On the Practically Reachable Subspace of the Discrete-Time Systems

By

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Abstract

Spanning vectors of the practically reachable subspace of a single input discrete-time system are studied. Reachable subspace is the space spanned by the vectors of the form $b, Ab, A^2b, \cdots$. If A^kb lies on the subspace S_k spanned by the vectors $b, Ab, A^2b, \cdots, A^{k-1}$, then $A^{k+l}b$ also lies on the same subspace S_k for any $l \geq 0$, and so, S_k is the reachable subspace. On checking, if A^kb lies on S_k by numerical calculation, we usually assume that A^kb lies on S_k in practice, if the distance δ of A^kb from S_k is small. As the distance of $A^{k+l}b$ from S_k is not guaranteed to be small in this case, it must be examined if $A^{k+l}b$ can be assumed to lie on S_k, in practice or not.

Concerning the distance of A^{k+l} from S_k, the following results are obtained. i) When $k<n-1$, where n is the dimension of the state, $A^{k+l}b$ can have an arbitrary value for $n\text{-}k\text{-}1>l>1$. ii) If the maximal number of practically independent vectors are taken from the set of vectors $b, Ab, \cdots, A^{n-1}b$, and if the absolute values of the eigenvalues are less than unity, then the distance between $A^{k+l}b$ and the subspace S spanned by these practically independent vectors does not become larger than a certain value $M\delta$ for any l. The condition that S becomes practically reachable subspace is also studied.

1. Introduction

Consider a discrete-time system given by

$$x(\tau+1)=Ax(\tau)+bu(\tau) \tag{1}$$

where x and b are n-vectors, u is a scalar and A is a non-singular $n \times n$ matrix. As is well known, the reachable subspace is spanned by the vectors $b, Ab, A^2b, \cdots$. If A^kb is expressed by a linear combination of $b, Ab, A^2b, \cdots, A^{k-1}b$, then for any $l \geq 0$, $A^{k+l}b$ can be expressed by a linear combiation of these vectors. In this case, the reachable subspace is given by the space $S(b, Ab, \cdots, A^{k-1}b)$ spanned by the vectors $b, Ab, \cdots, A^{k-1}b$, and the algorithm to get the reachable subspace is given as follows. Suppose

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$b, Ab, \cdots, A^{k-1}b$ are independent. Add a new vector A^kb to the set and check if $b, Ab, \cdots, A^kb$ are independent. If they are not, then $b, Ab, \cdots, A^{k-1}b$ are the spanning vectors of the reachable subspace. If they are independent, add another new vector $A^{n+1}b$ to the set and continue the procedure as before. This procedure will end, at most, before $A^{n+1}b$.

From the theoretical viewpoint, checking that the Grammian is zero is useful for testing if a set of k ($k>n$) vectors is linearly independent. In computer applications, however, to check if the Gramian precisely equals zero is not practical because of the rounding off errors arising in the computations.

Suppose $b, Ab, \cdots, A^{k-1}b$ are linearly independent. The height h_k of A^kb from $S(b, Ab, \cdots, A^{k-1}b)$ is given by

$$h_k{}^2=\frac{|G(b, Ab, \cdots, A^{k-1}b, A^kb)|}{|G(b, Ab, \cdots, A^{k-1}b)|} \tag{2}$$

where $|G(\cdot)|$ is the Gramian.[2] Thus for computer applications, we usually set a small number $\delta>0$, and if

$$|h_k|<\delta \tag{3}$$

holds, we then suppose that A^kb is practically on the subspace $S(b, Ab, \cdots, A^{k-1}b)$, and that $b, Ab, \cdots, A^{k-1}b, A^kb$ are linearly dependent. In this case, we usually assume that the reachable subspace is practically $S(b, \cdots, A^{k-1}b)$. But strictly speaking, since the vector A^kb is approximately on the $S(b, Ab, \cdots, A^kb)$, it is nessceary to prove that the height of $A^{k+l}b$ from $S(b, Ab, \cdots, A^{k-1}b)$ is also small for any $l>0$. In some cases, this expectation is violated, unfortunately, as is shown in this paper, and for some vectors like $A^{k+l}b$, the height from $S(b, \cdots, A^{k-1}b)$ can take an arbitrarily large value.

2. Fundamental Relations

Theorem 1.

