# Wave Propagation in a Linear Thermo-Magneto-Elastic Material

By

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**Abstracts**

Acceleration wave propagation in an isotropic homogeneous linear thermo-magneto-elastic material is investigated. A thermo-electrical interaction is taken into account in the heat equation to restrain the infinite velocity of thermal disturbances. There exist four types of coupled waves: an electro-acoustical wave, two thermo-acoustical waves and an electro-magnetic wave. The amplitudes of all the waves, in general, decay exponentially in time, but the damping constants depend, in different ways, on the propagation directions and the thermo-magneto-elastic properties of the material. Especially, according as the angles between the initial magnetic field and propagation directions vary from 0 to $\pi/2$, the damping constant of the electro-acoustical wave decreases to 0, and those of the thermo-acoustical waves increase. On the other hand, the damping constant of the electro-magnetic wave depends also on the angle between the initial magnetic field and the direction of its amplitude, and it gives rise to the elliptical polarization.

## 1. Introduction

The object of this paper is to investigate the acceleration waves in a linear thermo-magneto-elastic material. The basic field equations proposed by Kaliski[1)] are considered, where the heat equation brings on finite propagation velocities of thermal disturbances.

Other sets of basic field equations of thermo-magneto-(electro-)elastic materials have been considered by Kaliski and Petykiewicz[2)], Jordan and Eringen[3)], Tiersten[4)] and McCarthy[5)]. Wave propagations in these materials have been analyzed by Paria[6),7)], Willson[8)] Kaliski and Nowacki[9)], Purushothama[10)], Cander[11)], Moon and Chattopadhyay[12)] and McCarthy[13),14)]. According to the dependences of the heat flux upon the temperature gradient, the deformation gradient, the electro-magnetic fields etc., the heat equations employed in the above articles can be classified into the following three types: linear type[2),6)-12)], non-linear type[3),4),13),14)], and memory type[5)]. These three types of heat equations are straight-forward

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generalizations of Fourier's law such that they are reduced to it as special cases. Hence, the sets of the field equations, including the above, as well as the case of Fourier's law, have parabolic-hyperbolic characters. That is to say, they bring on infinite propagation velocities of thermal disturbances, and also mechanical and electro-magnetic disturbances induced by the thermal ones. On the other hand, the set of field equations considered by Kaliski[1] is a hyperbolic type, so that the behaviors of the waves will be essentially different from the above cases.

In section 2, the field equations are reviewed and the definition of acceleration waves is given. Section 3 shows the existence of four types of coupled waves and their velocities. In section 4, the differential equations for the amplitudes of the waves are obtained. Here, it is shown that all waves, in general, decay exponentially in time, with different damping constants. The influence of the material properties and the propagation directions of the waves on these damping constants is also discussed here.

## 2. Basic Equations and Definition of Acceleration Waves

An isotropic homogeneous linear thermo-magneto-elastic material is, according to Kaliski[1], defined by the field equations

$$\operatorname{rot} \boldsymbol{h} = \boldsymbol{j} + \dot{\boldsymbol{D}}, \quad \operatorname{div} \boldsymbol{D} = 0, \tag{2.1}$$

$$\operatorname{rot} \boldsymbol{E} = -\dot{\boldsymbol{b}}, \quad \operatorname{div} \boldsymbol{b} = 0, \tag{2.2}$$

$$\rho \ddot{\boldsymbol{u}} = \operatorname{div} \boldsymbol{s} + (\boldsymbol{j} \times \boldsymbol{B}_0), \tag{2.3}$$

$$\tau \dot{\boldsymbol{q}} + \boldsymbol{q} = -K \operatorname{grad} T + \pi \boldsymbol{j}, \tag{2.4}$$

$$\rho T_0 \dot{s} = -\operatorname{div} \boldsymbol{q}, \tag{2.5}$$

$$\boldsymbol{s} = 2G\boldsymbol{e} + \lambda(\operatorname{tr} \boldsymbol{e})\mathbf{1} - \alpha_0 T \mathbf{1}, \tag{2.6}$$

$$\boldsymbol{b} = \mu \boldsymbol{h}, \tag{2.7}$$

$$\boldsymbol{D} = \varepsilon[\boldsymbol{E} + (\dot{\boldsymbol{u}} \times \boldsymbol{B}_0)], \tag{2.8}$$

$$\phi \boldsymbol{j} = \eta[\boldsymbol{E} + (\dot{\boldsymbol{u}} \times \boldsymbol{B}_0)] + \frac{\kappa}{K}\boldsymbol{q}, \tag{2.9}$$

$$\rho s = \alpha_0(\operatorname{tr} \boldsymbol{e}) + \beta_0 T, \tag{2.10}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.11}$$

$$\phi = 1 + \frac{\kappa \pi}{K}. \tag{2.12}$$

The external forces and the heat supplies are assumed to be absent and a superposed dot denotes the differentiation with respect to time. In the above equations, the unknown quantities are the magnetic field $\boldsymbol{h}$, the electrical current density vector $\boldsymbol{j}$, the magnetic induction field $\boldsymbol{b}$, the electric field $\boldsymbol{E}$, the electric induction field $\boldsymbol{D}$, the displacement $\boldsymbol{u}$, the stress $\boldsymbol{s}$, the heat flux $\boldsymbol{q}$, the temperature $T$ and the entropy $s$. The other quantities may be assumed to be constant from the homogeneity and the linearity of the material. Eliminating $\boldsymbol{b}$, $\boldsymbol{D}$, $\boldsymbol{s}$ and $s$ from (2.1)–(2.10) gives another set of equations:

$$\operatorname{rot} \boldsymbol{h} = \boldsymbol{j}+\varepsilon[\dot{\boldsymbol{E}}+(\ddot{\boldsymbol{u}}\times \boldsymbol{B}_0)]\,, \quad \operatorname{div}[\boldsymbol{E}+(\dot{\boldsymbol{u}}\times\boldsymbol{B}_0)] = 0\,, \tag{2.13}$$

