán constant.

2.3 The formal solutions for the profiles of ϕ, $\hat{H}$ and W_i.

Equating Eqs. (5) and (11) and integrating once with respect to η, the following profile available in the viscous sublayer for ϕ is as follows:

$$\phi=\frac{1}{B}\exp\left[B\frac{Cf}{2}Re_\delta\int_0^\eta\frac{d\eta}{\tilde{\mu}}\right]-1 \tag{17}$$

where $\tilde{\mu}=\mu/\mu_e$ and,

$$Re_\delta=\frac{\rho_e U_e\delta}{\mu_e}.$$

On the other hand, in the turbulent region, the integration of an equation resulting from the combination of Eqs. (5) and (14) gives formally the following profile for ϕ:

$$\eta=\eta_s\exp[a_w(\zeta-\zeta_s)], \tag{18}$$

where $(\zeta-\zeta_s)$ is the following function of ϕ

$$\zeta-\zeta_s=\int_{\phi_s}^{\phi}\sqrt{\frac{\tilde{\rho}}{1+B\phi}}\,d\phi \tag{19}$$

and $\tilde{\rho}$ the normalized density of the fluid (ρ/ρ_e), the subscript s denotes the edge of the viscous sublayer, and a_w the parameter relating to Cf as follows:

$$a_w=\kappa\sqrt{\frac{2}{Cf}}.$$

Eq. (18) is practically available to calculate the ϕ profile only after the integration (19) is performed. This will be performed later.

To solve the profiles of H and W_i, the distributions of Pr and Sc_i must be known Here it was considered to replace them with their representative values Pr^*, and Sc_i^*, and to introduce the following representative values:

$$\overset{*}{Pr}=\frac{\int_0^{\phi_s}\frac{BPr(\phi)}{1+B\phi}d\phi}{ln(1+B\phi_s)} \tag{20}$$

$$Sc_i^*=\frac{\int_0^{\phi_s}\frac{BSc_i(\phi)}{1+B\phi}d\phi}{ln(1+B\phi_s)} \tag{21}$$

In the case of subsonic flow, these values fit exactly for the purpose. Next it is assumed that Pr and Sc_i distribute linearly for ϕ. Then, the above expressions for $\overset{*}{Pr}$ and Sc_i^* reduce to the following simpler forms:

$$\overset{*}{Pr}=\frac{Pr_s-Pr_w+\left(Pr_w-\frac{Pr_s-Pr_w}{B\phi_s}\right)ln\,(1+B\phi_s)}{ln(1+B\phi_s)} \tag{22}$$

$$Sc_i^*=\frac{Sc_{is}-Sc_{iw}+\left(Sc_{iw}-\frac{Sc_{is}-Sc_{iw}}{B\phi_s}\right)ln\,(1+B\phi_s)}{ln(1+B\phi_s)} \tag{23}$$

These expressions are used in this study not only for subsonic flow but also in a superson c flow case. The expressions do not exactly fit the supersonic case, but the error caused by this may not exceed so much the one caused by the assumption of the linear dependencies of Pr and Sc_i on ϕ. In a case without transpiration, incidentally, the expressions (22) and (23) reduce to the follouing very simple forms:

$$\overset{*}{Pr}=\frac{1}{2}(Pr_w+Pr_s), \tag{24}$$

$$Sc_i^*=\frac{1}{2}(Sc_{iw}+Sc_{is}). \tag{25}$$

From Eqs. (5) (6), (11) and (12), the profile of H in the sublayer can be expressed in terms of $\overset{*}{Pr}$ introduced above. That is

$$(1+B\phi)^{\overset{*}{Pr}}=\frac{1}{B(\overset{*}{Pr}-1)-\frac{K\overset{*}{Pr}}{\overset{*}{Pr}-2}}\times[B(\overset{*}{Pr}-1)(1+B\hat{H})$$
$$+K\overset{*}{Pr}(1+B\phi)\{1-\frac{\overset{*}{Pr}-1}{\overset{*}{Pr}-2}(1+B\phi)\}], \tag{26}$$

where

$$K=\frac{Cf}{2St}\;\frac{(\overset{*}{Pr}-1)U_s^2}{\overset{*}{Pr}(\hat{h}_{wad}-\hat{h}_w)}\;. \tag{27}$$

Similarly, from Eqs. (5), (7), (11) and (13), the result for W_i in the sublayer is obtained in the following form:

$$(1+B\phi)^{Sc_t^*}=1+BW_i. \tag{28}$$

On the other hand, in the turbulent region the profiles of H and W_i are obtained by using Eqs. (5) through (7) and (14) through (16). They are:

$$1+B\hat{H}=\frac{1+B\hat{H}_e}{1+B}(1+B\phi). \tag{29}$$

$$1+BW_i=\frac{1+BW_{ie}}{1+B}(1+B\phi). \tag{30}$$

2.4 Momentum Integral Equation,

The momentum integral equation for a compressible, turbulent boundary layer on a flat plate with transpiration and with a zero pressure gradient may be written in the following form:

$$\frac{dRe_\theta}{dRe_x}=\frac{Cf}{2}+\frac{(\rho v)_w}{\rho_e U_e}=\frac{Cf}{2}(1+B) \tag{31}$$

where Re_θ and Re_x are the Reynolds numbers based on the momentum thickness of the boundary layer and on the longitudinal distance x along the wall, respectively. The velocity U_e, the fluid density ρ_e and the viscosity μ_e in the main flow are introduced into both Re_θ and Re_x.

The integration of Eq. (31) with respect to x may lead to the following equation (32), because, in a turbulent boundary layer, the relative change in Re_x, $(\Delta Re_x/Re_x)$ causes a much smaller relative change in Cf, $(\Delta Cf/Cf)$.

$$Re_\theta=\frac{Cf}{2}(1+B)\,Re_x, \tag{32}$$

where either B or $(\rho v)_w$ has been assumed to distribute uniformly along the entire region of x considered.