For any set of vectors $\xi_1, \xi_2, \cdots, \xi_{n+1}$, if $\xi_1, \xi_2, \cdots, \xi_n$ are independent, there exists a pair (A, b) that satisfies

$$\xi_i=A^{i-1}b \qquad (i=1, 2, \cdots, n+1) \tag{4}$$

Proof

Denote the matrix whose columns are $\xi_1, \cdots, \xi_n$ by $[\xi_1, \cdots, \xi_n]$. Then, Eq. (4) is equivalent to

$$b=\xi_1 \tag{5}$$

$$A[\xi_1, \cdots, \xi_n]=[\xi_2, \cdots, \xi_{n+1}] \tag{6}$$

As $\xi_1, \xi_2, \ldots, \xi_n$ are independent, $[\xi_1, \ldots, \xi_n]$ is non-singular. Thus, A is given by

$$A=[\xi_2, \cdots, \xi_{n+1}]\,[\xi_1, \cdots, \xi_n]^{-1} \tag{7}$$

Q.E.D.

The condition that $\xi_1, \ldots, \xi_n$ are independent can be slightly loosened.

Theorem 2.

For any set of m $(m>n)$ independent vectors $\xi_1, \xi_2, \cdots, \xi_m$, and for any set of scalars $c_1, c_2, \cdots, c_m$, there exists a pair (A, b) satisfying the relations

$$\xi_i=A^{i-1}b \qquad (i=1, 2, \cdots, m) \tag{8}$$

$$A^m b=\sum_{j=1}^{m} c_j\xi_j \tag{9}$$

Proof

The Eqs. (8) (9) are equivalent to the relation

$$[\xi_2, \cdots, \xi_m, \sum_{j=1}^{m} c_j\xi_j]=A[\xi_1, \cdots, \xi_{m-1}, \xi_m] \tag{10}$$

Take a set of vectors $\zeta_1, \zeta_2, \cdots, \zeta_{n-m}$ by which $\xi_1, \xi_2, \cdots, \xi_m, \zeta_1, \cdots, \zeta_{n-m}$ are independent. For any $\eta_1,\eta_2, \ldots, \eta_{n-m}$

$$A=[\xi_2, \cdots, \xi_m, \sum_{j=1}^{m} c_j\xi_j, \xi_1, \cdots, \xi_{n-m}]\,[\xi_1, \cdots, \xi_m, \zeta_1, \cdots, \zeta_{n-m}]^{-1} \tag{11}$$

satisfies Eqs. (8) and (9).

Q.E.D.

In the proof of theorem 2, we can make a polynonial of the order m

$$\psi(\lambda)=\lambda^m-\sum_{j=1}^{m} c_j\lambda^{j-1} \tag{12}$$

which is the minimal polynomial of A for the vector b. As the characteristic polynominal is divisable by the minimal polynomial, m out of n eigen-values of A can also be taken arbitrarily. Therefore, the following theorem holds.

Theorem 3.

Given an arbitrary set of m $(m\leq n)$ independent vectors $\xi_1, \xi_2, \cdots, \xi_m$ and m scalars $\lambda_1, \lambda_2, \cdots, \lambda_m$, there exists a pair (A, b) that satisfies the following conditions:

i) Eq. (8) holds.

ii) m eigen-values are $\lambda_1, \lambda_2, \cdots, \lambda_m$.

iii) $A^m b$ is linearly dependent on $\xi_1, \xi_2, \cdots, \xi_m$.

In the case $m=n$, A is unique.

From theorem 1. the following theorem is obtained directly.

Theorem 4.

For any k $(k>n)$ and positive δ and L, there exists a triplet (A, b, l) $(0<l\leq n-k)$ that satisfies

$$\frac{|G(b, Ab, \cdots, A^{k-1}b, A^k b)|}{|G(b, Ab, \cdots, A^{k-1}b)|} < \delta^2 \tag{13}$$

$$\frac{|G(b, Ab, \cdots, A^{k-1}b, A^{k+l}b)|}{|G(b, Ab, \cdots, A^{k-1}b)|} > L^2 \tag{14}$$