$$\operatorname{rot} \boldsymbol{E} = -\mu\dot{\boldsymbol{h}}\,, \quad \operatorname{div} \boldsymbol{h} = 0\,, \tag{2.14}$$

$$\rho\ddot{\boldsymbol{u}} = G\nabla^2\boldsymbol{u}+(\lambda+G)\operatorname{grad}\cdot\operatorname{div}\boldsymbol{u}-\alpha_0 \operatorname{grad} T+(\boldsymbol{j}\times\boldsymbol{B}_0)\,, \tag{2.15}$$

$$\tau\dot{\boldsymbol{q}}+\boldsymbol{q} = -K\operatorname{grad} T+\pi\boldsymbol{j}\,, \tag{2.16}$$

$$\alpha_0 T_0 \operatorname{tr}\dot{\boldsymbol{e}}+\beta_0 T_0 \dot{T} = -\operatorname{div}\boldsymbol{q}\,, \tag{2.17}$$

$$\phi\boldsymbol{j} = \eta[\boldsymbol{E}+(\dot{\boldsymbol{u}}\times\boldsymbol{B}_0)]+\frac{\kappa}{K}\boldsymbol{q}\,. \tag{2.18}$$

Here, we define the acceleration wave in this material as a propagating surface which has the following properties:

(i) *$\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, $\boldsymbol{u}_{,i}$, $\boldsymbol{h}$, $\boldsymbol{E}$, $T$, $\boldsymbol{q}$ and $\boldsymbol{j}$ are continuous everywhere.*

(ii) *the first derivatives of them, except $\boldsymbol{u}$, suffer jump discontinuities acrcss the surface, but are continuous everywhere else.*

Henceforth, we are concerned only with plane waves. The compatibility conditions of the first and second order for a continuous function $f(\boldsymbol{x}, t)$ across the surface are then given by

$$[f_{,i}] = \bar{f}n_i\,, \quad [\dot{f}] = -U\bar{f}, \tag{2.19}$$

$$[f_{,ij}] = \bar{\bar{f}}n_in_j\,, \quad [\dot{f}_{,i}] = (-U\bar{\bar{f}}+\dot{\bar{f}})n_i\,, \quad [\ddot{f}] = U^2\bar{\bar{f}}-2U\dot{\bar{f}}, \tag{2.20}$$

$$\bar{f}\equiv [f_{,i}]n_i\,, \quad \bar{\bar{f}}\equiv [f_{,ij}]n_in_j\,, \tag{2.21}$$

where a bracket denotes the jump of the quantity within it, and $\boldsymbol{n}$ and $U$ are, respectively, the unit normal vector and the normal velocity of the surface.

## 3. Velocities of Acceleration Waves

Taking the jumps of (2.13)–(2.18) across an acceleration wave by the compatibility conditions (2.19)–(2.21), we have

$$\bar{\boldsymbol{h}}\times\boldsymbol{n}-\varepsilon U\bar{\boldsymbol{E}}-\varepsilon U(\bar{\boldsymbol{v}}\times\boldsymbol{B}_0) = \boldsymbol{0}\,, \tag{3.1}$$

$$\bar{\boldsymbol{E}}\times\boldsymbol{n}+\mu U\bar{\boldsymbol{h}} = \boldsymbol{0}\,, \tag{3.2}$$

$$(\rho U^2-G)\bar{\boldsymbol{v}}-(\lambda+G)(\bar{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n}-\alpha_0 U\bar{T}\boldsymbol{n} = \boldsymbol{0}\,, \tag{3.3}$$

$$\tau_0 U\bar{\boldsymbol{q}}-K\bar{T}\boldsymbol{n} = \boldsymbol{0}\,, \tag{3.4}$$

$$\alpha_0 T_0\bar{\boldsymbol{v}}\cdot\boldsymbol{n}-\beta_0 T_0 U\bar{T}+\bar{\boldsymbol{q}}\cdot\boldsymbol{n} = 0\,, \tag{3.5}$$

or

$$R_{\alpha\beta}a_\beta = 0\,, \qquad (\alpha,\ \beta = 1,\ 2,\ \cdots,\ 13) \tag{3.6}$$

where $$a_\beta \equiv (\boldsymbol{h}, \boldsymbol{E}, \boldsymbol{v}, \boldsymbol{q}, T)\,, \tag{3.7}$$

$$\|R_{\alpha\beta}\| \equiv \begin{Vmatrix} -\boldsymbol{n}\times & -\varepsilon U\boldsymbol{1} & \varepsilon U\boldsymbol{B}_0\times & \boldsymbol{0} & \boldsymbol{0} \\ \mu U\boldsymbol{1} & -\boldsymbol{n}\times & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & (\rho U^2-G)\boldsymbol{1}-(\lambda+G)\boldsymbol{n}\otimes\boldsymbol{n} & \boldsymbol{0} & -\alpha_0 U\boldsymbol{n} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \tau U\boldsymbol{1} & -K\boldsymbol{n} \\ \boldsymbol{0} & \boldsymbol{0} & \alpha_0 T_0\boldsymbol{n} & \boldsymbol{n} & -\beta_0 T_0 U \end{Vmatrix}, \tag{3.8}$$

$$\boldsymbol{n}\times \equiv \begin{Vmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{Vmatrix}, \tag{3.9}$$

and $\boldsymbol{v}$ means $\dot{\boldsymbol{u}}$.

The velocity of the acceleration wave is given by $U$, satisfying

$$\det \|R_{\alpha\beta}\| = 0 \tag{3.10}$$

and then the right nullvector of $\|R_{\alpha\beta}\|$ indicates the coupled fields of the wave.

