On Pole Assignment and Stabilization for the Heat Equation

By

Teruo HAMATSUKA*, Abdul-Aziz MO'OMEN** and Hajime AKASHI*

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Abstract

For one type of infinite dimensional linear systems, specifically the heat equation, the possibility of assigning the infinite set of poles of a closed loop system formed by means of a suitable linear state feedback operator is discussed. Necessary and sufficient conditions are derived for the existence of a feedback opeartor to shift all the eigenvalues of the controllable system, and to assign an arbitrary finite set of poles of the closed loop system. As an application of this result, it is shown that an open loop controllable system can be stabilized in a desired order of convergence by a suitable choice of the feedback operator.

1. Introduction

We consider the heat conduction problem which is normally described by the partial differential equation of the parabolic type:

$$(1) \qquad \frac{\partial}{\partial t}\theta(\eta, t) = \frac{\partial^2}{\partial \eta^2}\theta(\eta, t), \qquad 0\leq\eta\leq 1,\ t\geq 0$$

with the adiabatic boundary conditions

$$\theta_\eta(\eta, t) = 0 = \theta_\eta(1, t), \qquad t\geq 0$$

and initial temperature data

$$\theta(\eta, 0) = \theta_0(\eta) \qquad 0\leq\eta\leq 1 .$$

This equation governs the evolution of the temperature distribution of a homogeneous insulated rod held to zero temperature at the ends $\eta=0$, $\eta=1$, where η denotes the position along the rod. Also, we assume zero temperature gradients at the ends.

* The authors are with the Department of Precision Mechanics, Faculty of Engineering, Kyoto University, Kyoto 606, Japan.

** ABDUL-AZIZ MO'OMEN is with the Department of Precision Mechanics, Faculty of Engine-rering, Kyoto Univesity, Kyoto 606, Japan, on leave from the Department of Mechanical engine-ering, Ain-Shams University, Abbasia, Cairo, Egypt.

One way that we might generalize our view of the "formal" (or "classical") description of the system by equation (1), would be to set up the abstract version which is completely equivalent. There are several good reasons why it is actually preferable, not the least of which is the possibility of making use of the functional analysis methods. Another reason becomes apparent when we regard the control problem for such systems which we wish to motivate, especially for introducing the control forces and monitoring some feedback.

Let the Hilbert psace $\mathcal{X}=L_2(0, 1)$ and consider the linear system $\mathcal{L}$

$$(2) \qquad \frac{d}{dt}u(t) = Au(t)+Bf(t)\,, \qquad t\geq 0\,,$$

with $A\approx\frac{\partial^2}{\partial x^2}$, and the domain $\mathcal{D}(A)$ of A given by

$$\mathcal{D}(A) = [u\,|\,u,\ u' \text{ a.c.},\ u',u''(\cdot)\in\cdot L_2(0,\ 1),\ \text{and } u'(0) = u'(1)=0]\,.$$

Also, A: $\mathcal{X}\supset\mathcal{D}(A)\rightarrow\mathcal{X}$ is the infinitesimal generator of a strongly continuous semigroup of bounded operators $T(t)$ on $\mathcal{X}$, $t\geq 0$. B is a bounded operator from the Hilbert space $\mathcal{U}$ (the control space) to the Hilbert space $\mathcal{X}$ (the state space). If, however, either dim $\mathcal{U}=m$ or (range of $B)=m$, we write $\mathcal{L}$ as $\mathcal{L}_m$

$$(3) \qquad \frac{d}{dt}u(t) = Au(t)+\sum_{i=1}^{m} b_i f_i, \text{ (the independent vector } b_i\ \mathcal{X}, \text{ and } f_i = \text{scalar)}$$