Proof

As $k<n$, there exists an $l>0$ with which $k<k+l\leq n$. Take a set of vectors $\xi_1, \xi_2, \cdots, \xi_{k+1}, \xi_{k+l+1}$ which satisfies

$$\frac{|G(\xi_1, \xi_2, \cdots, \xi_{k+1})|}{|G(\xi_1, \xi_2, \cdots, \xi_k)|} < \delta^2 \tag{15}$$

$$\frac{|G(\xi_1, \xi_2, \cdots, \xi_k, \xi_{k+l+1})|}{|G(\xi_1, \xi_2, \cdots, \xi_k)|} > L^2 \tag{16}$$

$\xi_{k+2}, \cdots, \xi_{k+l}, \xi_{k+l+2}, \cdots, \xi_{n+1}$ can be taken arbitrarily, as long as $\xi_1, \cdots, \xi_n$ are independent. It is clear that the pair (A, b)

$$\begin{aligned} b &= \xi_1 \\ A &= [\xi_2, \cdots, \xi_{n+1}]\,[\xi_1, \cdots, \xi_n]^{-1} \end{aligned} \tag{17}$$

satisfies Eqs. (13) and (14).

Theorem 4 says nothing about the eigen-values of A, so the absolute value of the eigen-values of A thus obtained may be greater than unity.

Theorem 5.

For any positive δ, L, for any k $(k>n-1)$ and for any set of n scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$, there exists a triplet that satisfies the following conditions:

i) Eqs. (13) and (14) hold true.

ii) $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigen-values of A

Proof

Let

$$\begin{aligned} \psi(\lambda) &\triangleq (\lambda-\lambda_1) \cdots\cdots (\lambda-\lambda_n) \\ &= \lambda^n - \sum_{j=1}^{n} c_j \lambda^{j-1} \end{aligned} \tag{18}$$

As $k<n-1$, there exists an $l>0$ such that $k<k+l\leq n-1$. Take a set of independent vectors that satisfy Eqs. (15) and (16). Let

$$\xi_{n+1} = \sum_{j=1}^{n} c_j \xi_j \tag{19}$$

It is easily verified that the pair (A, b)

$$A = [\xi_2, \cdots, \xi_n, \xi_{n+1}]\,[\xi_1, \cdots, \xi_m]^{-1} \tag{20}$$

$$b = \xi_1 \tag{21}$$

satisfies Eqs. (13) and (14), and that eigen-values of A are $\lambda_1, \cdots, \lambda_n$.

Q.E.D.

It is seen by theorems 4 and 5 that even if Eq. (13) may hold for any small δ, $A^{k+l}b$ is not necessarily near enough to $S(p, Ab, \cdots, A^{k-1}b)$ for all $l>0$ if $k<n$.

3. Number of Vectors to Construct Practically Reachable Subspace

Define the practically reachable subspace as follows.

Definition (Practically Reachable Subspace)

The practically reachable subspace is the space spanned by a set of vectors $A^{k_1}b$, $A^{k_2}b, \cdots, A^{k_m}b$ which satisfies the following conditions.

i) $0=k_1<k_2<\cdots<k_m$

ii) $$\frac{|G(b, \cdots, A^{k_i}b)|}{|G(b, \cdots, A^{k_{i-1}}b)|}>\delta^2 \quad \text{for all } i \tag{22}$$

iii) for any $k_{i-1}<l<k_i$

$$\frac{|G(b, \cdots, A^{k_{i-1}}b, A^l b)|}{|G(b, \cdots, A^{k_{i-1}}b)|}<\delta^2 \tag{23}$$

iv) for any $l>k$

$$\frac{|G(b, \cdots, A^{k-1}b, A^l b)|}{|G(b, \cdots, A^{k-1}b)|}<\delta^2 \tag{24}$$

End of definition

If the input to the system is not very large, the state of the system given by Eq. (1) moves only on the neighbourhood of the practically reachable subspace.

As is stated in chapter 2, if $A^k b$ is dependent on $b, \cdots, A^{k-1}b$, then $A^{k+l}b$ is also dependent on $b, \cdots, A^{k-1}b$. Therefore, to check if $A^{k+l}b$ is independent of $b, \cdots, A^{k-1}b$, is not necessary. In case $A^k b$ is only practically dependent on $b, \cdots, A^{k-1}b$, $A^{k+l}b$ is not necessarily practically dependent on the same set of vectors. There is the possibility that for some $l>0$, the vector $A^{k+l}b$ may be completely independent of $b, \cdots, A^{k-1}b$. Therefore, to get a set of vectors which spans the practically reachable subspace, it is necessary to check at least up to the $A^n b$ term in the sequence $b, Ab, \cdots$, (By thorem 3). Even if it is known that the eigen-values of A lie inside the unit circle centered at the origin, to check at least up to $A^{n-1}b$ is necessary, which is the result of theorem 5.