*Electro-acoustical wave*

Taking the scalar and vector products of (3.3) and (3.4) by $\boldsymbol{n}$ and combining (3.5), we get

$$(\rho U^2-G)\bar{\boldsymbol{v}}\times\boldsymbol{n} = \boldsymbol{0}\,, \tag{3.11}$$

$$\bar{\boldsymbol{q}}\times\boldsymbol{n} = \boldsymbol{0}\,, \tag{3.12}$$

$$\|R'\|\begin{pmatrix} \bar{\boldsymbol{v}}\cdot\boldsymbol{n} \\ \bar{\boldsymbol{q}}\cdot\boldsymbol{n} \\ T \end{pmatrix} = \boldsymbol{0}\,, \tag{3.13}$$

where

$$\|R'\| = \begin{Vmatrix} \rho U^2-2G-\lambda & 0 & -\alpha_0 U \\ 0 & \tau U & -K \\ \alpha_0 T_0 & 1 & -\beta_0 T_0 U \end{Vmatrix}. \tag{3.14}$$

Now, if we assume that

$$\bar{\boldsymbol{v}}\times\boldsymbol{n} \neq 0 \tag{3.15}$$

at a wave, then we must have

$$U^2 = U_{EA}^2 \equiv \frac{G}{\rho}\,(=c_T^2)\,. \tag{3.16}$$

The velocity $U_{EA}$ is equal to that of a usual transverse wave. In general, det $\|R'\|$ does not vanish for $U=U_{EA}$, so that (3.12) and (3.13) then imply

$$\bar{\boldsymbol{v}}\cdot\boldsymbol{n} = T = 0\,, \qquad \bar{\boldsymbol{q}} = \boldsymbol{0} \tag{3.17}$$

for this wave. To investigate the other coupled fields, we shall first obtain the general relations between $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{v}}$ which are valid for any wave. That is, eliminating $\bar{\boldsymbol{h}}$ from (3.1) and (3.2), and taking the scalar and vector products of the result by $\boldsymbol{n}$, by use of the notation $c^2=(\varepsilon\mu)^{-1}$, we have

$$\bar{\boldsymbol{E}}\cdot\boldsymbol{n} = (\bar{\boldsymbol{v}}\times\boldsymbol{n})\cdot\boldsymbol{B}_0\,, \tag{3.18}$$

$$\left(1-\frac{U^2}{c^2}\right)\bar{\boldsymbol{E}}\times\boldsymbol{n} = \frac{U^2}{c^2}[(\bar{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{B}_0-(\boldsymbol{B}_0\cdot\boldsymbol{n})\bar{\boldsymbol{v}}]\,. \tag{3.19}$$

For this wave, (3.19) reduces to (3.16) and (3.17) to

$$\left(1-\frac{U_{EA}^2}{c^2}\right)\bar{\boldsymbol{E}}\times\boldsymbol{n} = -\frac{U_{EA}^2}{c^2}(\boldsymbol{B}_0\cdot\boldsymbol{n})\bar{\boldsymbol{v}}\,. \tag{3.19'}$$

Since we may assume $U_{EA}^2/c^2\ll 1$, it is found from (3.2) and (3.19′) that $\bar{\boldsymbol{h}}$ and the transverse component of $\bar{\boldsymbol{E}}$ are negligibly small. These quantities completely vanish when $\boldsymbol{B}_0$ is perpendicular to $\boldsymbol{n}$. However, (3.17) and (3.18) imply that the longitudinal component of $\bar{\boldsymbol{E}}$ vanishes when $\boldsymbol{B}_0$ is located on the plane spanned by $\bar{\boldsymbol{v}}$ and $\boldsymbol{n}$. Finally, we can state that this wave is, in general, the coupled wave of an electrical longitudinal wave and a mechanical transverse wave.

*Thermo-acoustical waves*

If we assume that

$$\bar{\boldsymbol{v}}\cdot\boldsymbol{n}\,,\quad \bar{\boldsymbol{q}}\cdot\boldsymbol{n}\,,\quad T\neq 0\,, \tag{3.20}$$

then (3.13) imposes that

$$\det\|R'\| = 0 \tag{3.21}$$

or

$$(U^2-c_H^2)(U^2-c_L^2)-\gamma U^2 = 0\,, \tag{3.22}$$

where

$$c_H^2 \equiv \frac{K}{\tau\beta_0 T_0}\,,\quad c_L^2 \equiv \frac{\lambda+2G}{\rho}\,,\quad \gamma \equiv \frac{\alpha_0^2}{\rho\beta_0} \tag{3.23}$$

and where $c_H$ and $c_L$ are the velocities of a usual thermal wave and a longitudinal wave, respectively.

The solutions of (3.22) are given by

$$U^2 = U_{TA}^2 \equiv \frac{1}{2}[c_H^2+c_L^2+\gamma\pm\{(c_H^2+c_L^2+\gamma)^2-4c_H^2c_L^2\}^{1/2}]\,. \tag{3.24}$$

The velocities of the fast and slow waves coincide, respectively, with those of the two thermo-acoustical waves in the linear thermo-elastic material governed by the modified heat equation with finite wave velocities. (Cf. Tokuoka 15)). In general, these velocities are not equal to $U_{EA}$, and then from (3.11), we have

$$\bar{\boldsymbol{v}}\times\boldsymbol{n}=\mathbf{0} \tag{3.25}$$

and from (3.18) and (3.19), we have

$$\bar{\boldsymbol{E}}=-\frac{U_{TA}^2/c^2}{1-U_{TA}^2/c^2}(\bar{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{B}_0\times\boldsymbol{n}\,. \tag{3.26}$$

Since we may assume $U_{TA}^2/c\ll 1$, (3.26) implies that the electric field $\bar{\boldsymbol{E}}$ has only a negligibly small transverse component, which completely vanishes when $\boldsymbol{n}$ is parallel to $\boldsymbol{B}_0$. The magnetic field $\bar{\boldsymbol{h}}$ is also negligible from (3.2) and (3.26). Thus, we can state that these waves are, in general, the coupled waves of a thermal wave and a mechanical longitudinal wave.

Next, we shall investigate the difference between the fast and slow waves. From (3.22) we get the inequalities:

$$U_{TAf}^2>c_H^2,\ c_L^2\,,\quad U_{TAs}^2<c_H^2,\ c_L^2\,, \tag{3.27}$$

which indicate that the fast and slow waes are, respectively, faster and slower than both the usual longitudinal and thermal waves.

