A Decomposition Method for Mixed-Integer Linear Programming Problems with Angular Structure

By

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Abstract

An algorithm is presented for solving mixed-integer linear programming problems with an angular structure, based on the decomposition technique of Dantzig and Wolfe. The subproblem is a mixed-integer problem of a smaller size than that of the original one. A sufficient condition for optimality is obtained. In the case where the optimality condition is not satisfied, a search for improving the solution is being continued within a restricted extent. By examining illustrative examples, it is observed that the present algorithm is efficient because it has less computing time than the conventional branch and bound method.

1. Introduction

This paper develops a decomposition method for solving mixed-integer linear programming problems with an angular structure. So far, there have been only a few studies on the method of decomposing integer programming problems. Benders[1] has presented a partitioning method for solving mixed-integer problems. In this method, the original problem is partitioned into a relaxed pure-integer problem and a linear programming problem, and then the two problems are solved iteratively. The pure-integer problem has the same size as the original problem. Consequently, this method seems to be effective only for a problem with a relatively small number of integer variables.

Sweeney and Murphy[2] have proposed a method for decomposing pure-integer linear programming problems having a block angular structure. In this method, subproblems of a small size are solved to obtain the optimal and several post-optimal solutions for each block. A master problem is then constructed containing one variable for each of the subproblem solutions. If the number of the subproblem solutions is

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large enough, the optimal solution to the master problem gives the combination of the subproblem solutions, satisfying the optimality condition for the original problem. However, the procedure can not be applied to mixed-integer linear programs, because the master problem becomes nonlinear in this case.

This paper is a modification of our previous studies.[3,4] The purpose of this series of our studies is to present a method for solving mixed-integer linear programs with an angular structure. The continuous variables are contained in both block and coupling constraints of the problem. On the other hand, the integer variables are contained only in their respective block constraints but have interrelations among them through the coupling constraint for the continuous variables.

The basic idea of the proposed algorithm is to decompose the original problem in the same way as the Dantzig-Wolfe decomposition principle in the linear programs, and to solve a restricted auxiliary problem as well as subproblems iteratively. The auxiliary problem is a linear program. The subproblems are mixed-integer programs of smaller sizes than the size of the original one. A sufficient condition for optimality is obtained and the termination of the present algorithm is checked in two stages. That is to say, if the optimality test is satisfied, the procedure terminates and the optimal solution is obtained. If not, the search for improving the solution is continued within a restricted extent. If the procedure terminates with no improved solution, the best solution obtained so far is provided as a suboptimal solution. The procedure aims at obtaining a good suboptimal solution quickly from the viewpoint of practical use. The numerical results are shown so as to make comparisons between the present algorithm and the branch and bound method without decomposing the problem.

2. Problem Statement

Consider the following mixed-integer linear programming problem:

(P1) $$\min_{x(t),y(t)} z=\sum_{t=1}^{T} c(t)'y(t)+\sum_{t=1}^{T} a(t)'x(t) \quad \text{subject to}$$

$$\sum_{t=1}^{T} A(t)y(t)=b \tag{1}$$

$$\left.\begin{aligned} &B(t)y(t)+D(t)x(t)\geqq d(t) \\ &y(t)\geqq 0, \qquad x(t)\in X_t \end{aligned}\right\} \tag{2}$$

$$t=1, 2, \cdots, T$$

where,

$$X_t \triangleq \{x(t)|x_i(t)=0 \text{ or } 1, \quad i=1, 2, \cdots, m_t\} \tag{3}$$

In (P1), $y(t)$ is an $n_t\times 1$ continuous vector, while $x(t)$ is an $m_t\times 1$ integer vector. The vectors $c(t)$, $a(t)$, $d(t)$ and b are of the dimension $n_t\times 1$, $m_t\times 1$, $l_t\times 1$ and $l_0\times 1$, respectively. $A(t)$, $B(t)$ and $D(t)$ are the matrices of appropriate dimensions. A prime denotes the transposition of a vector or a matrix.

