A Uniformly Convergent Expression of the Magnetic Field Induced by a Helical Coil

By

Yoshio SONE

Department of Aeronautical Engineering, Kyoto University

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Abstract

A uniformly convergent series expression is given for the magnetic field which is induced by electric current in a straight helical coil with a uniform pitch.

1. Introduction

It is well known that the magnetic field which is induced by electric current in a straight helical coil with a uniform pitch is given in infinite series including the modified Bessel functions. The formulae, however, are conditionally convergent or divergent on the cylindrical surface around which the coil is wound, although the magnetic field is regular except at the coil. In this paper we try to reconstruct the series in such a form that the main singularity is isolated in a closed form and that the remaining series is uniformly convergent.

2. Magnetic Field by a Helical Coil

The fundamental equations governing the magnetic flux density $\boldsymbol{B}$ induced by the current density $\boldsymbol{J}$ are:

$$\operatorname{div} \boldsymbol{B}=0, \tag{1}$$

$$\operatorname{curl} \boldsymbol{B}=\mu_o \boldsymbol{J}, \tag{2}$$

where μ_o is the magnetic permeability. We here introduce a helical coordinate system $(\tilde{r}, \phi, \tilde{z})$ which is related with the cylindrical coordinate system (r, θ, z) by

$$\left.\begin{aligned}\tilde{r}&=r,\\ \phi&=\theta-kz,\\ \tilde{z}&=z,\end{aligned}\right\}\tag{3}$$

where k is a constant which determines the pitch of the helix ($\tilde{r}$=const., ϕ=const.). If we assume the helical symmetry of the field, each component of $\boldsymbol{B}$ and $\boldsymbol{J}$ in the cylindrical coordinate system, say f, satisfies

$$\frac{\partial f}{\partial \tilde{z}}=0. \tag{4}$$

Thus,

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial \phi}, \qquad \frac{\partial f}{\partial z}=-k\frac{\partial f}{\partial \phi}.$$

Then, Eq. (1) is rewritten as

$$\frac{\partial \tilde{r}B_r}{\partial \tilde{r}}+\frac{\partial B_\theta-k\tilde{r}B_z}{\partial \phi}=0, \tag{5}$$

where B_r, B_θ, and B_z are the r, θ, and z components of $\boldsymbol{B}$ respectively. Therefore, in case of helical symmetry there exists a scalar function ψ such that

$$B_r=\frac{1}{\tilde{r}}\frac{\partial \psi}{\partial \phi}, \qquad B_\theta-k\tilde{r}B_z=-\frac{\partial \psi}{\partial \tilde{r}}. \tag{6}$$

Putting

$$\zeta=B_z+k\tilde{r}B_\theta, \tag{7}$$

we have

$$\left.\begin{aligned}B_\theta&=\frac{k\tilde{r}\zeta-\dfrac{\partial \psi}{\partial \tilde{r}}}{1+k^2\tilde{r}^2},\\ B_z&=\frac{\zeta+k\tilde{r}\dfrac{\partial \psi}{\partial \tilde{r}}}{1+k^2\tilde{r}^2}.\end{aligned}\right\}\tag{8}$$

Further, the r, θ, and z components of curl $\boldsymbol{B}$ are given as:

$$\left.\begin{aligned}(\mathrm{curl}\,\boldsymbol{B})_r&=\frac{k}{x}\frac{\partial \zeta}{\partial \phi},\\ (\mathrm{curl}\,\boldsymbol{B})_\theta&=-k^2x\Delta^*\psi-k\frac{\partial}{\partial x}\left(\frac{\zeta}{1+x^2}\right),\\ (\mathrm{curl}\,\boldsymbol{B})_z&=-k^2\Delta^*\psi+\frac{k}{x}\frac{\partial}{\partial x}\left(\frac{x^2\zeta}{1+x^2}\right),\end{aligned}\right\}\tag{9}$$

where

$$\Delta^*\psi=\frac{1}{x}\frac{\partial}{\partial x}\left(\frac{x}{1+x^2}\frac{\partial \psi}{\partial x}\right)+\frac{1}{x^2}\frac{\partial^2 \psi}{\partial \phi^2},$$