From (3.24), (3.4) and (3.5), we have the following relations amongst the coupled fields $\bar{\boldsymbol{v}}$, $\bar{\boldsymbol{q}}$ and $\boldsymbol{T}$,

$$(U_{TA}^2-c_H^2)\,\bar{\boldsymbol{q}}\cdot\boldsymbol{n}=\alpha_0T_0c_H^2\bar{\boldsymbol{v}}\cdot\boldsymbol{n}\,, \tag{3.28}$$

$$\boldsymbol{T}=\frac{\rho}{\alpha_0U_{TA}}(U_{TA}^2-c_L^2)\bar{\boldsymbol{v}}\cdot\boldsymbol{n}\left(=\frac{\tau U_{TA}}{K}\bar{\boldsymbol{q}}\cdot\boldsymbol{n}\right). \tag{3.29}$$

We have a small $\gamma$ for usual materials, and hence, we consider the limit $\gamma\rightarrow 0$. When $c_L^2>c_H^2$, it follows from (3.22) that

$$U_{TAf}^2\rightarrow c_L^2\,,\quad U_{TAs}^2\rightarrow c_H^2 \tag{3.30}$$

as $\gamma\rightarrow 0$. Then, from (3.12) and (3.29), for the fast wave,we have

$$\boldsymbol{T}\rightarrow 0\,,\quad \bar{\boldsymbol{q}}\rightarrow\mathbf{0} \tag{3.31}$$

and from (3.25) and (3.28), for the slow wave, we have

$$\bar{\boldsymbol{v}}\rightarrow\mathbf{0}\,. \tag{3.32}$$

Thus, we can conclude that when $c_L^2>c_H^2$, the fast wave is a predominantly mechanical wave and the slow wave is a predominantly thermal wave. It is easily verified that the situation is reversed when $c_H^2>c_L^2$.

*Electro-magnetic wave*

If we assume that a wave has a different velocity from those of the electro-acoustical and thermo-acoustical waves, the above discussion imposes that

$$\bar{\boldsymbol{v}} = \bar{\boldsymbol{q}} = \mathbf{0}\,, \quad \bar{T} = 0\,. \tag{3.33}$$

Then (3.18) and (3.19) reduce to, respectively,

$$\bar{\boldsymbol{E}}\cdot\boldsymbol{n} = 0\,, \tag{3.34}$$

$$\left(1-\frac{U^2}{c^2}\right)\bar{\boldsymbol{E}}\times\boldsymbol{n} = \mathbf{0}\,. \tag{3.35}$$

Assuming that

$$\bar{\boldsymbol{E}}\times\boldsymbol{n}\neq\mathbf{0} \tag{3.36}$$

yields from (3.35)

$$U^2 = U_{EM}^2 \equiv \frac{1}{\varepsilon\mu}\,(=c^2) \tag{3.37}$$

and from (3.2)

$$\bar{\boldsymbol{h}}\neq\mathbf{0} \tag{3.38}$$

for this wave. Thus, this wave is the electro-magnetic wave with the velocity of light. Eqs. (3.2) and (3.34) imply that the magnetic field $\bar{\boldsymbol{h}}$ is perpendicular to the electric field $\bar{\boldsymbol{E}}$ and that their longitudinal components vanish.

## 4. Variation of the Amplitudes of Acceleration Waves

In order to analyze the growth or decay of the acceleration waves, we take the jumps of the differentiations of (2.13)–(2.18) with respect to time, and eliminate $\bar{\boldsymbol{j}}$ from the result. That is, for any wave, we have

$$P_{\alpha\beta}\dot{\bar{a}}_{\beta} = Q_{\alpha\beta}\bar{a}_{\beta} + UR_{\alpha\beta}\bar{\bar{a}}_{\beta}\,, \qquad (\alpha,\ \beta = 1,\ 2,\ \cdots,\ 13) \tag{4.1}$$

where

$$||P_{\alpha\beta}|| \equiv \begin{Vmatrix} \mathbf{0} & -\varepsilon U\mathbf{1} & \varepsilon U\boldsymbol{B}_0\times & \mathbf{0} & \mathbf{0} \\ \mu U\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\rho U^2+G)\mathbf{1}+(\lambda+G)\boldsymbol{n}\otimes\boldsymbol{n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau U\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\beta_0 T_0 U \end{Vmatrix}, \tag{4.2}$$

$$||Q_{\alpha\beta}|| \equiv \frac{U}{\phi}\begin{Vmatrix} \mathbf{0} & \eta\mathbf{1} & -\eta\boldsymbol{B}_0\times & (\kappa/K)\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\eta U\boldsymbol{B}_0\times & \eta U(\boldsymbol{B}_0\times)^2 & -(\kappa/K)U\boldsymbol{B}_0\times & \mathbf{0} \\ \mathbf{0} & \pi\eta\mathbf{1} & -\pi\eta\boldsymbol{B}_0\times & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \end{Vmatrix} \tag{4.3}$$

and $||R_{\alpha\beta}||$ is defined by (3.8). In the above calculation, we have used (2.19)-(2.21) and the relation $R_{\alpha\beta}\dot{a}_\beta=0$ derived from (3.6). Taking the scalar products of (4.1) by the left nullvectors $\boldsymbol{b}^\gamma$ of $||R_{\alpha\beta}||$ eliminates the second term of the right member of (4.1) and yields

$$(b^\gamma_\alpha P_{\alpha\beta})\dot{a}_\beta = (b^\gamma_\alpha Q_{\alpha\beta})a_\beta\,, \tag{4.4}$$

which are the set of the ordinary differential equations for the amplitudes of the wave under consideration.