Suppose we are free to modify (2) by setting

$$f(t) = Fu(t)+g(t)\,, \qquad t\geq 0$$

where $g(t)$ is a new external input, and F: $\mathcal{X}\rightarrow\mathcal{U}$[3] is a bounded linear operator (e.g. an integral operator). We refer to F as the *state feedback*. The obvious result of introduicng a state feedback is to achieve stability, or to speed up re· sponse. In the finite dimensional case, it is always possible to stabilize a completely controllable system by the state feedback, or to improve its stability by assigning the closed-loop poles to locations in the left-half complex plane [10]. It is impossible to generalize this result directly to the infinite dimensional case due to a number of difficiulties. For example, the spectrum of an operator on infinite dimensional space consists, generally, of something more than eigenvalues. Also, the number of the elements of the spectrum of the operator in the half plane $Re.\ s<-\varepsilon$, $\varepsilon>0$, may not even be countable. We point out that our selection for the heat equation reduces some of these difficulties. However, the spectrum of the operator A are not finite. The problem of stabilization for controllable linear systems in the infinite dimensional case, and in the framework pertinent to the present paper, was first considered by Slemrod, in two papers [7], [8]. Slemrod's approach-motivated mainly by hyperbolic systems and carried out for Hilbert spaces $\mathcal{X}$ and $\mathcal{U}$-parallels the Lyapunove type of argument used in [5] in finite dimension, by relying on Hale's generalization of LaSalle's invariance principle for Banach spaces. In particular, the stabilizability of $\mathcal{L}$ is sought in [7] and [8] under the assumption of its approximate controllability- or the D_ε-assumption-plus some further hypotheses. Also, slemrod considered only the problem of stabilization without being concerned about assigning the poles of the closed loop feedback system. Triggiani, in his paper [9], tried to extend the finite dimensional theory by decomposing the state space into two suitable subspaces (invariant under A), and studying the projection of the original system on such spaces. However, he restricted himself to the case where only many finite eigenvalues of A lie in the right half plane, and was then able to reduce the problem to the finite dimensional case. In [4], the authors made no such assumptions. However, their main result (theorem 3) included the crucial condition: $\sigma(A+b\otimes c)\cap\sigma(A)=\phi$. This condition is introduced with no explanation about its realization. Unfortunately, the examples given there did not treat such a condition. Here, we examine this condition well, and introduce an alternative proof for theorem (1) [4] so as to avoid making use of the discretized system.

Some basic notions are defined in Section (2). Section (3) explains the reduction to a single-input system, with some discussion about the rank one operator which is used as a stayte feedback operator in this paper. In Sections (4) and (5), we treat and solve two important problems in the case of infinite dimensional and of long-standing interest in control theory: pole assignment and stabilizability by the state feedback respectively.

2. Preliminaries

We have assumed that A is the densely defined self-adjoint heat diffusion operator. As is well known [1], A has a pure point spectrum with no finite limit point, i.e.

$$\sigma(A)=\sigma_p(A)=\{\lambda=\lambda_n=-n^2\pi^2, \qquad n=0, 1, 2, \cdots\}$$

and

$$\phi_n(x)=\sqrt{2}\cos n\pi x$$

is an eigen function corresponding to eigenvalue λ_n. It is easy to prove that A generates a compact semigroup $T(t)$ such that

$$T(t)u=\sum_0^\infty e^{-n^2\pi^2 t}(u, \phi_n)\phi_n$$

where

$$(u, \phi_n) = \int_0^1 u(x)\overline{\phi_n(x)}\,dx\,.$$

The resolvent operator is compact for all $\lambda \in \rho(A)$; and

$$R(\lambda, A)u = [\lambda I - A]^{-1}u = \sum \frac{1}{\lambda - \lambda_n}(u, \phi_n)\phi_n\,.$$

It is known that the system (2) has a solution if $f(t)$ is sufficiently smooth in $[0, \infty)$, for instance continuously differentiable, and it is always so assumed.

Definition 1. By a system [A,B] we mean (2), a state u in $\mathcal{X}$ is called *controllable* if for any $\varepsilon > 0$, there is $f \in C^1((0, t), \mathcal{U})$ such that the solution of (2) with $u(0) = 0$ satisfies

$$\|u(t) - u\| < \varepsilon \text{ for some } t > 0\,.$$

where $C^1((0, t), \mathcal{U})$ is the set of control which is continuously differentiable. The set of every controllable state of $[A, B]$-denoted by $\mathcal{X}_c(A, B)$-is said to be the *controllable subspace.* $[A, B]$ is controllable if $\mathcal{X}_c = \mathcal{X}$. It is well known that

$$\mathcal{X}_c(A, B) = \overline{\bigcup_{t \geq 0} T(t) \text{ Range } B}$$

where $\overline{\qquad}$ denotes the closure.

3. Reduction to Single Input-System

The first step in our procedure will be to reduce the problem to the case where B is a rank one operator. Our later discussion will be devoted to the system $\mathcal{L}_m$ subjected to a feedback operator of rank one denoted by $b \otimes c$.

Theorem 1. If $[A, B]$ is controllable, then there is a vector $b \in \text{Range } B$ such that $[A, B]$ is controllable.