2.5 Skin friction coefficient

Rubesin and Pappas, and Denison proposed the following parameter $(\zeta_e-\zeta_s)$ to express Re_θ:

$$\zeta_e-\zeta_s=\int_{\phi_s}^{1}\sqrt{\frac{\tilde{\rho}}{1+B\phi}}\,d\phi, \tag{33}$$

which is obtained by putting the upper limit of the integration of Eq. (19) equal to unity. In the present problem, Re_θ can be expressed in terms of $(\zeta_e-\zeta_s)$ as follows [6]:

$$Re_\theta=\frac{\exp[a_w(\zeta_e-\zeta_s)]}{a_w}(1+B)^{1/2}Re_\delta\eta_s. \tag{34}$$

The insertion of Eq. (34) into Eq. (32) gives the following result:

$$\frac{\zeta_e-\zeta_s}{\sqrt{Cf}}=\frac{1}{\sqrt{2}}ln\left[\kappa\sqrt{\frac{Cf}{2}(1+B)}\ \frac{Re_x}{Re_\delta\eta_s}\right]. \tag{35}$$

For later use, here is considered an incompressible, adiabatic case without transpiration. Specifying the case with subscript 0, Eq. (35) reduces to

$$\frac{1-\phi_{s0}}{\sqrt{Cf_0}}=\frac{1}{\sqrt{2}}ln\left[\kappa\sqrt{\frac{Cf_0}{2}}\ \frac{Re_x}{Re_{\delta 0}\eta_{s0}}\right], \tag{36}$$

where the relations $\zeta_{e_0}=1$ and $\zeta_{s_0}=\phi_{s0}$. obtained easily from Eq. (33), have been inserted. The following relations are easily confirmed to hold

$$Re_{\delta_0}\eta_{s_0}=y^+_{s_0}\sqrt{\frac{2}{Cf_0}}\ \cdot \quad \phi_{s_0}=y^+_{s_0}\sqrt{\frac{Cf_0}{2}}\ , \tag{37}$$

where

$$y^+_s=\frac{\rho_e y_{s_0}U_e}{\mu_e}\sqrt{\frac{Cf_0}{2}} \tag{38}$$

By making use of the above relations and by setting $\kappa=0.4$ and $y^+_{s_0}=13.1$. Eq. (36) becomes

$$\frac{2}{Cf_0}=4.38+2.50\ ln\left[Re_x\frac{Cf_0}{2}\right], \tag{39}$$

This relationship is akin to Schoenherr's expression [8] for the total skin friction, while Eq. (39) is for the local skin friction. Eq. (39) agrees however with the measured values in the wide range of Re_x. namely, for $10^6\leq Re_x\leq 10^8$ [6].

Here are introduced the following two parameters.

$$\Omega=\left(\frac{Cf}{Cf_0}\right)_{Re_x}, \qquad \psi=\left(\frac{Cf}{Cf_0}\right)_{Re_\theta}. \tag{40}$$

The subscripts Re_x and Re_θ mean to compare the two boundary layers with and without transpiration respectively at the same x, and at the positions of different x' s, but having the same momentum thickness as each other. From Eqs. (35), (36) and (40), the following equations can be reduced respectively for Ω and ψ:

$$\frac{1}{\sqrt{\Omega}}=\frac{1-\phi_{s_0}}{\zeta_e-\zeta_s}+\frac{\sqrt{Cf_0}}{\sqrt{2}\kappa(\zeta_e-\zeta_s)}ln\left[\sqrt{\Omega(1+B)}\ \frac{Re_{\delta_0}\eta_{s_0}}{Re_\delta\eta_s}\right] \tag{41}$$

$$\frac{1}{\sqrt{\psi}}=\frac{1-\phi_{s_0}}{\zeta_e-\zeta_s}-\frac{\sqrt{Cf_0}}{\sqrt{2}\kappa(\zeta_e-\zeta_s)}ln\left[\sqrt{\psi(1+B)}\ \frac{Re_\delta\eta_s}{Re_{\delta_0}\eta_{s_0}}\right]. \tag{42}$$

2.6 Auxiliary Relationships

The unknowns appearing in Eqs. (41) and (42) just given are $(\zeta_e-\zeta_s)$, y_s^+, ϕ_s, u_s^+ and $Re_e\eta_s$. The calculation of the first is the main concern of this report, and will be explained later. In the reference [6] several possible expressions were examined for each y_s^+, ϕ_s, u_s^+ and $Re_e\eta_s$, and a set of expressions was found giving the best final results. They are:

$$\frac{Re_{e_0}\eta_{s_0}}{Re_e\eta_s}=\frac{\bar{\rho}_s^2 y_{s_0}^+}{\bar{\rho}_w^{3/2}\bar{\mu}_w y_s^+}\sqrt{\frac{Cf}{Cf_0}} \tag{43}$$

$$\phi_s=u_{s_0}^+\bar{\rho}_w^{-1/2}\sqrt{\frac{Cf}{2}}, \tag{44}$$

$$y_s^+=\left(\frac{\bar{\rho}_s}{\bar{\rho}_w}\right)^{3/2}\frac{y_{s_0}^+}{1+5.15\bar{\rho}_w^{-1/2}\sqrt{\frac{Cf}{2}}} \tag{45}$$

$$u_s^+=u_{s_0}^+=y_{s_0}^+=13.1 \tag{47}$$

Eq. (43) was obtained by making use of the Howarth assumption that $\rho_s\mu_s=\rho_w\mu_w$. Eq. (45) was basically derived from Eq. (17) theoretically, but modified a little by paying attention to the form of the damping function for the mixing length in the sublayer [9]. Eq. (46) was found by optimization of the results. These expressions are also used in the present study.