*Electro-acoustical wave*

Taking the vector product of the part $\alpha=7, 8, 9$ of (4.1) by $\boldsymbol{n}$ yields

$$\dot{\bar{\boldsymbol{v}}}\times\boldsymbol{n} = -\frac{\eta}{2\rho\phi}(\boldsymbol{B}_0\times\bar{\boldsymbol{E}})\times\boldsymbol{n}+\frac{\eta}{2\rho\phi}[\boldsymbol{B}_0\times(\boldsymbol{B}_0\times\bar{\boldsymbol{v}})]\times\boldsymbol{n}\,, \tag{4.5}$$

where we have used (3.16) and (3.17). By means of (3.18) and (3.19′), we can eliminate $\bar{\boldsymbol{E}}$ from (4.5), and we finally have

$$\dot{\bar{\boldsymbol{v}}}\times\boldsymbol{n} = -\delta_{EA}\bar{\boldsymbol{v}}\times\boldsymbol{n} \tag{4.6}$$

or

$$\bar{\boldsymbol{v}} = \bar{\boldsymbol{v}}_0\exp(-\delta_{EA}t)\,, \tag{4.7}$$

where

$$\delta_{EA}\equiv\frac{\eta}{2\rho\phi(1-U_{EA}^2/c^2)}(\boldsymbol{B}_0\cdot\boldsymbol{n})^2$$

$$\approx\frac{\eta}{2\rho\phi}(\boldsymbol{B}_0\cdot\boldsymbol{n})^2\,. \qquad (\geq 0) \tag{4.8}$$

The electric field $\bar{\boldsymbol{E}}$ is also expressed in the same form as (4.7) by (3.18) and (3.19′). Hence the electro-acoustical wave, in general, decays exponentially with respect to time. In particular, if $\boldsymbol{B}_0$ vanishes or is perpendicular to $\boldsymbol{n}$, the damping constant $\delta_{EA}$ vanishes and hence, the amplitudes of this wave remain constant. The decay of the mechanical transverse amplitude of this wave is a characteristic of the thermo-magneto-elastic material, because the transverse wave does not decay in the thermo-elastic material considered by Tokuoka[15]. It is also found from (4.8) that the electro-acoustical wave decays more rapidly in electric conductors than in dielectrics.

*Thermo-acoustical waves*

Taking the scalar product of the part $\alpha=7, 8, \cdots, 13$ of (4.1) by the vector

$$\boldsymbol{b}\equiv\left(\frac{\alpha_0T_0}{\rho U_{TA}^2-2G-\lambda}\boldsymbol{n}\,,\ \frac{1}{\tau U_{TA}}\boldsymbol{n}\,,\ -1\right) \tag{4.9}$$

yields
$$\frac{\alpha_0 T_0(\rho U_{TA}^2+2G+\lambda)}{\rho U_{TA}^2-2G-\lambda}\dot{\bar{\boldsymbol{v}}}\cdot\boldsymbol{n}+\dot{\bar{\boldsymbol{q}}}\cdot\boldsymbol{n}+\beta_0 T_0 U\dot{\bar{T}}$$
$$=\frac{\alpha_0 T_0 U_{TA}^2}{\phi(\rho U_{TA}^2-2G-\lambda)}\left[-\eta(\boldsymbol{B}_0\times\bar{\boldsymbol{E}})\cdot\boldsymbol{n}+\eta\,\{\boldsymbol{B}_0\times(\boldsymbol{B}_0\times\bar{\boldsymbol{v}})\}\cdot\boldsymbol{n}-\frac{\kappa}{K}(\boldsymbol{B}_0\times\bar{\boldsymbol{q}})\cdot\boldsymbol{n}\right]$$
$$+\frac{1}{\tau\phi}[\pi\eta\bar{\boldsymbol{E}}\cdot\boldsymbol{n}-\pi\eta(\boldsymbol{B}_0\times\bar{\boldsymbol{v}})\cdot\boldsymbol{n}-\bar{\boldsymbol{q}}\cdot\boldsymbol{n}]\,. \tag{4.10}$$

Eliminating $\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{q}}$ and $\bar{T}$ from (4.10) by means of (3.26), (3.12), (3.28) and (3.29), we finally obtain

$$\dot{\bar{\boldsymbol{v}}}\cdot\boldsymbol{n}=-\delta_{TA}\bar{\boldsymbol{v}}\cdot\boldsymbol{n} \tag{4.11}$$

or
$$\bar{\boldsymbol{v}}=\bar{\boldsymbol{v}}_0\exp(-\delta_{TA}t)\,, \tag{4.12}$$

where
$$\delta_{TA}\equiv\frac{1}{2\phi}\left[\frac{\eta(U_{TA}^2-c_H^2)^2}{\rho(1-U_{TA}^2/c^2)}(\boldsymbol{B}_0\times\boldsymbol{n})\cdot(\boldsymbol{B}_0\times\boldsymbol{n})+\frac{\gamma c_H^2}{\tau}\right]\bigg/[(U_{TA}^2-c_H^2)^2+\gamma c_H^2]$$
$$\approx\frac{\eta M}{2\rho\phi}(\boldsymbol{B}_0\times\boldsymbol{n})\cdot(\boldsymbol{B}_0\times\boldsymbol{n})+\frac{N}{2\tau\phi}\,,\qquad(>0) \tag{4.13}$$

and where
$$M\equiv\frac{(U_{TA}^2-c_H^2)^2}{(U_{TA}^2-c_H^2)^2+\gamma c_H^2}\,,\quad N\equiv\frac{\gamma c_H^2}{(U_{TA}^2-c_H^2)^2+\gamma c_H^2}\,. \tag{4.14}$$

The other coupled fields $\bar{\boldsymbol{q}}\cdot\boldsymbol{n}$ and $\bar{T}$ also take the same form as (4.12) by (3.28) and (3.29). Hence, the thermo-acoustical waves decay exponentially with respect to time. In view of the first term on the right of (4.13), the damping constants decrease, in contrast to the case of the electro-acoustical wave, according as $\boldsymbol{n}$ becomes parallel to $\boldsymbol{B}_0$. When $\boldsymbol{n}$ is paprallel to $\boldsymbol{B}_0$, the initial magnetic field does not influence the decay of these waves. The second term on the right of (4.13) is equal, apart from the multiplicative constant $\phi^{-1}$, to the damping constants of the thermo-acoustical waves considered by Tokuoka[15]. The presence of the multiplicative constant $\phi^{-1}(<1)$, in other words, the presence of the thermo-electrical interactions, depresses the damping constants below their normal values. The influence of the electrical conductivity is similar to the case of the electro-acoustical wave.