$$x=k\tilde{r}.$$

If the current is in the helical direction (x=const., ϕ=const.), namely,

$$J_r : J_\theta : J_z = 0 : x : 1,$$

where J_r, J_θ, and J_z are r, θ, and z components of $\boldsymbol{J}$ respectively, we have

$$\frac{\partial \zeta}{\partial \phi} = \frac{\partial \zeta}{\partial x} = 0,$$

from Eqs. (2) and (9). With these and (4), ζ is seen to be a constant, i.e.,

$$\zeta = \zeta_o. \tag{10}$$

When a helical coil (with its pitch $2\pi/k$) which lies at $x=X$ and $\phi=\Phi$ carries a current I_o, the current density J_z is given

$$J_z = \frac{I_o k^2}{X} \delta(x-X)\delta(\phi-\Phi), \tag{11}$$

where δ is Dirac's delta function. From Eqs. (2), (9), and (11) we have

$$\Delta^*\left(\psi - \frac{\zeta_o}{2k}x^2\right) = -\frac{\mu_o I_o}{X}\delta(x-X)\delta(\phi-\Phi). \tag{12}$$

In order to get ψ, we have only to obtain the solution of the following equation:

$$\Delta^* \psi_\delta(x, \phi;\ X, \Phi) = \delta(x-X)\delta(\phi-\Phi). \tag{13}$$

The solution of Eq. (13) has been given in the following form.

$$\begin{aligned} \psi_\delta &= \frac{1}{\pi}\left\{ \frac{X\left(\frac{X^2}{2} + \ln X\right)}{2} + X^2 x \sum_{j=1}^{\infty} K_j'(jX) I_j'(jx) \cos j(\phi-\Phi) \right\}, \\ &\qquad\qquad x \leq X, \\ &= \frac{1}{\pi}\left\{ \frac{X\left(\frac{x^2}{2} + \ln x\right)}{2} + X^2 x \sum_{j=1}^{\infty} I_j'(jX) K_j'(jx) \cos j(\phi-\Phi) \right\}, \\ &\qquad\qquad x \geq X, \end{aligned} \tag{14}$$

where I_j and K_j are the modified Bessel functions defined in Ref. 1, and I_j' and K_j' are their derivatives with their argument. For $\boldsymbol{B}$ we need the derivatives of ψ, namely,

$$\left.\begin{aligned} \frac{\partial \psi_\delta}{\partial x} &= \frac{1}{\pi}\frac{X^2(1+x^2)}{x}\sum_{j=1}^{\infty} jK_j'(jX) I_j(jx) \cos j(\phi-\Phi), \quad x \leq X, \\ &= \frac{1}{\pi}\left[\frac{1}{2}X\left(x+\frac{1}{x}\right) + \frac{X^2(1+x^2)}{x}\sum_{j=1}^{\infty} jI_j'(jX) K_j(jx) \cos j(\phi-\Phi)\right], \\ &\qquad\qquad x \geq X, \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} \frac{1}{x}\frac{\partial \psi_\delta}{\partial \phi} &= -\frac{X^2}{\pi}\sum_{j=1}^{\infty} jK_j'(jX) I_j'(jx) \sin j(\phi-\Phi), \quad x \leq X, \\ &= -\frac{X^2}{\pi}\sum_{j=1}^{\infty} jI_j'(jX) K_j'(jx) \sin j(\phi-\Phi), \quad x \geq X. \end{aligned}\right\} \tag{16}$$

