Distortion of Wave-Form of Weak and Short Wave in Eulerian Fluid

By

Tatsuo TOKUOKA*

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Abstract

The distortion of wave-form of a wave propagating in Eulerian fluid is studied theoretically. The geometrical acoustics is applied for the weak and short wave. The displacement vector of the wave is assumed to be a product of the amplitude vector and the wave-form which is a function of eikonal. The amplitude, the wave number and the frequency are assumed to vary gradually. From the first approximation of the equation of motion, the existence of a longitudinal wave is proved and the propagation velocity is obtained. From its second approximation, the evolutional equation of amplitude is reduced. By that equation, the distortion of the wave-form is analyzed. It may be flatter or steeper according to the material function, the curvature of the wave-form and the direction of the displacement of the wave.

1. Introduction

Usually, the wave is represented by a *sinusoidal wave,* a *characteristic surface* and a *singular surface.* For the linear wave phenomena, the method of a sinusoidal wave can be applied, but the method can not be used for the non-linear one. The method of characteristic manifold is a convenient analysis to treat a wave, and it is applied to the field of variety.[1,2] The theory of a singular surface[3] might be the most powerful method, and by it we can obtain not only the propagation velocity and the direction of the amplitude, but also the growth and decay of the amplitude. (E.g., refer to Chen[4].) However, in this theory, the amplitude is defined by the *jump* of the acceleration of material particle, and we can not discuss the *wave-form* which is the distribution of the amplitude with respect to the phase.

There is a method of *geometrical optics*,[5] where the electromagnetic wave is regarded as a plane sinusoidal wave in a small region, but its wave-length and propagation direction may vary gradually.

In this paper, the *geometrical acoustics* will be applied to the waves propagating

* Department of Aeronautical Engineering

in *Eulerian fluid*. The wave is assumed to be weak and have a short wave-length and an arbitrary wave-form. The distortion of the wave-form will be analyzed.

2. Equation of Motion

The Eulerian fluid, *i.e.*, the barotropic fluid, has the constitutive equation

$$\boldsymbol{T} = -p(\rho)\mathbf{1}, \tag{2.1}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, p is the pressure and a function of the mass density ρ and $\mathbf{1}$ is the unit tensor. The material function $p(\rho)$ is assumed to be continuous and continuously differentiable to any order if desired.

The Cauchy equation of motion

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) \tag{2.2}$$

must be satisfied by any continuum, where $\boldsymbol{b}$ is the body force density, $\boldsymbol{v}$ is the velocity vector of material particle, and the summation convention and Carterian coordinates are assumed.

A material particle has a point $\boldsymbol{X}$ in a reference configuration $\boldsymbol{\kappa}$ and a point $\boldsymbol{x}$ in the current configuration $\boldsymbol{\chi}$. Its motion is expressed by

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)\,. \tag{2.3}$$

The independent variables in (2.2) are $\boldsymbol{x}$ and t. Let us change them into $\boldsymbol{X}$ and t. Multiplying the Jacobian

$$J = \det\left(\frac{\partial x_i}{\partial X_j}\right) \tag{2.4}$$

by both sides of (2.2), we have

$$\frac{\partial T_{\kappa ik}}{\partial X_k} + \rho_\kappa b_i = \rho_\kappa \ddot{x}_i\,, \tag{2.5}$$

where

$$T_{\kappa ik} = J\frac{\partial X_k}{\partial x_j} T_{ij} \tag{2.6}$$

is the Piola-Kirchhoff stress tensor,

$$\rho_\kappa = \rho J \tag{2.7}$$

is the mass density in $\boldsymbol{\kappa}$, $\ddot{x}_i$ is the acceleration, a dot denotes the material time derivative which holds $\boldsymbol{X}$ constant, and the identity

$$\frac{\partial}{\partial X_k}\left(J\frac{\partial X_k}{\partial x_j}\right)\equiv 0 \tag{2.8}$$

and the inverse relation of (2.3)

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t) \tag{2.9}$$

were used. Equation (2.7) denotes the conservation of mass.

Substituting (2.1), (2.4), (2.6) and (2.7) into (2.5), and neglecting the body force, we have

$$p'(\rho)\frac{\partial^2 x_p}{\partial X_q X_r}\frac{\partial X_q}{\partial x_i}\frac{\partial X_r}{\partial x_p} = \ddot{x}_i\,, \tag{2.10}$$

where

$$p'(\rho) = \frac{\mathrm{d}p(\rho)}{\mathrm{d}\rho}\,. \tag{2.11}$$

3. Geometrical Acoustics

Here, we assume that the wave is weak and has a short wavelength. Then, the concept of geometrical acoustics can be applied.