The non-dimensional quantities $M$, $N$, which depend on the thermo-elastic properties of the material, have respectively different values for the fast and slow waves. Consequently, the damping constants of the two waves also have different values. We get from (4.14)

$$M+N=1 \tag{4.15}$$

for each wave, and from (3.24) and (4.14)

$$M_f=N_s\,,\quad M_s=N_f\,. \tag{4.16}$$

The variation of $M$ and $N$ with respect to the parameters:

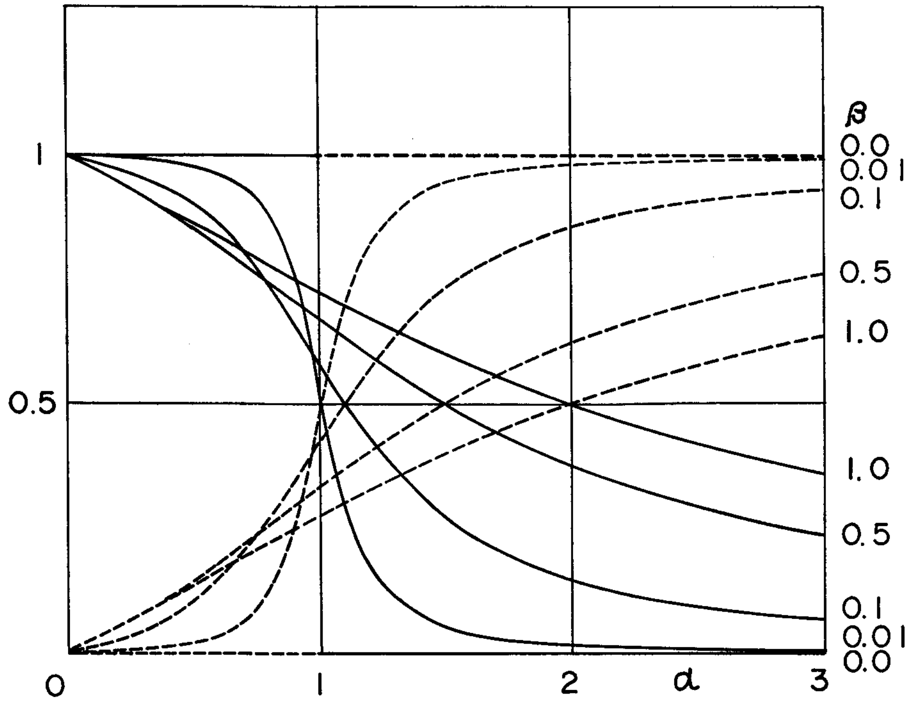


Fig. 1. Variation of $M$, $N$, where solid and broken lines denote, respectively, $M_f = N_s$ and $M_s = N_f$.

$$\alpha \equiv \frac{c_H^2}{c_L^2}, \quad \beta \equiv \frac{\gamma}{c_L^2} \tag{4.17}$$

is shown in Fig. 1. Next, according to the discussion in the last section, we shall consider the limit $\gamma \to 0$ for $c_L^2 > c_H^2$. From Fig. 1, we find that

$$M_f,\ N_s \to 1 \quad \text{and} \quad M_s,\ N_f \to 0 \tag{4.18}$$

as $\gamma \to 0$. Then (4.13) implies that

$$\delta_{TAf} \to \frac{\eta}{2\rho\phi}(\boldsymbol{B}_0 \times \boldsymbol{n}) \cdot (\boldsymbol{B}_0 \times \boldsymbol{n}) \quad \text{and} \quad \delta_{TAs} \to \frac{1}{2\tau\phi}. \tag{4.19}$$

Thus, for the damping constant of the predominantly mechanical wave, the term including the electrical conductivity and the initial magnetic field is essential. For that of the predominantly thermal wave, the term including the relaxation time of thermal conductivity is essential. It is easily verified that the situation is also valid when $c_H^2 > {}^2_L$.

*Electro-magnetic wave*

Taking the product of (4.1) by the tensor

$$||B|| \equiv \left\| \begin{array}{ll} -\dfrac{1}{\varepsilon c}\boldsymbol{n}\times & \boldsymbol{1}-\boldsymbol{n}\otimes\boldsymbol{n} \quad \dfrac{(c^2-c_H^2)[\boldsymbol{B}_0\otimes\boldsymbol{n}-(\boldsymbol{B}_0\cdot\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n}]}{\rho[(c^2-c_L^2)(c^2-c_H^2)-\gamma c^2]}+\dfrac{(\boldsymbol{B}_0\cdot\boldsymbol{n})(\boldsymbol{n}\times)^2}{\rho(c^2-c_T^2)} \\ \dfrac{\alpha_0 c[\boldsymbol{B}_0\otimes\boldsymbol{n}-(\boldsymbol{B}_0\cdot\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n}]}{\rho\tau\beta_0 T_0[(c^2-c_L^2)(c^2-c_H^2)-\gamma c^2]} & \dfrac{-\alpha_0 c^2[\boldsymbol{B}_0-(\boldsymbol{B}_0\cdot\boldsymbol{n})\boldsymbol{n}]}{\rho\beta_0 T_0[(c^2-c_L^2)(c^2-c_H^2)-\gamma c^2]} \end{array} \right\| \tag{4.20}$$

yields by use of (3.34),

$$\boldsymbol{n}\times\dot{\bar{\boldsymbol{E}}}+\mu c\dot{\bar{\boldsymbol{h}}}-\mu c(\dot{\bar{\boldsymbol{h}}}\cdot\boldsymbol{n})\boldsymbol{n} = -\frac{\eta}{\varepsilon\phi}\boldsymbol{n}\times\bar{\boldsymbol{E}}-\frac{\eta c^2(c^2-c_H^2)}{\rho\phi[(c^2-c_L^2)(c^2-c_H^2)-\gamma c^2]}[\{\boldsymbol{n}\cdot(\boldsymbol{B}_0\times\bar{\boldsymbol{E}})\}\boldsymbol{B}_0$$
$$-(\boldsymbol{B}_0\cdot\boldsymbol{n})\{\boldsymbol{n}\cdot(\boldsymbol{B}_0\times\bar{\boldsymbol{E}})\}\boldsymbol{n}]-\frac{\eta c^2}{\rho\phi(c^2-c_T^2)}(\boldsymbol{B}_0\cdot\boldsymbol{n})[\boldsymbol{n}\times\{\boldsymbol{n}\times(\boldsymbol{B}_0\times\bar{\boldsymbol{E}})\}]. \tag{4.21}$$