With ψ_s just given, ψ in Eq. (9) is given as:

$$\psi=\frac{\zeta_o}{2k}x^2-\frac{\mu_o I_o}{X}\psi_s. \tag{17}$$

The constant ζ_o is related to the uniform field in the z direction. When no uniform field is applied (thus $\boldsymbol{B}\to 0$ as $x\to\infty$), ζ_o is determined as:

$$\zeta_o=\frac{\mu_o I_o k}{2\pi}. \tag{18}$$

3. Uniformly Convergent Expression

The infinite series in these well known formulae (14), (15), and (16) are conditionally convergent or divergent on the surface $x=X$, although the magnetic field is regular there except on the coil ($x=X$, $\phi=\Phi$), where it is singular and infinite. In order to avoid this, we try to reconstruct the series. Namely, we rewrite the series, say $\sum s_j$, in Eqs. (14), (15), and (16) in the following form:

$$\sum_j s_j=\sum_j (s_j-\bar{s}_j)+\sum_j \bar{s}_j, \tag{19}$$

where $\bar{s}_j$ is the first one or two terms of the asymptotic expansion of s_j as $j\to\infty$. Then, as will be seen below, the first series is more rapidly and uniformly convergent and therefore is continuous, and the second one can be summed up in a closed form, where the main singularity of $\sum s_j$ is isolated. The results are given as follows.

$$\left.\begin{aligned}\psi_s&=\frac{1}{\pi}\left[\frac{1}{2}X\left(\frac{X^2}{2}+\ln X\right)+X^2xG_I\right], && x\leq X,\\ &=\frac{1}{\pi}\left[\frac{1}{2}X\left(\frac{x^2}{2}+\ln x\right)+X^2xG_O\right], && x\geq X,\end{aligned}\right\} \tag{20}$$

$$\begin{aligned}G_I=&\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}}{4Xx}\ln[1-2e^{\varepsilon}\cos(\phi-\Phi)+e^{2\varepsilon}]\\ &+\sum_{j=1}^{\infty}\left[K_j'(jX)I_j'(jx)+\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}e^{j\varepsilon}}{2Xxj}\right]\cos j(\phi-\Phi),\end{aligned} \tag{21}$$

$$\begin{aligned}G_O=&\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}}{4Xx}\ln[1-2e^{-\varepsilon}\cos(\phi-\Phi)+e^{-2\varepsilon}]\\ &+\sum_{j=1}^{\infty}\left[I_j'(jX)K_j'(jx)+\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}e^{-j\varepsilon}}{2Xxj}\right]\cos j(\phi-\Phi),\end{aligned} \tag{22}$$

where

$$\eta=\sqrt{1+x^2}+\ln\frac{x}{1+\sqrt{1+x^2}},\qquad H=\sqrt{1+X^2}+\ln\frac{X}{1+\sqrt{1+X^2}},$$

$$\varepsilon=\eta-H.$$

$$\begin{aligned}\frac{\partial\psi_s}{\partial x}&=\frac{1}{\pi}\frac{X^2(1+x^2)}{x}G_I^1, && x\le X,\\ &=\frac{1}{\pi}\frac{X(1+x^2)}{x}\left(\frac{1}{2}+XG_O^1\right), && x\ge X,\end{aligned}\tag{23}$$

$$\begin{aligned}G_I^1=&-\frac{(1+X^2)^{1/4}}{2X(1+x^2)^{1/4}}\left\{\frac{e^{\epsilon}\cos(\phi-\Phi)-e^{2\epsilon}}{1-2e^{\epsilon}\cos(\phi-\Phi)+e^{2\epsilon}}\right.\\ &\left.-\frac{u(t)-v(T)}{2}\ln\left[1-2e^{\epsilon}\cos(\phi-\Phi)+e^{2\epsilon}\right]\right\}\\ &+\sum_{j=1}^{\infty}\left[jK_j'(jX)I_j(jx)\right.\\ &\left.+\frac{(1+X^2)^{1/4}}{2X(1+x^2)^{1/4}}e^{j\epsilon}\left(1+\frac{u(t)-v(T)}{j}\right)\right]\cos j(\phi-\Phi),\end{aligned}\tag{24}$$

$$\begin{aligned}G_O^1=&\frac{(1+X^2)^{1/4}}{2X(1+x^2)^{1/4}}\left\{\frac{e^{-\epsilon}\cos(\phi-\Phi)-e^{-2\epsilon}}{1-2e^{-\epsilon}\cos(\phi-\Phi)+e^{-2\epsilon}}\right.\\ &\left.-\frac{v(T)-u(t)}{2}\ln\left[1-2e^{-\epsilon}\cos(\phi-\Phi)+e^{-2\epsilon}\right]\right\}\\ &+\sum_{j=1}^{\infty}\left[jI_j'(jX)K_j(jx)\right.\\ &\left.-\frac{(1+X^2)^{1/4}}{2X(1+x^2)^{1/4}}e^{-j\epsilon}\left(1+\frac{v(T)-u(t)}{j}\right)\right]\cos j(\phi-\Phi),\end{aligned}\tag{25}$$