The Eulerian fluid is assumed to move steadily in the middle configuration χ_0 and

$$\boldsymbol{x}_0 = \boldsymbol{X}+\boldsymbol{U}t\,, \tag{3.1}$$

where $\boldsymbol{U}$ is a constant velocity. A wave is superposed on χ_0, that is,

$$\boldsymbol{x} = \boldsymbol{x}_0+\boldsymbol{u}(\boldsymbol{X}, t)\,, \tag{3.2}$$

where $\boldsymbol{u}$ is the displacement vector of the wave. By the assumption of geometrical acoustics, we may suppose that

$$\boldsymbol{u} = \boldsymbol{a}(\boldsymbol{X}, t)f(\psi(\boldsymbol{X}, t))\,, \tag{3.3}$$

where $\boldsymbol{a}$ is the *amplitude vector*, ψ is the *phase* and called the *eikonal*[5], and $f(\psi)$ indicates the *wave-form*. The eikonal is a rapidly varying function and

$$k_i = \frac{\partial\psi}{\partial X_i}\,, \quad \omega = -\dot{\psi} \tag{3.4}$$

are, respectively, the *wave-number vector* and the *circular frequency*. We have

$$\boldsymbol{k} = k\boldsymbol{n}\,, \quad \lambda = \frac{2\pi}{k}\,, \tag{3.5}$$

where k is the *wave-number*, $\boldsymbol{n}$ is the unit vector along the propagation direction and λ is the *wave-length*. The propagation velocity is given by

$$V=\frac{\omega}{k}, \tag{3.6}$$

which is assumed to be positive.

By the assumption of a weak wave we can put

$$|\boldsymbol{a}|\ll\lambda, \quad O\left(\frac{\boldsymbol{a}}{\lambda}\right)=\varepsilon\ll 1. \tag{3.7}$$

By the geometrical acoustics, quantities $\boldsymbol{a}$, k and ω are assumed to be constant in the region of magnitude λ. However, they may change, respectively, in the distances, l_a, l_k and l_ω, which are presumably in same order, and large with respect to λ, that is,

$$O\left(\frac{\lambda}{l_a}\right)\sim O\left(\frac{\lambda}{l_k}\right)\sim O\left(\frac{\lambda}{l_\omega}\right)=\varepsilon'\ll 1. \tag{3.8}$$

Then, we can estimate that

$$\left.\begin{aligned} &O\left(\frac{\partial\boldsymbol{a}}{\partial\boldsymbol{X}}\right)\sim O(\dot{\boldsymbol{a}})=\varepsilon\varepsilon', \\ &O\left(\frac{\partial^2\boldsymbol{a}}{\partial\boldsymbol{X}^2}\right)\sim O\left(\frac{\partial\dot{\boldsymbol{a}}}{\partial\boldsymbol{X}}\right)\sim O(\ddot{\boldsymbol{a}})\sim\frac{1}{\lambda}\varepsilon\varepsilon'^2, \\ &O\left(\frac{\partial k}{\partial\boldsymbol{X}}\right)\sim O(\dot{k})\sim O\left(\frac{\partial\omega}{\partial\boldsymbol{X}}\right)\sim O(\dot{\omega})=\frac{1}{\lambda^2}\varepsilon'. \end{aligned}\right\} \tag{3.9}$$

From (3.1)-(3.3) we can obtain

$$\left.\begin{aligned} &\frac{\partial x_p}{\partial X_q}=\delta_{pq}+a_pk_qf'+\frac{\partial a_p}{\partial X_q}f, \\ &\frac{\partial^2 x_p}{\partial X_q\partial X_r}=a_pk_qk_rf''+\left(\frac{\partial a_p}{\partial X_q}k_r+\frac{\partial a_p}{\partial X_r}k_q+a_p\frac{\partial k_q}{\partial X_r}\right)f'+\cdots, \\ &\frac{\partial X_q}{\partial x_i}=\delta_{qi}-a_qk_if'-\frac{\partial a_q}{\partial X_i}f+(\boldsymbol{a}\cdot\boldsymbol{k})a_qk_if'^2+\cdots, \end{aligned}\right\} \tag{3.10}$$

and

$$\ddot{x}_i=\omega^2a_if''-(2\omega\dot{a}_i+\dot{\omega}a_i)f'+\cdots, \tag{3.11}$$

where

$$f'=\frac{\mathrm{d}f(\psi)}{\mathrm{d}\psi}, \quad f''=\frac{\mathrm{d}^2f(\psi)}{\mathrm{d}\psi^2}. \tag{3.12}$$

and here and henceforth, the third and higher orders with respect to ε and ε'

are neglected.