The constraints of this problem have the so-called angular structure. Particularly, the integer vector $x(t)$ for each t is contained in the t-th block constraint (2) alone, but has interrelations among $x(t)$ for all other t through the coupling constraint (1) imposed on the continuous vector $y(t)$.

The present problem is a modification of that discussed previously.[3,4] In fact, if we put $a(t)=0$ for all t in the objective function, this problem is identical with (Pl) in Refs. 3 and 4. In this paper, we deal with this problem as a preliminary consideration of the more general problem in which the coupling constraint is also imposed on $x(t)$.

3. Construction of the Auxiliary Problem

The t-th block constraint (2) gives a feasible region for the continuous vector $y(t)$, if the integer vector $x(t)$ is assigned at an appropriate value in X_t. Let k_t be the number of feasible points of $x(t)$ satisfying (2). Then, the feasible region for $y(t)$ consists of the convex polyhedrons in R^{n_t} defined by

$$Y_t^k \triangleq \{y(t)|B(t)y(t)\geqq d(t)-D(t)x^k(t),\ y(t)\geqq 0,\ x^k(t)\in X_t\} \quad k=1,2,\cdots,k_t \tag{4}$$

Let Y_t^k be bounded for simplicity†, and let $\{{}^jy_t^k\}$ be the set of its extreme points. The convex combination of all extreme points of the t-th block forms the convex hull of the union of the polyhedrons given by (4), i.e.,

$$\bar{Y}_t \triangleq \{y(t)|y(t)=\sum_{j,k}{}^j\mu_t^k\,{}^jy_t^k,\ \sum_{j,k}{}^j\mu_t^k=1,\ {}^j\mu_t^k\geqq 0\} \tag{5}$$

which contains the feasible region for $y(t)$.

Then the problem (Pl) is to choose $x^k(t)$, or equivalently Y_t^k, for each t so that the resultant combination of Y_t^k can attain the satisfaction of the coupling constraint (1) and the minimization of the objective function z. For this purpose, we define an auxiliary problem in terms of all the extreme points of all the blocks. By substituting $y(t)\in\bar{Y}_t$ into the first term of the objective function z and the coupling constraint (1), we have the following linear programming problem:

(P2) $\quad \min_{\lambda} w=f'\lambda \quad$ subject to

$$F\lambda=b \tag{6}$$

† Even if the set Y_t^k is not bounded, the following procedure can still be useful by introducing the extreme ray of Y_t^k into (5).

$$E\lambda = e \tag{7}$$

$$\lambda \geqq 0 \tag{8}$$

where λ and f are the $\nu\times 1$ vectors whose components are ${}^j\mu_t{}^k$ and $c(t)'\,{}^jy_t{}^k$ respectively, ν being the number of all the combinations among the superscripts j, k and the subscript t. F is the $l_0\times\nu$ matrix with the columns $A(t)\,{}^jy_t{}^k$, and E denotes the $T\times\nu$ matrix given by

$$E \triangleq \mathrm{diag}(e_1', e_2', \cdots, e_T') \tag{9}$$

In (7) and (9), e_t ($t=1, 2, \cdots, T$) and e denote the vectors, of which all the components are equal to unity.

The value of $y(t)$, obtained by solving (P2), belongs to the set $\bar{Y}_t$, which gives a larger region than the feasible one for the t-th block. Therefore, in order to hold the feasibility, we impose the following condition on the basic feasible solution of (P2):

C1. The extreme points $\{{}^jy_t{}^k\}$ of the t-th block, which correspond to the basic variables $\{{}^j\mu_t{}^k\}$ of (P2), must belong to the same polyhedron $Y_t{}^k$.

Since ν is large, solving (P2) directly may be impossible. Therefore, we deal with the restricted auxiliary problem (RAP for short), whose variables are restricted to candidates for the basic variables in (P2). In this case, condition C1 suggests that RAP is constructed with the variables corresponding to the extreme points determined by selecting one polyhedron for each block.