2.7 Stanton numbers for heat and mass transfers.

As seen from Eq. (9), $\hat{H}$ and $\hat{H}_e$ include the Stanton number in the form of $(Cf/2St)$. Thus, the ratio can be determined simply by matching Eq. (26) with Eq. (29) at $\phi=\phi_s$. The final form of the expression is given as follows:

$$\frac{2St}{Cf}=\frac{B}{\{(1+B)(1+B\phi_s)^{\overset{*}{Pr}-1}-1\}}\Big\{\frac{\hat{h}_e-\hat{h}_w}{\hat{h}_{wad}-\hat{h}_w}$$
$$+\frac{2(1+B)}{B_2}\Big[1-\frac{\overset{*}{Pr}-1}{\overset{*}{Pr}-2}(1+B\phi_s)+\frac{1}{\overset{*}{Pr}-2}(1+B\phi_s)^{\overset{*}{Pr}-1}\Big]\frac{U_e^2}{2(\hat{h}_{wad}-\hat{h}_w)}\Big\}. \tag{47}$$

The ratio of the Stanton number for a mass transfer to Cf is obtained in a similar way. That is

$$\frac{2St_m}{Cf}=\frac{B}{\{(1+B)(1+B\phi_s)^{\overset{*}{Sc_i}-1}-1\}}. \tag{48}$$

2.8 The evaluation of $(\zeta_e-\zeta_s)$

The integration of Eq. (33) should be performed for the evaluation of $(\zeta_e-\zeta_s)$. For carrying out the integration, ρ must be given in terms of ϕ. In the case of subsonic fiow, this has been done allowing more than two for the number of species, and the integral (33) has been expressed successfully by elliptic integrals. In the case including supersonic flow, the effect of kinetic heating on the ρ profile must be included additionally.

The integration starts from $\phi=\phi_s$, so that the expression of ρ only in the turbulent region is sufficient for the present demand. As seen from Eqs. (29) and (30) $\hat{H}$ and ω_i affecting the value of ρ in the turbulent region can be given by the linear function of ϕ. For simplicity, they are rewritten as follows:

$$\hat{h}_t=\chi+\theta\phi, \tag{49}$$

$$\omega_i=\varepsilon_i+\sigma_i\phi. \tag{50}$$

χ, θ, ε_i and σ_i can be determined by the above matching of Eqs. (26) and (28) with their counterparts Eqs. (29) and (30) at the sublayer edge $\phi=\phi_s$.

By making use of these forms of the profiles of $\hat{h}_t$ and ω_i and by a thermally and calorifically ideal gas assumption, ρ can be expressed in the following way:

$$\bar{\rho}=\bar{\rho}_w\frac{1+F\phi}{(1+E\phi-E_u\phi^2)(1+G\phi)}, \tag{51}$$

where

$$\bar{\rho}_w=\frac{\chi+\theta-\frac{U_e^2}{2}-\sum_i^n h_{i_0}(\varepsilon_i+\sigma_i)}{\chi-\sum_i^n h_{i_0}\varepsilon_i}\;\frac{\sum_i^n R_i(\varepsilon_i+\sigma_i)}{\sum_i^n R_i\varepsilon_i}\;\frac{\sum_i^n Cp_i\varepsilon_i}{\sum_i^n Cp_i(\varepsilon_i+\sigma_i)},$$

$$E=\frac{\theta-\sum_i^n h_{i_0}\sigma_i}{\chi-\sum_i^n h_{i_0}\varepsilon_i}, \qquad E_u=\frac{U_e^2}{2(\chi-\sum_i^n h_{i_0}\varepsilon_i)}$$

$$F=\frac{\sum_i^n Cp_i\sigma_i}{\sum_i^n Cp_i\varepsilon_i}, \qquad G=\frac{\sum_i^n R_i\sigma_i}{\sum_i^n R_i\varepsilon_i}. \tag{52}$$

In Eqs. (52), n represents the total number of chemical species existing in the fluid, and R_i, Cp_i, and H_{i_0} are the properties of the species i: the gas constant, the specific heat at constant pressure, and the heat of formation at a reference state. Insertion of Eq. (51) into Eq. (33) gives the following final form of the integral:

$$\zeta_e-\zeta_s=\int_{\phi_s}^1\frac{\bar{\rho}_w^{1/2}\delta_F(1+F\phi)}{\sqrt{(1+B\phi)(1+E_1\phi)(1+E_2\phi)(1+F\phi)(1+G\phi)}}d\phi, \tag{53}$$

where E_1 and E_2 are related to E and E_u defined above as follows:

$$E_1+E_2=E, \qquad E_1E_2=-E_u,$$

and δ_F is a sign parameter such that $\delta_F=\pm1$, depending on $(1+F\phi)\gtrless 0$.

The above integral can be expressed with the first and the third kinds of Legendre-Jacobis standard forms of the elliptic integral. While the integration should be the main part of the theory, the explanation of the manipulation how to reduce the final expression is cumbersome, so that it is given in the Appendix.

2.9 The properties of the fluid

Usual gas does not obey so well the calorifically ideal gas assumption. Considering this fact, the following averaged specific heat Cp_i^* has been used for Cp_i in Eq. (52):

$$Cp_i^*=\int_0^{T^*} Cp_i dT, \quad T^*=\frac{h_{t_e}+h_w-\frac{U_e^2}{2}-\sum_i^n h_{i_0}(\omega_{i_e}+\omega_{i_w})}{\sum_i^n Cp_i^*(\omega_{i_e}+\omega_{i_w})} \tag{54}$$

For the determination of Cp_i^*, an iterative calculation is nesessary, but it converges very rapidly. Cp_i in Eq. (54) is calculated from its polynomial expression given by Prothero [10]. The calculation of Pr_w, Pr_e, Sc_{i_w}, and Sc_{i_e} in Eqs. (22) and (23) needs the viscosity and the thermal conductivity of the mixture, and the effective binary diffusion coefficient D_i between species i and the mixture of the remaining species. For the evaluation of the first two properties, Wilkes equation was used. A modified Stefan-Maxwell equation was used for the calculation of Di, and is described as follows:

$$D_i=\frac{\mu M_i\sum_j\frac{1}{M_iM_i\mathscr{D}_{ij}}(\omega_i\omega_{j-\infty}-\omega_j\omega_{i-\infty})}{\rho\left(\omega_i\sum_j\frac{\omega_{j-\infty}}{M_j}-\frac{\omega_{j-\infty}}{M}\right)}, \tag{55}$$

where $\mathscr{D}_{ij}$ is the diffusivity of a species pair i and j in a binary mixture, M and M_i are the molecular weights of the mixture and of the species i, respectively, and the suffix $-\infty$ denotes the condition in the injectant reservoir. The value of $\mathscr{D}_{ij}$, the viscosity and the thermal conductivity of species i were calculated with rigorous kinetic theory [9].