From (2.4), (2.7) and (3.10) we have

$$\begin{aligned} J &= 1+(\boldsymbol{a}\cdot\boldsymbol{k})f'+(\mathrm{div}\,\boldsymbol{a})f+\cdots\cdots, \\ \rho &= \rho_\kappa[1-(\boldsymbol{a}\cdot\boldsymbol{k})f'-(\mathrm{div}\,\boldsymbol{a})f+(\boldsymbol{a}\cdot\boldsymbol{k})^2f'^2+\cdots\cdots] \\ &= \rho_\kappa+\Delta\rho, \end{aligned} \tag{3.13}$$

and we can obtain

$$\begin{aligned} p'(\rho) &= p'(\rho_\kappa+\Delta\rho) = p'_\kappa+p''_\kappa\Delta\rho+\cdots\cdots \\ &= p'_\kappa-p''_\kappa\rho_\kappa(\boldsymbol{a}\cdot\boldsymbol{k})f'+\cdots\cdots, \end{aligned} \tag{3.14}$$

where

$$p'_\kappa = \left.\frac{\mathrm{d}p(\rho)}{\mathrm{d}\rho}\right|_{\rho=\rho_\kappa}, \quad p''_\kappa = \left.\frac{\mathrm{d}^2p(\rho)}{\mathrm{d}\rho^2}\right|_{\rho=\rho_\kappa}, \tag{3.15}$$

and second and higher order terms with respect to ε and ε' are neglected.

Substituting (3.10) and (3.14) into the left hand side of (2.10) we have

$$\begin{aligned} p'_\kappa\Big\{(\boldsymbol{a}\cdot\boldsymbol{k})k_if''+\Big[k_p\frac{\partial a_p}{\partial X_i}+k_i(\mathrm{div}\,\boldsymbol{a})+\frac{\partial k_i}{\partial X_p}a_p \\ -2(\boldsymbol{a}\cdot\boldsymbol{k})^2k_if''\Big]\Big\}-p''_\kappa\rho_\kappa(\boldsymbol{a}\cdot\boldsymbol{k})^2k_if'f''. \end{aligned} \tag{3.16}$$

4. Propagation Velocity

Taking up the order ε/λ in (2.10), (3.11) and (3.16), we have

$$p'_\kappa(\boldsymbol{a}\cdot\boldsymbol{n})k^2\boldsymbol{n} = \omega^2\boldsymbol{a}, \tag{4.1}$$

where $f''\neq0$ and (3.5) were assumed. If $\boldsymbol{a}\cdot\boldsymbol{n}=0$, we have $\boldsymbol{a}=\mathbf{0}$ which means no existence of any wave. Then, assuming $\boldsymbol{a}\cdot\boldsymbol{n}\neq0$, we have

$$\boldsymbol{a}=a\boldsymbol{n}, \tag{4.2}$$

which means a *longitudinal wave,* where a is the magnitude of the amplitude. The propagation velocity (3.6) is given by

$$V = (p'_\kappa)^{1/2}. \tag{4.3}$$

We can consider that V and $\boldsymbol{n}$ are constant because the state χ_0 is homogeneous. Equation (4.3) gives the propagation velocity with respect to the reference configuration $\boldsymbol{\kappa}$. That with respect to the current configuration χ is given by

$$V\boldsymbol{n}+\boldsymbol{U}. \tag{4.4}$$

5. Distortion of Wave-Form

Taking up the order ε^2/λ in (2.10), (3.11) and (3.16), we have

$$p'_\kappa\left[k\left(\frac{\partial a}{\partial X_i}+\frac{\partial a}{\partial X_j}n_j n_i\right)+\frac{\partial k}{\partial X_j}n_j a n_i-2k^3a^2n_i f''\right]$$
$$-p''_\kappa\rho_\kappa k^3a^2n_i f''=-2\omega\dot{a}n_i-\dot{\omega}an_i\,, \tag{5.1}$$

where $\varepsilon\sim\varepsilon'$ and $f'\neq 0$ were assumed. Multiplying (5.1) by n_i, and using (3.6) and (4.3), we have

$$2\omega\left(\dot{a}+V\frac{\partial a}{\partial X_j}n_j\right)+\left(\dot{\omega}+V^2\frac{\partial k}{\partial X_j}n_j\right)a=(2p'_\kappa+p''_\kappa\rho_\kappa)k^3a^2f''\,. \tag{5.2}$$