4. Optimality Condition

Assume that a basic feasible solution of (P2), satisfying C1, is obtained with the basic variables $\{{}^j\mu_t{}^K\}$. Although K is fixed at a specified number for each t, the dependence of K on t is omitted in the notation. Let w^* be the value of the objective function of (P2) for this solution. Let π be the simplex multipliers associated with this basis. Partition π as

$$\pi = (\pi_0', \pi_1, \pi_2, \cdots, \pi_T)' \tag{10}$$

where the $l_0\times 1$ vector π_0 corresponds to the constraint (6), and the $T\times 1$ vector $(\pi_1, \pi_2, \cdots, \pi_T)'$ to (7). Then, the relative cost factor for the variable ${}^j\mu_t{}^k$ is given by[5)]

$${}^j\bar{f}_t{}^k = [c(t)' - \pi_0' A(t)]\,{}^jy_t{}^k - \pi_t \tag{11}$$

We are now to derive the optimality condition for the solution of (P1) given by

$$\left.\begin{array}{l} x^*(t),\, y^*(t);\quad t=1, 2, \cdots, T \\ \text{where} \\ x^*(t) \triangleq x^K(t), \qquad y^*(t) \triangleq \sum_j {}^j\mu_t{}^K\,{}^jy_t{}^K \end{array}\right\} \tag{12}$$

By using the simplex multiplier π_0, we define the following subproblem for each block:

(P3) $\min_{x(t),y(t)} \bar{z}_t(\pi_0)=[c(t)'-\pi_0'A(t)]y(t)+a(t)'x(t)$ subject to (2)

Note that (P3) is a mixed-integer problem of a smaller size than that of the original (P1). By solving (P3), $\bar{z}_t^*$ is obtained as the minimum objective value. Then we define the following index for checking the optimality:

$$\bar{f}(t)\triangleq\bar{z}_t^*-\pi_t-a(t)'x^*(t) \tag{13}$$

and establish the following theorem:

Theorem 1. The solution (12) is optimal for (P1), if

$$\bar{f}(t)\geqq 0 \qquad \text{for all } t \tag{14}$$

Proof. Obviously the solution (12) is feasible for (P1). We assume that the solution (12) is not optimal for (P1). Then, there exists a solution of (P1), i.e., $x^0(t)$, $y^0(t)$; $t=1, 2, \cdots, T$ which satisfies (1), (2) and the following inequality:

$$\sum_t c(t)'y^0(t)+\sum_t a(t)'x^0(t)<\sum_t c(t)'y^*(t)+\sum_t a(t)'x^*(t) \tag{15}$$

Using (1) we have

$$\sum_t \{[c(t)'-\pi_0'A(t)]y^0(t)+a(t)'x^0(t)\}$$
$$<\sum_t \{[c(t)'-\pi_0'A(t)]y^*(t)+a(t)'x^*(t)\} \tag{16}$$

Hence there exists at least one block $t_1\in\{1, 2, \cdots, T\}$, for which the following relation holds:

$$[c(t_1)'-\pi_0'A(t_1)]y^0(t_1)+a(t_1)'x^0(t_1)$$
$$<[c(t_1)'-\pi_0'A(t_1)]y^*(t_1)+a(t_1)'x^*(t_1) \tag{17}$$

Subtracting π_{t_1} from both sides of (17) and using (11) and (12) leads to

$$[c(t_1)'-\pi_0'A(t_1)]y^0(t_1)-\pi_{t_1}+a(t_1)'x^0(t_1)$$
$$<[c(t_1)'-\pi_0'A(t_1)]y^*(t_1)-\pi_{t_1}+a(t_1)'x^*(t_1)$$
$$=[c(t_1)'-\pi_0'A(t_1)]\sum_j {}^j\mu_{t_1}{}^K\, {}^jy_{t_1}{}^K-\pi_{t_1}\sum_j {}^j\mu_{t_1}{}^K+a(t_1)'x^*(t_1)$$
$$=\sum_j {}^j\bar{f}_{t_1}{}^K\, {}^j\mu_{t_1}{}^K+a(t_1)'x^*(t_1) \tag{18}$$

Since ${}^j\bar{f}_{t_1}{}^K$ is the relative cost factor for the basic variable ${}^j\mu_{t_1}{}^K$ in (P2), it vanishes. Therefore,

$$[c(t_1)'-\pi_0'A(t_1)]y^0(t_1)-\pi_{t_1}+a(t_1)'x^0(t_1)<a(t_1)'x^*(t_1) \tag{19}$$

which contradicts (14). Q.E.D.