Eliminating $\bar{\boldsymbol{h}}$ form (4.21) by means of (3.2), we finally obtain the differential equation for $\bar{\boldsymbol{E}}$:

$$\dot{\bar{\boldsymbol{E}}} = -\left[\frac{\eta}{2\varepsilon\phi}-\frac{\eta c^2}{2\rho\phi(c^2-c_T^2)}(\boldsymbol{B}_0\cdot\boldsymbol{n})^2\right]\bar{\boldsymbol{E}}$$
$$+\frac{\eta c^2(c^2-c_H^2)}{2\rho\phi[(c^2-c_L^2)(c^2-c_H^2)-\gamma c^2]}[(\boldsymbol{B}_0\times\boldsymbol{n})\cdot\bar{\boldsymbol{E}}]\boldsymbol{B}_0\times\boldsymbol{n}, \tag{4.22}$$

where we have used the relation $\bar{\boldsymbol{E}}=\boldsymbol{n}\times(\bar{\boldsymbol{E}}\times\boldsymbol{n})$ derived from (3.34). In order to solve (4.22), we shall first assume that $\boldsymbol{n}$ is not parallel to $\boldsymbol{B}_0$, and introduce the unit vectors being perpendicular to $\boldsymbol{n}$:

$$\boldsymbol{e}_1=\frac{\boldsymbol{B}_0\times\boldsymbol{n}}{|\boldsymbol{B}_0\times\boldsymbol{n}|}, \quad \boldsymbol{e}_2=\frac{\boldsymbol{n}\times(\boldsymbol{B}_0\times\boldsymbol{n})}{|\boldsymbol{n}\times(\boldsymbol{B}_0\times\boldsymbol{n})|}. \tag{4.23}$$

By taking the scalar products of (4.22) by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, we have, respectively,

$$\dot{\bar{\boldsymbol{E}}}\cdot\boldsymbol{e}_1 = -\delta_{EM1}\bar{\boldsymbol{E}}\cdot\boldsymbol{e}_1 \quad \text{and} \quad \dot{\bar{\boldsymbol{E}}}\cdot\boldsymbol{e}_2 = -\delta_{EM2}\bar{\boldsymbol{E}}\cdot\boldsymbol{e}_2, \tag{4.24}$$

or, by solving them

$$\bar{\boldsymbol{E}}\cdot\boldsymbol{e}_1 = (\bar{\boldsymbol{E}}_0\cdot\boldsymbol{e}_1)\exp(-\delta_{EM1}t) \quad \text{and} \quad \bar{\boldsymbol{E}}\cdot\boldsymbol{e}_2 = (\bar{\boldsymbol{E}}_0\cdot\boldsymbol{e}_2)\exp(-\delta_{EM2}t), \tag{4.25}$$

where

$$\delta_{EM1} \equiv \frac{\eta}{2\varepsilon\phi}-\frac{\eta}{2\rho\phi(1-c_T^2/c^2)}(\boldsymbol{B}_0\cdot\boldsymbol{n})^2$$
$$-\frac{\eta(1-c_H^2/c^2)}{2\rho\phi[(1-c_L^2/c^2)(1-c_H^2/c^2)-\gamma/c]^2}(\boldsymbol{B}_0\times\boldsymbol{n})\cdot(\boldsymbol{B}_0\times\boldsymbol{n})$$
$$\approx \frac{\eta}{2\varepsilon\phi}-\frac{\eta}{2\rho\phi}\boldsymbol{B}_0\cdot\boldsymbol{B}_0 \tag{4.26}$$

$$\delta_{EM2} \equiv \frac{\eta}{2\varepsilon\phi}-\frac{\eta}{2\rho\phi(1-c_T^2/c^2)}(\boldsymbol{B}_0\cdot\boldsymbol{n})^2 \approx \frac{\eta}{2\varepsilon\phi}-\frac{\eta}{2\rho\phi}(\boldsymbol{B}_0\cdot\boldsymbol{n})^2.$$

The other coupled field $\bar{\boldsymbol{h}}$ is obtained from (3.2) and (4.25), i.e.

$$\bar{\boldsymbol{h}}\cdot\boldsymbol{e}_1 = (\bar{\boldsymbol{h}}_0\cdot\boldsymbol{e}_1)\exp(-\delta_{EM2}t) \quad \text{and} \quad \bar{\boldsymbol{h}}\cdot\boldsymbol{e}_2 = (\bar{\boldsymbol{h}}_0\cdot\boldsymbol{e}_2)\exp(-\delta_{EM1}t)\,. \tag{4.27}$$

We may assume that $\epsilon^{-1}-|\boldsymbol{B}_0|^2\rho^{-1}>0$ for usual cases, so that we see from (4.26)

$$\delta_{EM2}\geq\delta_{EM1}>0\,. \tag{4.28}$$

Thus, we can conclude that the electro-magnetic wave also decays exponentially with respect to time, and that the electric field and the magnetic field become polarized, respectively, in the direction of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ during propagation. For example, we shall consider the electro-magnetic wave which is circularly polarized at $t=0$. Then, the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ components of any electric field of the wave at $t=0$ satisfy the equation of the circle:

$$(\bar{\boldsymbol{E}}_0\cdot\boldsymbol{e}_1)^2+(\bar{\boldsymbol{E}}_2\cdot\boldsymbol{e}_2)^2 = \bar{\boldsymbol{E}}_0\cdot\bar{\boldsymbol{E}}_0\,. \tag{4.29}$$