3. The presentation and discussion of the results.

The results of Cf, St, and St_m to be shown here are plotted against the following three parameters b, b_h and b_m respectively:

$$b=\frac{(\rho v)_w}{\rho_e U_e}\frac{2}{Cf_1} \qquad b_h=\frac{(\rho v)_w}{\rho_e U_e}\frac{1}{St_1} \qquad b_m=\frac{(\rho v)_w}{\rho_e U_e}\frac{1}{2St_{m_1}} \quad .$$

The suffix 1 used in the above definitions denotes the boundary layer without transpiration and having the same main flow/wall temperature ratio (T_w/T_e) as that in the boundary layer to be con sidered. Gf_1, St_1, and St_{m_1}, are calculated with Eqs. (35), (47) and (48) by setting $B=0$, $Re_e\eta_e=Re_{e_1}\eta_{e_1}$, and $Cf=Cf_1$. The calculated values are shown in Figs. 1 and 2, together with the computed results by Landis and Mills, and the experimental results by Spalding and *Chi* [11]. The present results show reasonable

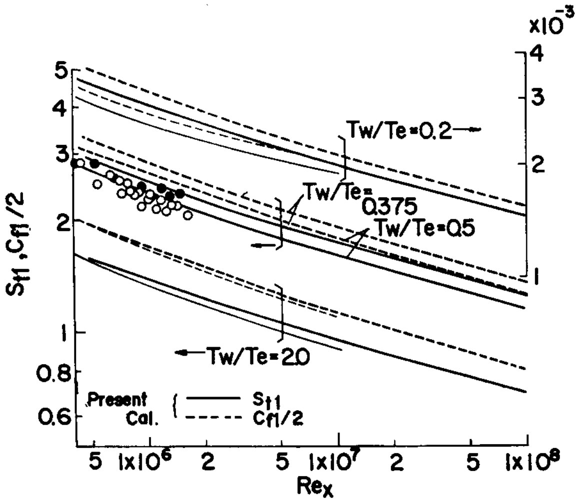

Fig. 1 Effect of Temperature ratio on Cf_1 and St_1(——, $Cf_1/2$ and St_1 calculated numerically by Landis and Mills, ○● Spalding and Chi for Tw/Te=0.375 and 0.5, respectively).

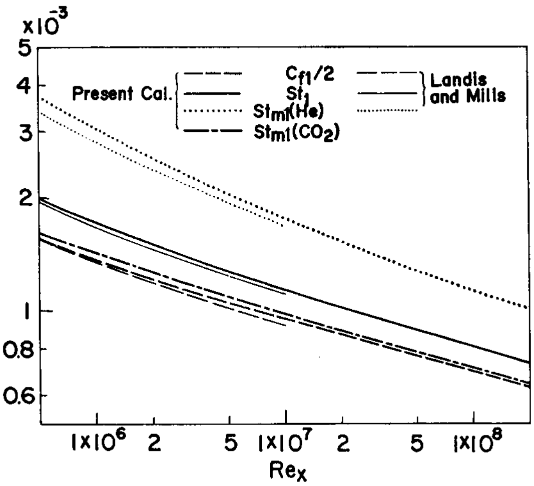

Fig. 2 Comparison of Cf_1, St_1 and St_{m1} presently obtained with the numerical results of Landis and Mills at $Tw/Te=0.9$.

values, but critically speaking, both the present theory and the calculation by Landis and Mills seem to show a slightly larger dependence on the temperature ratio.

Figures 3 through 5 show the present results for a subsonic flow situation. These figures clearly show the higher effectiveness of Helium gas transpiration to reduce Cf, St and St_m. Their normalized values (Cf/Cf_1), (St/St_1) and (St_m/St_{m_1}) are found to depend rather slightly on both the temperature ratio and the Reynolds number.

To compare with the experimental results at an isothermal situation, the two calculated curves at $(T_w/T_e)=0.67$ and 1.67 for Helium gas transpiration already shown in Fig. 3 are replotted in Fig. 6 together with several theoretical results. The present

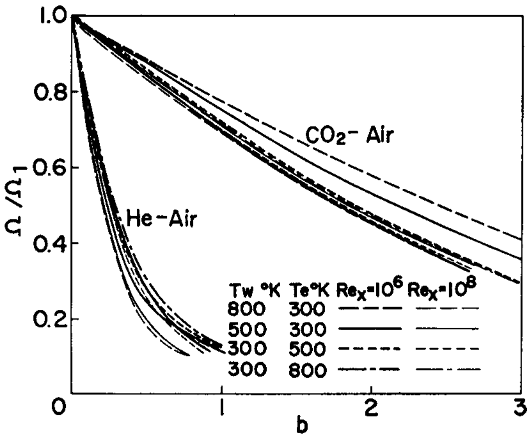

Fig. 3. Effects of temperature difference and Reynolds number on the skin friction coefficient.

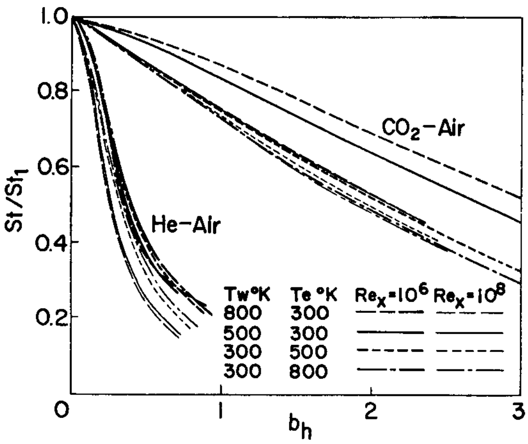

Fig. 4. Effects of temperature difference and Reynolds number on the Stanton number for heat transfer.