Next, we estimate the difference between the optimal objective value z_{opt} for (P1) and the current objective value obtained by solving (P2), i.e.,

$$w^* + \sum_t a(t)'x^*(t) \triangleq z^* \tag{20}$$

Introducing an $l_0 \times 1$ vector ξ, we consider the Lagrangean relaxation of (P1) relative to the coupling constraint (1), i.e.,

$$\begin{aligned}\min_{x(t),y(t)} v(\xi) &= \sum_t c(t)'y(t) + \sum_t a(t)'x(t) + \xi'[b - \sum_t A(t)y(t)] \\ &= \sum_t \bar{z}_t(\xi) + \xi'b \qquad \text{subject to (2)}\end{aligned}$$

It follows from Geoffrion[6] that

$$z_{opt} \geqq v(\xi) \qquad \text{for all } \xi \tag{21}$$

If we put $\xi = \pi_0$, a lower bound of z_{opt} is obtained in terms of the solution of the subproblem (P3). Accordingly,

$$\sum_t \bar{z}_t^* + \pi_0'b \leqq z_{opt} \leqq z^* \tag{22}$$

Then, by defining a quantity representing the difference between z_{opt} and z^* as

$$\varepsilon \triangleq w^* + \sum_t a(t)'x^*(t) - (\sum_t \bar{z}_t^* + \pi_0'b) \tag{23}$$

the following theorem is established.

Theorem 2. $\bar{f}(t) = 0$ for all $t \Longleftrightarrow \varepsilon = 0$

Proof. Applying the dual theorem of linear programming to (23) yields

$$\begin{aligned}\varepsilon &= \pi_0'b + \sum_t \pi_t + \sum_t a(t)'x^*(t) - (\sum_t \bar{z}_t^* + \pi_0'b) \\ &= -\sum_t [\bar{z}_t^* - \pi_t - a(t)'x^*(t)] = -\sum_t \bar{f}(t)\end{aligned} \tag{24}$$

For each t, the value of $\bar{f}(t)$ obtained from the solution of (P3) is never greater than that for the current basic feasible solution of (P2). Thus $\bar{f}(t)$ must be non-positive for all t. Therefore, it follows from (24) that ε is non-negative and that ε vanishes if and only if $\bar{f}(t) = 0$ for all t. Q.E.D.

Remarks 1. The optimality condition mentioned above is derived by comparing the current solution of (P2) with all the extreme points of the polyhedrons $Y_t^k (k=1, 2, \cdots, k_t)$ contained in the set $\bar{Y}_t$ for each t. However, since $\bar{Y}_t$ is larger than the feasible region for the t-th block, the optimality test for the current solution need not be checked for all nonbasic variables corresponding to the extreme points in $\bar{Y}_t$. Thus, it follows that the condition mentioned above is sufficient for the optimality, but not necessary.

The results of this section show the procedure for obtaining the optimal solution of (Pl) as follows:

Step I. Select one polyhedron Y_t^K for each t, and construct a RAP with the variables satisfying Cl.

Step II. Solve RAP to obtain w^* and π_0.

Step III. Solve (P3) and calculate $\bar{f}(t)$ for all t.

Step IV. Check condition (14). If (14) holds, the solution (12) is optimal. If not, return to Step I.

The detailed procedure of Step I is described in Section 5, particularly for the case where the optimality condition (14) does not hold.