Proof. Since $\{b_1, \cdots, b_m\}$ are the basis of Range B:

$$b_i = \sum_{k=1}^{\infty} (b_i, \phi_k)\phi_k\,, \quad \text{and}$$

$$T(t)b_i = \sum_{k=1}^{\infty} e^{-n^2\pi^2 t}(b_i, \phi_k)\phi_k$$

Note that the controllability condition

$$\mathcal{X} = \mathcal{X}_c = \overline{\bigcup_{t \geq 0} T(t) \text{ Range } B}$$

$$= \overline{\bigcup_{t \geq 0} \text{Span } \{\sum_{k=1}^{\infty} e^{-k^2\pi^2 t}(b_i, \phi_k)\phi_k\}}$$

implies that for each k, $(u_i, \phi_k) \neq 0$ for one i at least, since

$$e^{-i^2\pi^2 t} \neq e^{-j^2\pi^2 t} \qquad \text{for } i \neq j,\ t>0 .$$

Thus, there exist numbers $\alpha_1, \cdots, \alpha_m$ such that $\sum_{i=1}^{m} \alpha_i (b_i, \phi_k) \neq 0$ for all k. This implies $b = \sum_{i=1}^{m} \alpha_i b_i$ is a cyclic vector for $T(t)$ and $[A, B]$ is controllable. The above theorem guarantees that we can replace a multi-input controllable system by a single-input controllable system with no loss of generality. Thus, in our case, we can replace (3) by the system:

$$\frac{d}{dt} u(t) = Au(t) + f(t) b , \qquad b \in \mathcal{X} . \tag{3'}$$

where $f(t)$ is a scalar valued function. For a state feedback:

$$f(t) = (u(t), c) + g(t) , \qquad \text{where } c \in \mathcal{X} \text{ and}$$
$$(u(t), c) = \int_0^1 u(x) \overline{c(x)} dx .$$

The state feedback system will be

$$\frac{d}{dt} u(t) = (A + b \otimes c) u(t) + g(u) b .$$

where $$(b \otimes c) u = (u, c) b = b(x) \int_0^1 u(y) \overline{c(y)}\, dy .$$

We call the operator $(b \otimes c)$ by the rank one operator since its range is one dimensional subspace.

We conclude this section by stating some properties for the rank one operator.

1. $b \otimes c$: rank one operator, i.e., the dimension of

$$\text{Range } b \otimes c = 1 .$$

2. $b \otimes c$: is compact, and then, from the Fredholm alternative [6]:

$$\sigma(b \otimes c) = \{0\} \cup \sigma_p(b \otimes c)$$

where $\sigma(b \otimes c)$ denotes the spectrum of $(b \otimes c)$, while $\sigma_p(b \otimes c)$ is its point spectrum. Both can be easily computed as:

$$\sigma(b \otimes c) = \{0, (b, c)\}$$

3. b is an eigenvector corresponding to the eigenvalue (b, c).
4. Span $\{b\}$ is $(b \otimes c)$-invariant.
5. Ker $b \otimes c = \{x \in \mathcal{X};\ (x, c) = 0\} = \mathcal{C}$
6. Every x which is orthogonal to c is an eigenvector corresponding to the eigenvalue zero.

$$\mathcal{N} \triangleq \text{Ker}\, b\otimes c = \mathcal{C}, \qquad \mathcal{b} \triangleq \text{span}\, b\,.$$

4. Pole Assignment

The following theorem allows us to shift all the eigenvalues of operator A by the state feedback.

Theorem 2. Suppose that $[A, B]$ is controllable, then

$$\sigma(A+b\otimes c)\cap\sigma(A) = \phi$$

if, and only if $\quad (c, \phi_k)\neq 0, \quad \text{for all } k = 1, 2, \cdots$

Furthermore, $\quad \sigma(A+b\otimes c) = \sigma_p(A+b\otimes c) = \{\mu\in\mathcal{C}\,|\,(R_\mu b, c) = 1\}$

Proof: We note that the controllability of $[A, B]$ is equivalent to

$$(b, \phi_k)\neq 0, \qquad \text{for all } k = 1, 2, \cdots$$

Suppose that $\quad -n^2\pi^2 u = (A+b\otimes c)u, \quad$ or

$$-n^2\pi^2 u = Au+(u, c)b \tag{4}$$

for some $n=1, 2, \cdots$. Substituting the representations

$$u = \sum (u, \phi_k)\phi_k, \quad b = \sum (b, \phi_k)\phi_k$$

into (4), we get

$$-n^2\pi^2(u, \phi_k) = -k^2\pi^2(u, \phi_k)+(u, c)(b, \phi_k), \quad \text{or}$$
$$(k^2-n^2)\pi^2(u, \phi_k) = (u, c)(b, \phi_k),$$

Since, $k^2-n^2=0$ for $k=n$, we have

$$(u, c) = 0 \qquad \text{if } (b, \phi_k)\neq 0\,.$$

Then, $\quad (u, \phi_k) = 0 \qquad \text{for } k\neq n, \text{ and } u_n = 0,$

if $(c, \phi_n)\neq 0$. This implies $u=0$, and $-n^2\pi^2\in\rho(A+b\otimes c)$ (the resolvent set of the operator $(A+b\otimes c)$. Supposing $(c, \phi_k)=0$ for some k, then

$$-k^2\pi^2\in\sigma(A)\cap\sigma(A+b\otimes c),$$

since $\quad (A+b\otimes c)\phi_k = -k^2\pi^2\phi_k+(\phi_k, c)b = -k^2\pi^2\phi_k\,.$

This proves the main part of the theorem. The remaining part is feasible in proving.