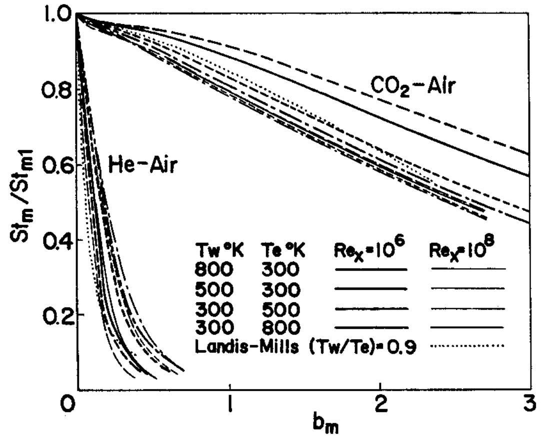

Fig. 5. Effects of temperature difference and Reynolds number on the Stanton number for mass transfer.

results are considered to be superior to the theory of Rubesin and Pappas. This may be mainly due to the refined expressions (43) through (46) for the thickness of the sublayer [6]. The present results go along well with the theoretical results of Economos and of Landis and Mills. The computation time is believed to be even shorter when compared to that required in the calculation by Economos method. The experiment by Pappas and Okuno shown in the above figure was carried out on a

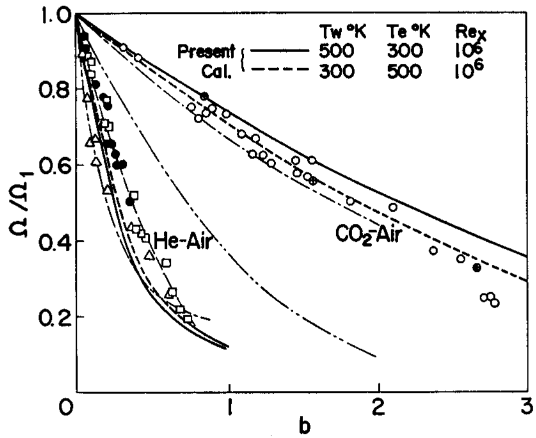

Fig. 6. Comparison of the present present theory with other theories and experimental results (······theory by Economos, —·— numer cal result by Landis and Mills, —··—theory by Rubesin and Pappas, ○ ● Romanenko and Kharchenko, ⊕ Dunbar and Squire, △ Pappas and Okuno, □ Scott refered to in the reference [4]).

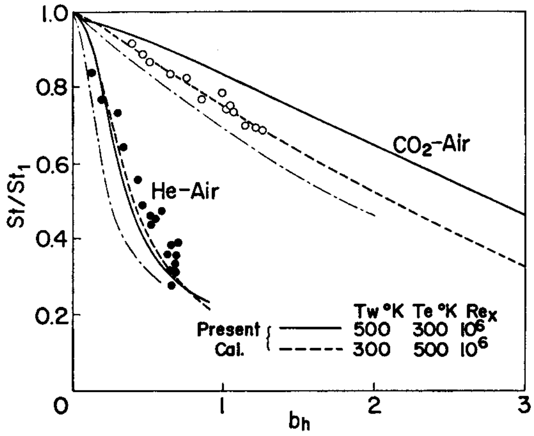

Fig. 7. Comparison of the present theory with other theoretical and experimental results (-·- numerical resul by Landis and Mills, ○ ● Romanenko and Kharchenko).

cone surface, but is usually compared with the flat plate data. Their results are not the local value but the results of the total skin friction coefficient, C_F. In relation to this, it may be appropriate to add the comment that Eq. (32) holds exactly for C_F if B or the mass blowing rate is uniform over the range of x considered. Thus, it may not be so illogical to compare their results with the present calculation.

In Fig. 7 are compared the results for St, shown in Fig. 2, with the experiments by Romanenko and Kharchenko [12], and the agreement between them is again found to be good. In Figs. 8 through 10, the present results are compared with the

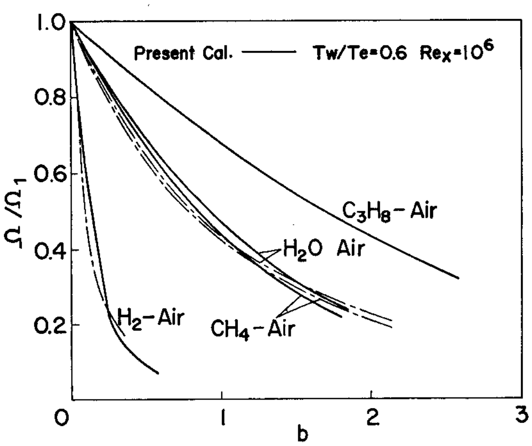

Fig. 8. Comparison of the present results with the numerical calculation by Landis and Mills (-·- Landis and Mills, $Tw/Te=0.9$).

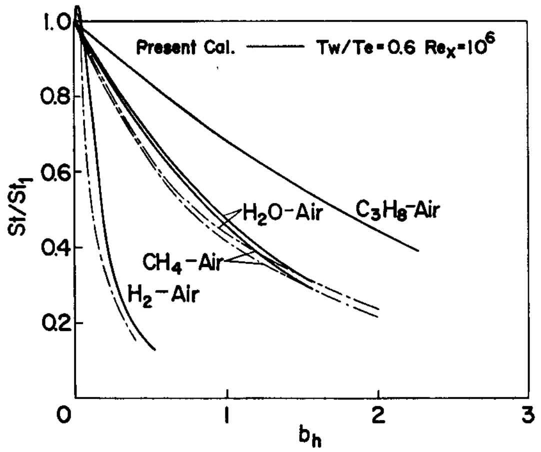

Fig. 9. Comparison of the present results with the numerical calculation Landis and Mills (-·- Landis and Mlls, $Tw/Te=0.9$).