5. The Procedure for Improving the Current Solution of RAP

In this section we consider the case where, as the result of the optimality test for a current basic feasible solution of RAP, the index $\bar{f}(t)$ is negative for some t (accordingly $\varepsilon>0$).

The block numbers t_i with $\bar{f}(t_i)<0$ are listed, by arranging them in the increasing order, as

$$H \triangleq \{t_i \mid t_i \in \{1, 2, \cdots, T\},\ \bar{f}(t_1) \leqq \bar{f}(t_2) \leqq \cdots \leqq \bar{f}(t_{i_f})<0\} \tag{25}$$

Then, the replacement of the polyhedrons is made for the block listed in H, and a new RAP is constructed with variables corresponding to the extreme points obtained by the new combination of polyhedrons. In this case, RAP does not necessarily have any improved solution because the current basic solution is replaced. Therefore, we must find out a new combination of polyhedrons, if it exists, which assures the improved solution of RAP. The search procedure for this purpose, however, may be tedious, if not impossible. Then we confine the search for improvement to the following restricted extent:

C2. The polyhedron, to which the extreme points corresponding to the current basic variables belong, is replaced one block at a time by a new polyhedron.

The search procedure under C2 is as follows. Let $x^{k*}(t^*)$ be the integer solution of the subproblem (P3) for a block $t^* \in H$. Then the current polyhedron for the t^*-th block is replaced by $Y_{t^*}^{k*}$. As for the other blocks, the current polyhedrons are not replaced. Consequently, we construct a new RAP in the form of the following linear program:

(P4) $\min_{y(t)} w=\sum_{t=1}^{T} c(t)'y(t)$ subject to (1) and

$$\left.\begin{array}{l} B(t)y(t)\geqq d(t)-D(t)\hat{x}(t) \\ y(t)\geqq 0 \end{array}\right\} \tag{26}$$

$$t=1, 2, \cdots, T$$

In this problem, the integer vector $\hat{x}(t)$ is fixed as follows:

$$\hat{x}(t)=\begin{cases} x^{k*}(t^*) & \text{for } t=t^* \\ \bar{x}(t) & \text{for } t\neq t^* \end{cases} \tag{27}$$

where $\bar{x}(t)$ is the integer solution associated with the current basic feasible solution of RAP. The solution of (P4) is obtained by the Dantzig-Wolfe decomposition technique.[5] If the solution of (P4) improves the current solution of RAP, i.e., $z<z^*$ where z^* is given by (20) and

$$z=w+\sum_t a(t)'\hat{x}(t) \tag{28}$$

proceed to the optimality test and check the solution of (P4) thus obtained. If not, the search procedure, i.e. solving (P4) is continued by replacing the polyhedron for another $t\in H$.

We consider the strategy for choosing the number t^* for which block the polyhedron is replaced. The block number t^* should be chosen according to the order listed in H. If the search procedure does not succeed even for the last block t_{i_f}, return to the first block in H and solve (P3) again, excluding the values of $x(t)$ obtained so far by solving (P3). That is to say, in place of (P3) solve the following mixed-integer program as the subproblem:

(P5) $\quad \min_{x(t),y(t)} \ \bar{z}_t(\pi_0)=[c(t)'-\pi_0'A(t)]y(t)+a(t)'x(t) \qquad$ subject to (2) and

$$x(t)\neq x^k(t) \qquad k=k_1, k_2, \cdots, k_\alpha \tag{29}$$

where $x^k(t)$ $(k=k_1,k_2,\cdots,k_\alpha)$ denote the integer solution of (P3) which have failed so far to improve the solution of RAP. If $\bar{f}(t)<0$ holds for the solution of (P5), proceed to the solving of (P4) by substituting the integer solution thus obtained into the $x^{k*}(t^*)$ in (27). On the other hand, if $\bar{f}(t)\geqq 0$ holds for the solution of (P5), remove the number t from the list H.

The procedure mentioned above is terminated when the list H becomes empty. Since the number of the solutions of (P3) with $\bar{f}(t)<0$ is finite, the procedure is completed in a finite number of iterations.