Here arises a problem. Is it necessary to check the practical dependencies of the vectors beyond the $A^n b$ term? What conditions are necessary for checking only up to the $A^n b$ term?

If some eigen-values λ_i of A are $|\lambda_i|>1$, any large value of $A^l b$ may appear for a large l. In a case where the eigen-vector of λ_i lies outside the subspace spanned by $b, Ab, \cdots, A^{k-1}b$, then even if

$$\frac{|G(b, Ab, \cdots, A^k b)|}{|G(b, Ab, \cdots, A^{k-1}b)}<\delta^2 \tag{24}$$

may hold for a very small positive δ,

$$\frac{|G(b, Ab, \cdots, A^{k-1}b, A^l b)|}{|G(b, Ab, \cdots, A^{k-1}b)|}>L^2 \tag{25}$$

holds for an arbitrarily large L, by taking l large enough if $|\lambda_i|>1$. Therefore, hereafter we will assume that all the eigen-values of A lie inside the unit disc centred at the origin.

Theorem 6.

Suppose every eigen-value λ_i of A satisfies Eq. (26).

$$|\lambda_i|<1 \tag{26}$$

For a set K_m of integers

$$K_m=\{k_i\},\ 1<k_1<k_2\cdots<k_m\leq n \tag{27}$$

and for some $\varepsilon>0$, if

$$\delta_2(k)\triangleq\frac{|G(\Xi_{N-K}, A^{k-1}b)|}{|G(\Xi_{N-K})|}<\frac{\varepsilon}{mM_{n-k}} \qquad k\in K_m \tag{28}$$

then

$$\frac{|G(\Xi_{N-K}, A^{n+l}b)|}{|G(\Xi_{N-K})|}<\varepsilon \tag{29}$$

for all $l\geq 0$.

Some notations used above are as follows:

Ξ_{N-K} is the set of vectors $\{A^{k-1}b, k\notin K_m,\ k=1, 2, \cdots, n\}$.

$|G(\Xi_{N-K})|$ is the Grammian of Ξ_{N-K}.

$M_j=\max\limits_l \binom{j+l-1}{l}\binom{n+l}{l+j}\rho^{j+l}$

$\rho=\max\limits_i |\lambda_i|$

To prove theorem 6, the following lemmas are necessary.

Lemma 1.

Let

$$\xi_i=A^{i-1}b \qquad i=1, 2, \cdots \tag{30}$$

and $\xi_1, \cdots, \xi_n$ be independent.

Take the arbitrary vector ζ which is expanded as

$$S=p_1\xi_1+\cdots+p_n\xi_n \tag{31}$$

Then

$$\frac{|G(\Xi_{N-K}, \zeta)|}{|G(\Xi_{N-K})|} \leq [\sum |p_k|^2 \delta_2(k)]^2 \tag{32}$$

where

$$\delta_2{}^2(k) = \frac{|G(\Xi_{N-K}, \xi_k)|}{|G(\Xi_{N-K})|} \tag{33}$$

Proof

Denote the subspace spanned by Ξ_{N-K} as $S(\Xi_{N-K})$. ξ_k and ζ are expressed by the sum of two vectors.

$$\begin{aligned} &\xi_k = \bar{\xi}_k + \tilde{\xi}_k \\ &\zeta = \bar{\zeta} + \tilde{\zeta} \\ &\bar{\xi}_k,\ \bar{\zeta} \in S(\Xi_{N-K}), \qquad \tilde{\xi}_k,\ \tilde{\zeta} \in S(\Xi_{N-K})^{\perp} \\ &\bar{\xi}_k = 0 \qquad \text{for } k \in K_m \end{aligned} \tag{34}$$

where $S(\cdot)^{\perp}$ is the orthogonal complement of $S(\cdot)$

By Eqs. (31) and (34),

$$|\zeta|^2 = \frac{|G(\Xi_{N-K}, \zeta)|}{|G(\Xi_{N-K})|} = \left|[\sum_{k\in K_m} p_k \tilde{\xi}_k]\right|^2 \leq [\sum_{k\in K_m} |p_k|\, |\tilde{\xi}_k|]^2 \tag{35}$$

Q.E.D.