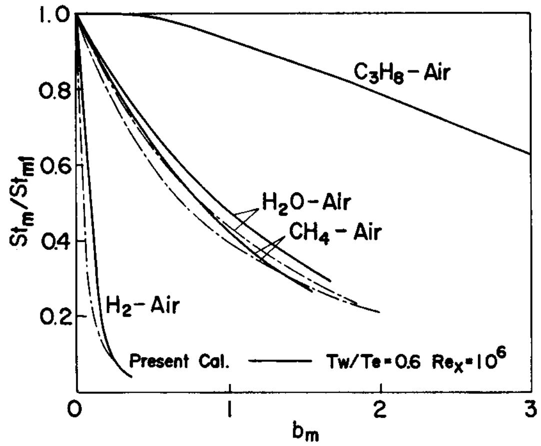

Fig. 10. Comparison of the present results with the numerical calculation by Landis Mills, (-·- Landis and Mills, $Tw/Te=0.9$).

numerical results by Landis and Mills for several cases of different injectants. Both results show fairly good agreement. The computation time is much shorter in the present calculation. Thus, the efficiency of the present method is believed to be sufficiently high. These figures again show the higher effectivess of lighter gas transpiration.

In Figs. 11 through 13 are compared the present results for a supersonic flow case with some available experimental results [13-15]. For Helium gas transpiration, the present results agree better with the experimental results than does the theory of Rubesin and Pappas. For Carbon-dioxide gas transpiration, the present results show a

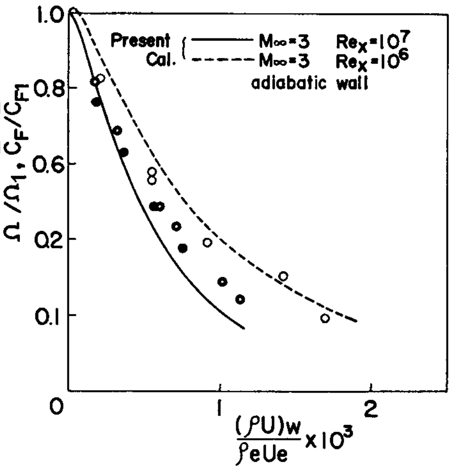

Fig. 11. Skin-friction coefficient with Helium transpiration into a compressible turbulent boundary layer. (Experiments by Pappas and Okuno.)

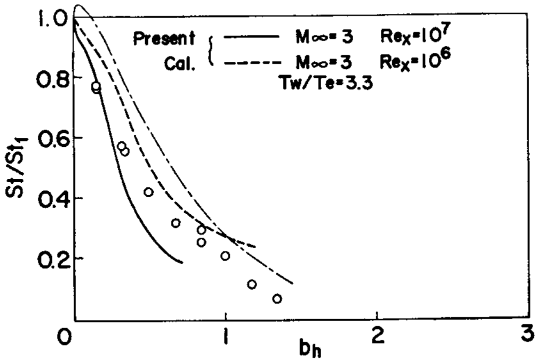

Fig. 12. Stanton number for heat transfer (-·- theory by Rubesin and Pappas, ○ Experiments by Leadon and Scott at $Re_x \cong 4 \times 10^6$).

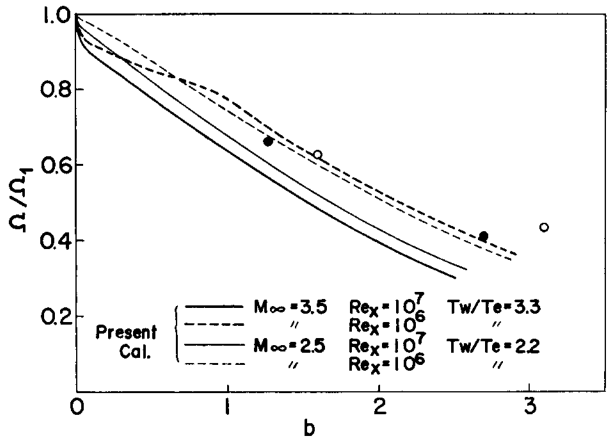

Fig. 13. Skin-friction coefficient (○ ● Experiments by Dunbar and Squire at $M_\infty=2.5$, and at $M_\infty=3.5$, respectively).

slightly poorer agreement with the experiments by Dunber and Squire, but are not so bad on the whole.

4. Concluding Remarks.

A refined classic theory has been developed for the calculation of Cf, St and St_m for a subsonic or supersonic, non-isothermal, turbulent boundary layer on a flat plate with foreign gas transpiration. The present results are in fairly good agreement with the experimental results found in reference. Any number of chemical species is allowed to compose the fluid in the present method. Thus it has a potentiality applicable to the ablation problem.

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Appendix

Here is introduced first a parameter P defined as

$$|P| = \text{the smallest one of } (|B|,\ |E_1|,\ |E_2|,\ |F|,\ |G|), \tag{A1}$$

and next are specified the recipocals of the remaining constants as A_1, A_2, A_3 and A_4 in a way satisfying the following inequality:

$$A_1 > A_2 > A_3 > A_4. \tag{A2}$$

Noting that either $|E_1|$ or $|E_2|$ should be smaller than unity except for the case when the main flow Mach number is infinitely large, $|P|$ is considered to be smaller than unity. Thus, $(1+P\phi)>0$. The Taylor expansion of $(1/\sqrt{1+P\phi})$ gives

$$\frac{1}{\sqrt{1+P\phi}} = 1 + \sum_{j=2}^{\infty} g_j \phi^{j-1} \tag{A3}$$

At the usual Mach number, the first ten terms of the expanded series are sufficient to get a converged value. Specifying j of the last of the retained terms in Eq. (A3) as M, Eq. (53) reduces to the following form:

$$\zeta_e - \zeta_s = \delta_F \bar{\rho}_w^{1/2} C_A [H(2) + (F+g_2) H(3) + \sum_{j=2}^{M} (F g_j + g_{j+1}) H(j+2)], \tag{A4}$$

where $C_A = \sqrt{A_1 A_2 A_3 A_4 \delta_A}$, $\delta_A = \pm 1$ depending on $(A_1 A_2 A_3 A_4) \gtrless 0$, and

$$H(j+2)=\int_{\phi_s}^{1}\frac{\phi^j d\phi}{\sqrt{\delta_A(\phi+A_1)(\phi+A_2)(\phi+A_3)(\phi+A_4)}}. \tag{A5}$$