where

$$t=\frac{1}{\sqrt{1+x^2}},\qquad T=\frac{1}{\sqrt{1+X^2}},$$

$$u(t)=\frac{3t-5t^3}{24},\qquad v(t)=\frac{-9t+7t^3}{24}.$$

$$\begin{aligned}\frac{1}{x}\frac{\partial\psi_s}{\partial\phi}&=-\frac{X^2}{\pi}G_I^2, && x\le X,\\ &=-\frac{X^2}{\pi}G_O^2, && x\ge X,\end{aligned}\tag{26}$$

$$\begin{aligned}G_I^2=&-\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}e^{\epsilon}\sin(\phi-\Phi)}{2Xx[1-2e^{\epsilon}\cos(\phi-\Phi)+e^{2\epsilon}]}\\ &+\sum_{j=1}^{\infty}\left[jK_j'(jX)I_j'(jx)\right.\\ &\left.+\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}e^{j\epsilon}}{2Xx}\right]\sin j(\phi-\Phi),\end{aligned}\tag{27}$$

$$\begin{aligned}G_O^2=&-\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}e^{-\epsilon}\sin(\phi-\Phi)}{2Xx[1-2e^{-\epsilon}\cos(\phi-\Phi)+e^{-2\epsilon}]}\\ &+\sum_{j=1}^{\infty}\left[jI_j'(jX)K_j'(jx)\right.\\ &\left.+\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}e^{-j\epsilon}}{2Xx}\right]\sin j(\phi-\Phi).\end{aligned}\tag{28}$$

The second terms in the square brackets under the $\sum$ signs in Eqs. (21), (22), (24), (25), (27), and (28) are the negatives of the asymptotic forms (as $j\to\infty$) of their preceding terms in the brackets. They cancel the leading terms of G_I, G_O, etc. after the summation is carried out. In their summation the following formulae are used.

$$\left.\begin{aligned}&\sum_{j=1}^{\infty}\frac{a^j}{j}\cos jb=-\frac{1}{2}\ln(1-2a\cos b+a^2),\\&\sum_{j=1}^{\infty}a^j\cos jb=\frac{a\cos b-a^2}{1-2a\cos b+a^2},\\&\sum_{j=1}^{\infty}a^j\sin jb=\frac{a\sin b}{1-2a\cos b+a^2}.\end{aligned}\right\}\tag{29}$$

Each term, say g_j, of the series in G_I, G_O, etc. can be estimated by the error estimate of the asymptotic formulae of the modified Bessel functions in Ref. 2 (cf. Appendix) as follows.

For G_I and G_O:

$$g_j=O[j^{-3}(1+c_1(j-1)|\varepsilon|)e^{-(j-1)|\varepsilon|}],\tag{30a}$$

for G_I^1 and G_O^1:

$$\frac{1}{x}g_j=O(j^{-2}e^{-(j-1)|\varepsilon|}),\tag{30b}$$

for G_I^2 and G_O^2:

$$g_j=O[j^{-2}(1+c_2(j-1)|\varepsilon|)e^{-(j-1)|\varepsilon|}],\tag{30c}$$

where c_1 and c_2 are some constants and X is fixed ($\neq 0$). It is seen that all the series are uniformly convergent and therefore are continuous. The following facts, however, should be noted. The leading terms of G_I, G_O, etc. are their main nonanalytic terms, but not all the nonanalyticities are isolated; the infinite series have discontinuity in some of their derivatives on the surface $x=X$.