The *displacement derivative*

$$\frac{\delta a}{\delta t}=\dot{a}+V\frac{\partial a}{\partial X_j}n_j \tag{5.3}$$

denotes the time derivative of a measured by an observer riding on a wave propagating with velocity V and direction $\boldsymbol{n}$. By the assumption ω/k=constant we have

$$\dot{\omega}-\frac{\omega}{k}\dot{k}=0\,,\quad \frac{\partial\omega}{\partial X_j}-\frac{\omega}{k}\frac{\partial k}{\partial X_j}=0, \tag{5.4}$$

which gives

$$\dot{\omega}+V^2\frac{\partial k}{\partial X_j}n_j=0, \tag{5.5}$$

where (3.4) were used. Then, we have the *evolutional equation* of the magnitude of the amplitude

$$\frac{\delta a}{\delta t}=\alpha f''a^2\,, \tag{5.6}$$

where

$$\alpha=\frac{\left(p'_\kappa+\frac{1}{2}\rho_\kappa p''_\kappa\right)\omega^2}{V^3}\,. \tag{5.7}$$

By the forms (3.3) and (4.2) we can put

$$a>0 \tag{5.8}$$

without loss of generality. The reason is that, if we assume that $a=0$ at a time, then from (5.6) we have at that time

$$\frac{\delta a}{\delta t}=\frac{\delta^2 a}{\delta t^2}=\frac{\delta^3 a}{\delta t^3}=\cdots\cdots=0, \tag{5.9}$$

which indicate that $a=0$ at all time. Then we can say that, if $\alpha f''>0$, a is growing, and if $\alpha f''<0$, it is decaying.

From (3.4), (3.6) and (5.3) we have

$$\frac{\delta\psi}{\delta t}=-\omega+Vk=0\,. \tag{5.10}$$

Thus, if we regard (5.6) as a differential equation, $\alpha f''$ can be regarded as a constant. We have

$$a(t)=\frac{a(0)}{1-a(0)\alpha f''t}, \tag{5.11}$$

where $a(0)$ is the initial amplitude at $t=0$. If $\alpha f''>0$, $a(t)$ increases to infinity at time

$$t_\infty=\frac{1}{a(0)\alpha f''}, \tag{5.12}$$

while, if $\alpha f''<0$, $a(t)$ decreases monotonically to zero as $t\to\infty$. However, the blowout at t_∞ can not be considered, because the assumption of weakness of the wave will not hold in that case.

Let us consider the case $\alpha>0$. The amplitude with $f''>0$, *i.e.*, the convex to below increases, while that with $f''<0$, *i.e.*, the convex to up decreases. Therefore, we can say that the uneven wave-form for $f>0$ is flatter and that for $f<0$ is steeper. In the case of $\alpha<0$, we have inverse relations.

6. Discussions

(1) Let us consider a material which has the material function

$$p(\rho)=K\rho^\gamma, \tag{6.1}$$

where K and γ are positive material constants. We have

$$\alpha=\frac{\gamma(\gamma+1)\omega^2 p_\kappa}{2\rho_\kappa V^3}, \tag{6.2}$$

which is positive. We then have the former case depicted in the last part of Section 5.

(2) The acceleration of a material particle (3.11) is equal to the acceleration of the displacement of the wave $\ddot{u}$. We obtain

$$\frac{\delta \bar{u}}{\delta t} = \frac{\delta a}{\delta t}\omega^2 f''. \tag{6.3}$$

where $\bar{u}=\ddot{\boldsymbol{u}}\cdot\boldsymbol{n}$, and the higher order terms were neglected. Therefore, we have

$$\frac{\delta \bar{u}}{\delta t} = \beta \bar{u}^2, \tag{6.4}$$

where

$$\beta = \frac{p'_\kappa + \frac{1}{2}\rho_\kappa p''_\kappa}{V^3} \tag{6.5}$$

Then we can say that, if $\beta>0$, the acceleration increases, while if $\beta<0$, it decreases.

(3) When the pressure is considered as a function of the specific volume

$$\nu = \frac{1}{\rho} \tag{6.6}$$

that is,

$$p(\rho) = \hat{p}(\nu), \tag{6.7}$$

we can easily obtain

$$\beta = \frac{\hat{p}''_\kappa}{2\rho_\kappa^3 V^3} \tag{6.8}$$

where

$$\hat{p}'_\kappa = \left.\frac{d\hat{p}(\nu)}{d\nu}\right|_{\nu=\nu_\kappa}, \quad \hat{p}''_\kappa = \left.\frac{d^2\hat{p}(\nu)}{d\nu^2}\right|_{\nu=\nu_\kappa}, \tag{6.9}$$

and $\nu_\kappa = 1/\rho_\kappa$.

The theory of a singular surface can be applied to the growth and decay of the acceleration wave in Eulerian fluid. Chen[6] reported an evolutional equation which has the same form as (6.4) and (6.8) for a plane wave. However, the amplitude defined there is not the acceleration of a material particle, but the jump of its acceleration on the singular surface.

References

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