Computation of $H(1)$, $H(2)$, $H(3)$ and $H(4)$ is sufficient to calculate $(\zeta_e-\zeta_s)$, because the following recurrence relation is available.

$$\begin{aligned}2\delta_A(j-1)H(j+2)&=2\sqrt{\delta_A(1+A_1)(1+A_2)(1+A_3)(1+A_4)}\\&-2\phi_s^{j-3}\sqrt{\delta_A(\phi_s+A_1)(\phi_s+A_2)(\phi_s+A_3)(\phi_s+A_4)}\\&-(2j-3)\delta_A(A_1+A_2+A_3+A_4)H(j+1)\\&-(2j-4)\delta_A(A_1A_2+A_1A_3+A_1A_4+A_2A_3+A_2A_4+A_3A_4)H(j)\\&-(2j-5)\delta_A(A_1A_2A_3+A_2A_3A_4+A_3A_4A_1+A_4A_1A_2)H(j-1)\\&-(2j-6)\delta_A A_1A_2A_3A_4H(j-2)\end{aligned} \tag{A6}$$

Here are introduced the following forms of integrals, which are used to express $H(1)$ through $H(4)$.

$$I_1=\int_{z_s}^{z_1}\frac{dz}{\sqrt{f(z^2)}} \tag{A7}$$

$$I_3(c)=\int_{z_s}^{z_1}\frac{dz}{(z^2-c^2)\sqrt{f(z^2)}} \tag{A8}$$

where $f(z^2)=(C_1^*-C_2^*z^2)(D_1^*-D_2^*z^2)$. Some manipulation is still necessary to transform the above integrals into the Legendre-Jacobi standard forms of an elliptic integral. Several variations of the final forms appear, depending on the relative magnitude among the coefficients C_1^*, C_2^*, D_1^*, and D_2^* in the above integrals, but the manipulation is straightforward. Thus, to save space here, an explanation will be given in the following, but only as regards how to express $H(1)$ through $H(4)$ with I_1 and I_3 (c).

To start with, the following two real numbers α and β are introduced:

$$\alpha=\frac{\delta_2\lambda_1(A_3+A_4)-\delta_1(A_1+A_2)}{2(\delta_1-\delta_2\lambda_1)} \tag{A9}$$

$$\beta=\frac{\delta_2\lambda_2(A_3+A_4)-\delta_1(A_1+A_2)}{2(\delta_1-\delta_2\lambda_2)} \tag{A10}$$

where λ_1 and λ_2 are the two solutions $(\lambda_1>\lambda_2)$ of the following quadratic equation for λ:

$$(A_3-A_4)\lambda^2-2[(A_1+A_2)(A_3+A_4)-2A_3A_4-2A_1A_2]\delta_A\lambda+(A_1-A_2)^2=0. \tag{A11}$$

$\delta_1=\pm1$ depending on $(\phi+A_1)(\phi+A_2)\lesseqgtr 0$, and $\delta_2=\pm1$ depending on $(\phi+A_3)(\phi+A_4)\gtrless 0$. There exists a relation among the sign parameters δ_1, δ_2 and δ_A as follows:

$$\delta_1\cdot\delta_2=\delta_A$$

Next, the following four constants are defined with λ_1 and λ_2 just introduced:

$$C_1=\frac{\lambda_1(\delta_1-\delta_2\lambda_2)}{\lambda_1-\lambda_2},\quad C_2=\frac{\lambda_2(\delta_1-\delta_2\lambda_1)}{\lambda_1-\lambda_2},\quad D_1=\frac{\delta_1-\delta_2\lambda_2}{\lambda_1-\lambda_2},\quad D_2=\frac{\delta_1-\delta_2\lambda_1}{\lambda_1-\lambda_2}.$$

The final form of $H(2)$ can now be expressed by the following unified form:

$$H(2)=\frac{1}{\alpha^*-\beta^*}\int_{z_s}^{z_1}\frac{dz}{\sqrt{f(z^2)}}=\frac{1}{\alpha^*-\beta_*}I_1, \tag{A12}$$

where

$$z_s=\frac{\phi_s-\alpha^*}{\phi_s-\beta^*},\quad \text{and}\quad z_1=\frac{1-\alpha^*}{1-\beta^*},$$

and

$$\alpha^*=\alpha,\ \beta^*=\beta,\ C_1^*=C_1,\ C_2^*=C_2,\ D_1^*=D_1,\ D_2^*=D_2,$$

when

$$\beta<\phi_s \text{ or } \beta>1,$$

or

$$\alpha^*=\beta\ \ \beta^*=\alpha,\ C_1^*=C_2,\ C_2^*=C_1,\ D_1^*=D_2,\ D_2^*=D_1,$$

when

$$\phi_s<\beta<1.$$

By making use of the same constants and the variable z, $H(1)$ and $H(3)$ can be rewritten in the following forms respectively:

$$H(1)=\frac{1}{\beta^*(\alpha^*-\beta^*)}I_1+\frac{\alpha^*}{\beta^{*3}}I_3\left(\frac{\alpha^*}{\beta^*}\right)+\frac{1}{\beta^{*2}}\int_{z_s}^{z_1}\frac{zdz}{\left(z^2-\frac{\alpha^{*2}}{\beta^{*2}}\right)\sqrt{f(z^2)}} \tag{A13}$$

$$H(3)=\frac{\alpha^*}{\alpha^*-\beta^*}I_1-I_3(1)-\int_{z_s}^{z_1}\frac{zdz}{(z^2-1)\sqrt{f(z)}}. \tag{A14}$$