When N helical coils each of which carries current I_0 are placed at $x=X$, $\phi=\frac{2\pi i}{N}$ ($i=1, 2, \cdots, N$), the solution ψ is given as the superposition of Eq. (17), namely

$$\psi=\frac{N\zeta_0}{2k}x^2-\frac{\mu_0 I_0}{X}\psi_{Ns},$$

$$\psi_{Ns}=\sum_{i=1}^{N}\psi_s\left(x, \phi;\ X, \frac{2\pi}{N}i\right).$$

The summation ψ_{Ns} can be given in a rather simpler form by the following steps. If we use Eqs. (14), (15), and (16) as ψ_s, $\frac{\partial\psi_s}{\partial x}$, and $\frac{1}{x}\frac{\partial\psi_s}{\partial\phi}$ respectively, and change the order of summation to make the sum with respect to i first, where the following formulae are applied:

$$\sum_{i=1}^{N} \cos j\left(\phi-\frac{2\pi}{N}i\right)=N\cos j\phi, \qquad j=Nk,$$
$$=0, \qquad \text{otherwise},$$
$$\sum_{i=1}^{N} \sin j\left(\phi-\frac{2\pi i}{N}\right)=N\sin j\phi, \qquad j=Nk,$$
$$=0, \qquad \text{otherwise}, \qquad (k=1, 2, \cdots),$$

then we find that ψ_{Ns}, $\frac{\partial\psi_{Ns}}{\partial x}$, and $\frac{1}{x}\frac{\partial\psi_{Ns}}{\partial\phi}$ are, respectively, N times Eqs. (14), (15), and (16) with the replacements Φ by 0 and j by Nj except j beneath the $\sum$ signs $[\sum_j f_j(j) \to \sum_j f_{Nj}(Nj)]$. From these forms their uniform convergent expressions corresponding to Eqs. (21), (22), (24), (25), (27), and (28) can be given very easily. The formulae thus obtained are convenient to see the asymptotic behavior for $N\to\infty$.

Finally, it is noted that some of the data in Ref. 3 were obtained by the application of the present work.

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Appendix, Asymptotic estimate of $I_j(jx)$, $K_j(jx)$, etc. [2]

The following asymptotic expansions hold uniformly with respect to x $(0\leq x\leq\infty)$ as $j\to\infty$.

$$I_j(jx)\sim\frac{1}{\sqrt{2\pi j}}\frac{e^{j\eta}}{(1+x^2)^{1/4}}\left[1+\frac{u(t)}{j}+O\left(\frac{1}{j^2}\right)\right],$$
$$K_j(jx)\sim\sqrt{\frac{\pi}{2j}}\frac{e^{-j\eta}}{(1+x^2)^{1/4}}\left[1-\frac{u(t)}{j}+O\left(\frac{1}{j^2}\right)\right],$$
$$I_j'(jx)\sim\frac{1}{\sqrt{2\pi j}}\frac{(1+x^2)^{1/4}}{x}e^{j\eta}\left[1+\frac{v(t)}{j}+O\left(\frac{1}{j^2}\right)\right],$$
$$K_j'(jx)\sim-\sqrt{\frac{\pi}{2j}}\frac{(1+x^2)^{1/4}}{x}e^{-j\eta}\left[1-\frac{v(t)}{j}+O\left(\frac{1}{j^2}\right)\right].$$

As for their products the following relations hold.

$$K_j'(jX)I_j'(jx)\sim-\frac{(1+X^2)^{1/4}(1+x^2)^{1/4}}{2Xx}\frac{e^{js}}{j}\left[1+\frac{v(t)-v(T)}{j}+O\left(\frac{1}{j^2}\right)\right],$$

$$K_j'(jX)I_j(jx) \sim -\frac{(1+X^2)^{1/4}}{2X(1+x^2)^{1/4}}\frac{e^{j\epsilon}}{j}\left[1+\frac{u(t)-v(T)}{j}+O\left(\frac{1}{j^2}\right)\right],$$

$$I_j'(jX)K_j(jx) \sim \frac{(1+X^2)^{1/4}}{2X(1+x^2)^{1/4}}\frac{e^{-j\epsilon}}{j}\left[1+\frac{v(T)-u(t)}{j}+O\left(\frac{1}{j^2}\right)\right].$$

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