In the case of replacing the polyhedron based upon condition C2, the polyhedron for the block with $\bar{f}(t)\geqq 0$ need not be replaced. Accordingly, the above algorithm gives a procedure examining all possibilities for improving the current basic solution of RAP by replacing a single polyhedron one by one. However, the simultaneous replacement of multiple polyhedrons is not considered in this procedure. Therefore, if the procedure

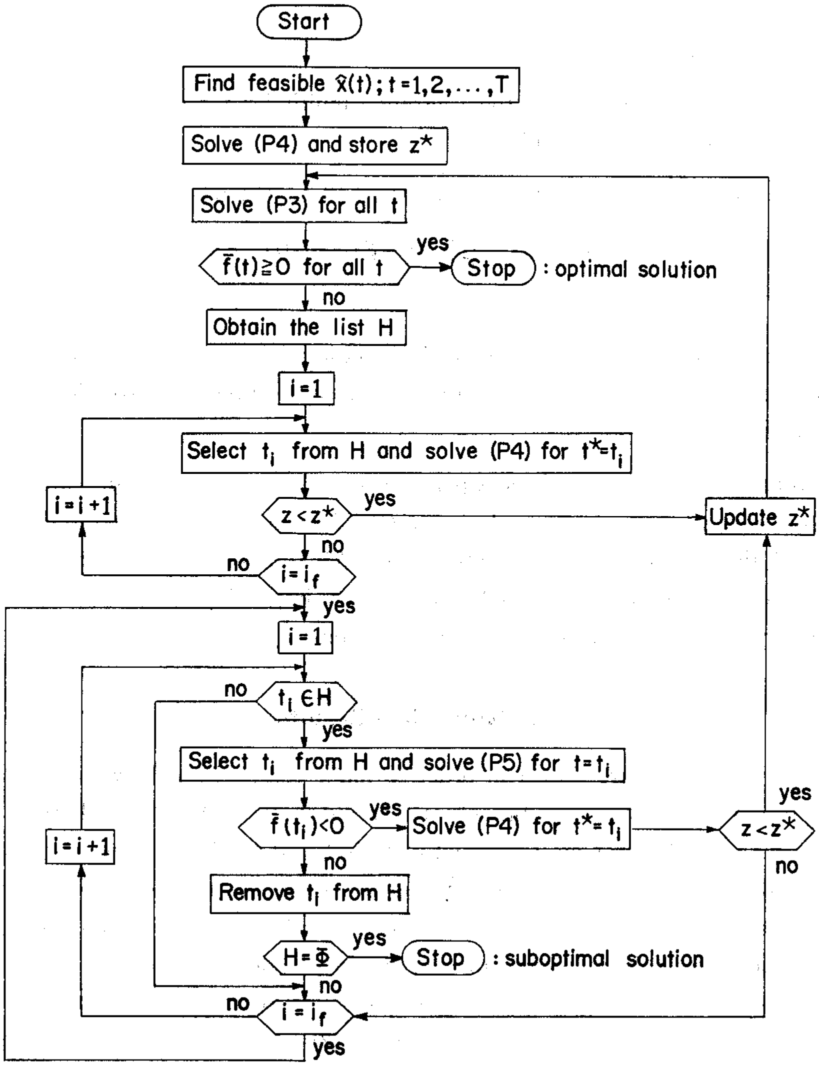

Fig. 1 Flowchart of the present algorithm

terminates with $H=\phi$, the best solution obtained so far is called a suboptimal solution. It is noted that the objective value for the suboptimal solution is less than $z_{opt}+\varepsilon$, where ε is given by (24).

The entire procedure of the algorithm mentioned above is shown by the flowchart in Fig. 1. Since the subproblem (P3) corresponds to (P5) without the constraint (29), both subproblems are solved by the same procedure due to the branch and bound method.