The third term of each of the above expressions can be intergrated easily and further manipulation may not be necessary. Similarly, $H(4)$ can be transformed into the following form:

$$H(4)=\frac{\beta^{*2}}{\alpha^*-\beta^*}I_1-2\beta^*I_3(1)-2\beta^*\int_{z_s}^{z_1}\frac{zdz}{(z^2-1)\sqrt{f(z^2)}}$$
$$-(\alpha^*-\beta^*)\int_{z_s}^{z_1}\frac{dz}{(z-1)^2\sqrt{f(z^2)}} \tag{A15}$$

The third term of the above expression can be integrated easily as mentioned above, and the manipulation of the last term needs the recurrence relation of the elliptic integral again. It reads in the present situation as follows:

$$(\alpha^*-\beta^*)\int_{z_s}^{z_1}\frac{dz}{(z-1)^2\sqrt{f(z^2)}}=\frac{\alpha^*-\beta^*}{(C_1^*-C_2^*)(D_1^*-D_2^*)}\Big\{-C_2^*D_2^*I_1$$
$$+(C_1^*D_2^*+C_2^*D_1^*-2C_2^*D_2^*)\Big(I_3(1)+\int_{z_s}^{z_1}\frac{zdz}{(z^2-1)\sqrt{f(z^2)}}\Big)$$
$$-\frac{\sqrt{f(z_1^2)}}{z_1-1}+\frac{\sqrt{f(z_s^2)}}{z_s-1}+C_2^*D_2^*\int_{z_s}^{z_1}\frac{z^2dz}{\sqrt{f(z^2)}}\Big\}, \tag{A16}$$

except for the cases when $C_1^*=C_2^*$ or when $D_1^*=D_2^*$. In these excetional cases, the recurrence relation reads as follows:

$$(\alpha^*-\beta^*)\int_{z_s}^{z_1}\frac{dz}{(z-1)^2\sqrt{f(z^2)}}$$
$$=\frac{\beta^*-\alpha^*}{6(C_2^*D_2^*-C_1^*D_1^*)}\{4C_2^*D_2^*I_1^*+2(5C_2^*D_2^*-C_1^*D_1^*)$$
$$\times\Big[I_3(1)+\int_{z_s}^{z_1}\frac{zdz}{(z^2-1)\sqrt{f(z^2)}}\Big]+\frac{2\sqrt{f(z_1^2)}}{(z_1-1)^2}-\frac{2\sqrt{f(z_s^2)}}{(z_s-1)^2}\Big\}. \tag{A17}$$

It may be sufficient only to show further manipulation for the last term of Eq. (A16) That can be transformed into the following form:

$$\int_{z_s}^{z_1}\frac{z^2dz}{\sqrt{f(z^2)}}=\frac{1}{2\sqrt{\delta_cC_2^*D_2^*}}\int_{w_s}^{w_1}\frac{wdw}{(w+A_1)(w+A_2')(w+A_3')}, \tag{A18}$$

where $w=z^2$, $\delta_c=\pm1$ depending on $C_2^*D_2^*\gtrless 0$, and for simplicity, a set of the constant $(-C_1^*/C_2^*)$, $(-D_1^*/D_2^*)$ and zero has been replaced by another set of constants A_1', A_2' and A_3' in a way that they satisfy an inequality $A_1'>A_2'>A_3'$. Here are defined two real constants μ and ν as follows:

$$\mu=\frac{\delta_c}{2\gamma_1}-\frac{1}{2}(A_2'+A_3'),\quad \nu=\frac{\delta_c}{2\gamma_2}-\frac{1}{2}(A_2'+A_3').$$

γ_1 and γ_2 in the above definitions are the two solutions of the following quadratic equation for γ:

$$(A_2'-A_3')^2\gamma^2-2\delta_c(A_2'+A_3'-2A_1')\gamma+1=0.$$

Eq. (A18) can finally be transformed into the following form:

$$2\sqrt{\delta_cC_2^*D_2^*}\int_{z_s}^{z_1}\frac{z^2dz}{\sqrt{f(z^2)}}=\frac{(\gamma_1-\gamma_2)}{(\mu^*-\nu^*)\sqrt{\delta_\gamma\gamma_1\gamma_2}}\Big\{\nu^*\int_{v_s}^{v_1}\frac{dv}{\sqrt{f'(v^2)}}$$
$$+(\nu^*-\mu^*)\Big[\int_{v_s}^{v_1}\frac{dv}{(v^2-1)\sqrt{f'(v^2)}}$$
$$+\int_{v_s}^{v_1}\frac{vdv}{(v^2-1)\sqrt{f'(v^2)}}\Big]\Big\}, \tag{A19}$$

where $f'(v^2)=(k_1-k_2v^2)(L_1-L_2v^2)$. The new variable v is related to w as follows:

$$v = \frac{w - \mu^*}{w - \nu^*},$$

and other constants are as follows:

$$\mu^* = \mu,\ \nu^* = \nu,\ k_1 = -1,\ k_2 = 1,\ L_1 = \gamma_2,\ L_2 = \gamma_1,$$

in the cases when $w_1 > \nu$ and $w_s > \nu$, or when $w_1 < \nu$ and $w_s < \nu$, and

$$\mu^* = \nu,\ \nu^* = \mu,\ k_1 = 1,\ k_2 = 1,\ L_1 = \gamma_1,\ L_2 = \gamma_2,$$

in the remaining case of the relative magnitude among w_1, w_s and ν. $\delta_r = \pm 1$, depending on the inequality $\gamma_1 \gamma_2 \gtrless 0$.