By use of (4.25) and (4.29), the components at the time $t$ are found to satisfy the equation of the ellipse:

$$\frac{(\bar{\boldsymbol{E}}\cdot\boldsymbol{e}_1)^2}{[\exp(-\delta_{EM1}t)]^2}+\frac{(\bar{\boldsymbol{E}}\cdot\boldsymbol{e}_2)^2}{[\exp(-\delta_{EM2}t)]^2} = \bar{\boldsymbol{E}}_0\cdot\bar{\boldsymbol{E}}_0\,, \tag{4.30}$$

where the ratio of the minor axis to the major axis is given by

$$r \equiv \exp\left[-\frac{\eta}{2\rho\phi}(\boldsymbol{B}_0\times\boldsymbol{n})\cdot(\boldsymbol{B}_0\times\boldsymbol{n})\,t\right]. \tag{4.31}$$

These equations show that the electro-magnetic wave which is circularly polarized at the first time becomes elliptically polarized during propagation, and that the intensity of the polarization increases exponentially with respect to time. It is also found that the intensity of the polarization at any time is greatest when $\boldsymbol{n}$ is perpendicular to $\boldsymbol{B}_0$.

When $\boldsymbol{n}$ is parallel to $\boldsymbol{B}_0$, the second term on the right of (4.22) vanishes. Therefore, any component of $\bar{\boldsymbol{E}}$ being perpendicular to $\boldsymbol{n}$ can be expressed in the same form as (4.25), where the two damping constants are consistent. Of course, the above elliptical polarization does not occur in this case.

Next, we shall mention the influence of the properties of the material on the decay of this wave. In view of (4.26), the damping constants do not depend on the thermo-elastic properties of the material in our approximation. Contrary to the cases of the electro-acoustical and thermo-acoustical waves, the damping constants decrease according as the magnitude of the initial magnetic field increases. The influence of the electrical conductivity and the thermo-electrical interactions is similar to the cases of the other three waves. Especially for electric conductors, the first terms on the right of (4.26) will be very large, and then the electro-magnetic wave decays instantaneously.

Finally in Fig. 2, we make a sketch of the variation of the five damping constants with respect to the angle between the propagation direction and the initial magnetic field.

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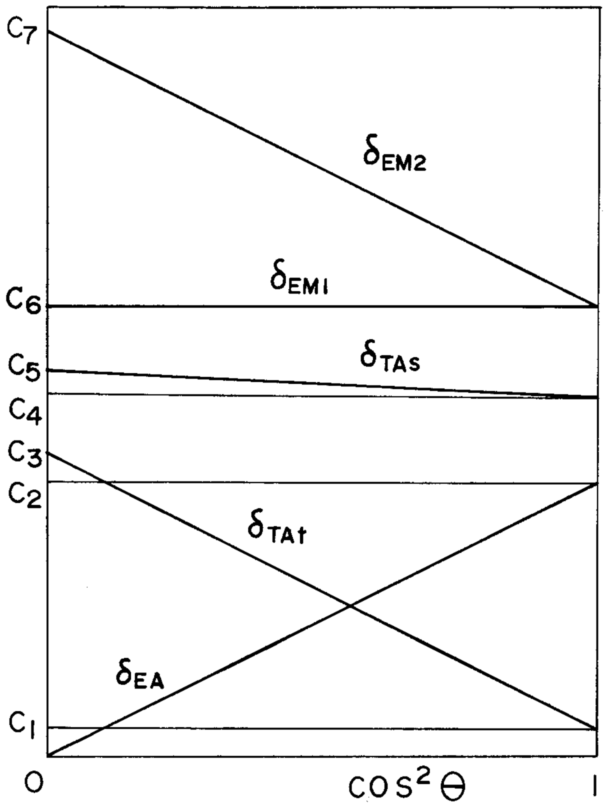


Fig. 2. Variation of the damping constants with respect to the angle $\theta$ between the propagation direction and the initial magnetic field, where $c_1=N_f/2\tau\phi$, $c_2=\eta|\boldsymbol{B}_0|^2/2\rho\phi$, $c_3=\eta M_f|\boldsymbol{B}_0|^2/2\rho\phi+N_f/2\tau\phi$, $c_4=N_s/2\tau\phi$, $c_5=\eta M_s|\boldsymbol{B}_0|^2/2\rho\phi+N_s/2\tau\phi$, $c_6=\eta/2\varepsilon\phi-\eta|\boldsymbol{B}_0|^2/2\rho\phi$, $c_7=\eta/2\varepsilon\phi$. We have assumed that $c_L^2>c_H^2$, $\eta/\varepsilon>1/\tau>\eta|\boldsymbol{B}_0|^2/\rho$.

## References


1) S. Kaliski; Proc. Vibr. Probl., 3, **6**, 231 (1965).
2) S. Kaliski and J. Petykiewicz; Proc. Vibr. Probl., 3, **1**, 81 (1960).
3) N.F. Jordan and A.C. Eringen; Int. J. Engng Sci., **2**, 59 (1964).
4) H.F. Tiersten; Int. J. Engng Sci., **9**, 587 (1971).
5) M.F. McCarthy; Int. J. Engng Sci., **12**, 45 (1974).

6) G. Paria; Proc. Cambridge Philos. Soc., **58**, 527 (1962).
7) G. Paria; Proc. Vibr. Probl., 1, **5**, 57 (1964).
8) A.J. Willson; Proc. Cambridge Philos. Soc., **59**, 483 (1963).
9) S. Kaliski and W. Nowacki; Int. J. Engng Sci. **1**, 163 (1963).
10) C.M. Purushothama; Proc. Cambridge Philos. Soc., **61**, 939 (1965).
11) S. Chander; Int. J. Engng Sci., **6**, 409 (1968).
12) F.C. Moon and S. Chattopadhyay; J. Appl. Mech., **41**, E, 3, 641 (1974).
13) M.F. McCarthy; Proc. Vibr. Probl., 4, **8**, 337 (1967), 4, **9**, 367 (1968).
14) M.F. McCarthy; Int. J. Engng Sci., **11**, 1301 (1973).
15) T. Tokuoka; J. Engng Math., **7**, 2, 115 (1973).