Remarks 2. We modify the problem (P1) by adding the coupling constraint such as

$$\sum_{t=1}^{T} G(t)x(t)=g \tag{30}$$

In this case, by introducing the Lagrange multiplier ξ associated with (30), the following Lagrange problem is obtained:

$$\begin{aligned}\min_{x(t),y(t)} u(\xi)&=\sum_t c(t)'y(t)+\sum_t a(t)'x(t)+\xi'[g-\sum_t G(t)x(t)]\\ &=\sum_t c(t)'y(t)+\sum_t [a(t)'-\xi'G(t)]x(t)+\xi'g \end{aligned} \tag{31}$$

subject to (1) and (2)

which has the same form as (P1). Thus, if the best possible choice for ξ is permissible, the proposed algorithm also provides the optimal or a suboptimal solution to such a modified problem.

Remarks 3. In the previous papers,[3,4] RAP was constructed in a way different from the above, when the integer solution of (P5) for some $t\in H$ is equal to the current integer solution. However, the feasibility of RAP thus obtained is not satisfactory. In this paper, the procedure is revised with regard to improving the feasibility, and finding a better combination of the subproblem solutions.

6. Numerical Results

Numerical results are shown in order to make comparisons between the present algorithm and the conventional branch and bound method (BBM for short). As an example, we treat the problem examined in Refs. 3 and 4. In the previous papers, we investigated the optimal problem of blending raw materials in order to manufacture a certain product, with special regard to the dynamical planning over the time period $[1, T]$. The value of the integer vector $x(t)$ depends on deciding whether each material is used in period t or not. The continuous vector $y(t)$ represents the quantity of each material used in period t. The block constraint (2) is the constraint imposed in each period, which means the required levels of the resultant qualities and a physical constraint on the equipments. On the other hand, the coupling constraint (1) means the condition necessary to continue the normal operation over the whole time period. Three cases are considered for the time period, namely, $T=2$, 3 and 4. In this case, the size of the problem is, corresponding to (P1),

$$m_t=n_t=5, \qquad l_t=24 \qquad \text{for all } t$$
$$l_0=5T \quad (=10, 15 \text{ and } 20)$$

The detailed description of (P1) is shown in the Appendix.

Table 1 summarizes the comparison of the computational results for the fifteen problems prepared. It is noted in item (ii) of this table that the linear programming

Table 1 Computational results

(a) $T=2$

Prob. No.	1	2	3	4	5
The present method					
(i)	4.3371*	4.7685	4.1987	4.3000	4.3027*
(ii)	96	118	162	101	63
(iii)	7.67	11.50	11.49	9.01	6.50
(iv)	0.0	4.00E-1	1.09E-1	2.00E-1	9.54E-7
BBM					
(i)	4.3371	4.7685	4.1987	4.3000	4.3027
(ii)	27	39	31	35	21
(iii)	13.00	18.88	15.70	17.44	10.37

(b) $T=3$

Prob. No.	6	7	8	9	10
The present method					
(i)	6.8000	7.0685	6.3827*	6.8000	6.5416*
(ii)	541	303	89	244	106
(iii)	44.27	25.72	12.18	19.83	10.32
(iv)	5.00E-1	5.00E-1	5.72E-6	5.00E-1	−2.86E-6
BBM					
(i)	6.8000	7.0685	6.3827	6.7999	6.5415
(ii)	135	55	71	97	137
(iii)	248.57	99.88	106.33	164.20	243.59

(c) $T=4$

Prob. No.	11	12	13	14	15
The present method					
(i)	9.0574	8.6042*	8.3974*	8.7000	8.7000
(ii)	384	220	330	356	391
(iii)	33.41	23.46	28.57	32.31	33.97
(iv)	4.00E-1	−6.68E-6	4.77E-6	1.00E-1	1.00E-1
BBM					
(i)	9.0574	8.6041	8.3974	8.7000	8.6999
(ii)	173	135	285	191	143
(iii)	627.65	517.35	1222.88	1023.90	669.18