By the Kayley-Hamilton theorem, it is known that $A^{n+l}b$ can be expanded as

$$A^{n+l}b = \kappa_{l,1}A^{n-1}b + \kappa_{l,2}A^{n-2}b + \cdots + \kappa_{l,n}b \tag{36}$$

When the eigen-values of A are known, the coefficients of the expansion $\kappa_{l,j}$ are given by the following lemma.

Lemma 2.[3]

The coefficients of expansion in Eq. (36) are given as

$$\kappa_{l,j} = (-1)^{j-1} \sum_{\substack{a_1+\cdots+a_n=j+l \\ 0\leq a_i \leq j+l}} \binom{\beta(a_1, \cdots, a_n)-1}{\beta(a_1, \cdots, a_n)-j} \lambda^{a_1} \cdots \lambda^{a_n} \tag{37}$$

where λ_i are eigen-values of A, and $\beta(a_1, \cdots, a_n)$ is the function which gives the number of non-zero $a_1, \cdots, a_n$.

Lemma 3.

The upper bounds of the absolute values of the coefficients of expansion in Eq. (37) are given by

$$|\kappa_{l,j}| \leq \binom{j+l-1}{l}\binom{n+j}{l+j}\rho^{j+l} \tag{38}$$

Proof of theorem 6

Let $\zeta = A^{n+l}b$ by lemma 1 and Eq. (36) be

$$h^2 \leq [\sum_{k\in K_m} |\kappa_{l,n-k}|\, \delta_2(k)]^2 \tag{39}$$

From Eq. (24) $\rho<1$,

$$\lim_{j\to\infty}\binom{j+l-1}{l}\binom{n+l}{l+j}\rho^{j+l}=0 \tag{40}$$

Therefore, M_j of theorem 6 exists for every j
From Eqs. (38), (36) and (39),

$$h^2\leq[\sum_{k\in K_m} M_k\delta_2(k)]^2 \tag{41}$$

Eq. (41) shows that if

$$\delta_2(k)<\varepsilon/mM_k \qquad k\in K_m \tag{42}$$

then $h^2<\varepsilon^2$.

Q.E.D.

Theorem 6 shows that if the set of vectors K_m is selected from the set of vectors b, Ab, $\cdots$, $A^{n-1}b$ so as to satisfy Eq. (38), then the subspace spanned by Ξ_{N-K} contains the practically reachable subspace, and to check the practical dependencies beyond the A^nb term is not necessary.

4. Numerical Example

Let

$$b=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad Ab=\begin{bmatrix}1\\\varepsilon\\0\end{bmatrix},\quad A^2b=\begin{bmatrix}2.5\\2\\2\end{bmatrix},\quad A^3b=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

then

$$A=\begin{bmatrix}1 & 1.5/\varepsilon & -1.5/\varepsilon-0.75\\ \varepsilon & -1+2/\varepsilon & -2/\varepsilon+1.5-1.25\varepsilon\\ 0 & 2/\varepsilon & -2/\varepsilon+0.5\end{bmatrix}$$

and the eigen-values of A are -0.5, $0.5\pm j0.5$. As M_2 defined in the theorem 6 is not very large, to check the practical independencies up to A^2b is sufficient. In this case, although b and Ab are practically dependent for small ε, the practically reachable subspace is a 2-dimentional subspace spanned by the vectors $(1, 0, 0)'$ and $(0, 1, 1)'$.

5. Conclusions

The following results are obtained.

i) If $k\leq n-1$, where n is the dimension of the state, then even if A^kb lies on S_k, the subspace spanned by the vectors b, Ab, $\cdots$, $A^{k-1}b$, practically, the distance of $A^{k+l}b$ from S_k may become arbitrarily large for such k that $k+l\leq n$. Therefore, to check the practical dependencies up to the $A^n b$ term is necessary at least.

ii) If $k < n-1$, and all the eigen-values of A are known to lie inside the unit disk centered at the origin, then, to check the practical dependencies up to the $A^{n-1}b$ term is nescesary.

iii) If Eq. (28) holds, then, to check the practical dependencies up to $A^{n-1}b$ is sufficient, and checking beyond this term is not necessary.

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