Notes: (i) Objective function. The value with an asterisk indicates that the optimality condition (14) holds.
(ii) Number of linear programming problems actually solved.
(iii) Computing time in seconds.
(iv) The value of ε given by (24).

problems solved in the present method have smaller sizes than those in BBM. Referring to Table 1, both algorithms require nearly the same computing time for $T=2$. However, as T increases, the present algorithm tends to have less computing time than BBM. Further, the solution obtained as the suboptimal solution in the present algorithm seems to be optimal, as compared with the result obtained by BBM. Therefore, the present algorithm is expected to be efficient from the standpoint of computing time. On the other hand, the value of ε is not so small, although the true difference is nearly zero. Accordingly, a more accurate estimation of the lower bound of the objective value needs to be developed in order to revise the present algorithm.

7. Conclusion

An algorithm has been developed for solving mixed-integer linear programming problems with an angular structure. The original decomposition technique of Dantzig and Wolfe is confined to solving linear programs with continuous variables only. However, by a modification of this technique, the present method can be applied to the mixed-integer problems given by (P1).

For further completion of the decomposition method, the following consideration is continued:

1. A more strict procedure for the case of failing in the optimality test (14).
2. A procedure for solving the problem (P1) subject to the additional constraint (30).

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Appendix

In this example, the coupling constraint (1) is given by

$$\sum_{\tau=1}^{t} y(\tau) \leqq b(t) \qquad t=1, 2, \cdots, T \tag{A. 1}$$

where $b(t)$ is a 5×1 vector. Then the vector b in (1) is

$$b=(b(1)', b(2)', \cdots, b(T)')' \tag{A. 2}$$

The block constraint (2) is given, in the same form for all t, by

$$By(t)+Dx(t) \geqq d \qquad t=1, 2, \cdots, T \tag{A. 3}$$

where

$$\left.\begin{array}{l}
B=\begin{pmatrix} B_1 \\ B_2 \\ I \\ -I \\ 0 \end{pmatrix} \qquad D=\begin{pmatrix} 0 \\ 0 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} \qquad d=\begin{pmatrix} d_1 \\ d_2 \\ 0 \\ 0 \\ -L \end{pmatrix} \\
D_1=-0.01\times \mathrm{diag}(2, 2, 2, 4, 2) \\
D_2=0.1\times \mathrm{diag}(4.35, 6.00, 5.55, 6.00, 3.60) \\
D_3=-(1.0, 1.0, 1.0, 1.0, 1.0) \\
d_1=(26.0, 2.3, 4.5, 1.15, 75.0, 1.0)' \\
d_2=-(28.0, 12.0, 0.65, 2.5, 1.4, 24.0, 1.0)' \\
B_1=\begin{pmatrix} 17.2 & 21.0 & 29.4 & 22.3 & 41.6 \\ 1.2 & 2.42 & 3.66 & 2.37 & 3.40 \\ 6.76 & 6.57 & 4.08 & 5.05 & 2.49 \\ 1.55 & 1.34 & 1.13 & 1.26 & 0.76 \\ 49.0 & 138.0 & 163.0 & 50.0 & 80.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix} \\
B_2=-\begin{pmatrix} 17.2 & 21.0 & 29.4 & 22.3 & 41.6 \\ 8.5 & 11.5 & 9.8 & 12.3 & 10.3 \\ 0.74 & 0.52 & 0.74 & 0.33 & 0.72 \\ 1.20 & 2.42 & 3.66 & 2.37 & 3.40 \\ 1.87 & 1.31 & 0.56 & 2.26 & 0.14 \\ 21.7 & 22.3 & 16.5 & 37.1 & 4.6 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix}
\end{array}\right\} \tag{A. 4}$$

In (A. 4), I is the 5×5 identity matrix, and 0 is a zero matrix or a zero vector of an appropriate dimension. The scalar parameter L is given as $L\leqq 5$.

We prepare fifteen numerical examples by varying the values of the parameters $c(t)$, $a(t)$, $b(t)$ and L, the details of which are omitted here.