Corollary 1. Suppose that the system $[A, B]$ is controllable, then

$$-n^2\pi^2\in\sigma(A)\cap\sigma(A+b\otimes c), \qquad \text{if } (c, \phi_n) = 0\,.$$

The proof ois obvious.

Thus, for the last step, we have the following theorem.

Theorem 3. Suppose $[A, B]$ is controllable. Let $\{\mu_k\}_{k=1,\cdots,N}$ for any finite number N. Then, there exists a vector $c\in\mathcal{X}$ such that

$$\{\mu_k\}_{k=1}^{n}\in\sigma(A+b\otimes c)\ , \quad \text{and;}$$
$$\{-n^2\pi^2\}_{n=1,2,\cdots}\in\sigma(A+b\otimes c)\ ,$$

except N number of eigenvalues, say, $-k^2\pi^2$, $k=i_{(1)}, i_{(2)}, \cdots, i_{(N)}$. That is, by $c\in\mathcal{X}$, any finite number of eigenvalues of A can be shifted.

Proof: Let $c=\sum_{k=0}^{N} c_k\phi_k$, $c_k=(c, \phi_k)=\sqrt{2}\int_0^1 c(x)\cos k\pi x dx$, and $h(\lambda)=(R_\lambda, b, c)$. Then, by simple computation, we have

$$h(\lambda) = (R_\lambda b, c) = \sum_{k=1}^{N-1} \frac{b_k\bar{c}_k}{\lambda+k^2\pi^2}\ .$$

For the proof, it is enough to show that we can evaluate c_k such that $h(\mu_k)=1$ by theorem (2). Note that $h(\lambda)$ is a rational function with numerators being a polynomial of degree $N-1$ and the denominator a polynomial of degree N. By the classical interpolation techniques we can calculate c_k such that $h(\mu_k)=1$, since $k^2\pi^2\neq l^2\pi^2$ for $k\neq l$. This completes the proof.

5. Stabilizability Problem of the Pair $[A, B]$

Find, if possible, a bounded linear operator F: $\mathcal{X}\to\mathcal{U}$, such that the strongly continuous semigroup of bounded operators $S(t)$, $t\geq 0$ in $\mathcal{X}$, generated by $A+BF$: $\mathcal{X}\supset \mathrm{D}\mathcal{D}(A)=\mathcal{D}(A+BF)\to\mathcal{X}$ [3, p. 630], satisfies

$$||S(t)x_0||\to 0 \qquad \text{as } t\to\infty, \text{ for all } x_0\in\mathcal{X}\,.$$

The pair $[A, B]$ is then called *stabilizable.*
If, in particular, the above is specialized as

$$||S(t)x_0||\leq M_{x_0}e^{-\delta t}\ , \qquad \delta>0,\ t\geq 0\ ,$$

($M_{x_0}A$ constant depending on x_0), or as

$$||S(t)x_0||\leq Me^{-\delta t}||x_0||\ , \qquad \delta>0,\ t\geq 0$$

(with M independet on x_0 and t), we can talk about the *exponential stabilizability* and the *proper exponential stabilizabilty* of the pair $[A, B]$, respectively. For the pair $[A, b_1, b_2, \cdots, b_m)]$ referring to $\mathcal{L}_m$, the operator C in the above conditions is replaced by the m-tuple $(c_1, \cdots, c_m)$, $c_i\in\mathcal{X}$.

The main result of this section is the following:

Theorem 4. Suppose $[A, B]$ is controllable. Then, there eists a $c\in\mathcal{X}$ such that the feedback system (3) is exponentially stable and the solution converges to zero in any desirable order, i.e.,

$$||S(t)u||\leq M_u e^{-Lt} \qquad \text{for any } L>0\,,$$

where M is a con$_u$stant which may depend on u.

Proof: Suppose that $(n-1)^2\pi^2\leq L<n^2\pi^2$. By theorem 3, we can select c, such that the eignevalues $\{k^2\pi^2\}_{k=0,1,\cdots,n-1}$ are replaced by

$$\{\mu_k\}_{k=0,1,\cdots,n-1},\ \mathrm{Re}\,\mu_k<-L\,.$$

Since $\sigma(A+b\otimes c)=\{\mu_1,\cdots,\mu_{n-1},n^2\pi^2,-(n+1)^2\pi^2,\cdots\}$, $'(A+b\otimes c)$ generates a compact semigroup $S(t)$ and has eigenfunctions $\{\psi_k\}_{k=0,1,\cdots}$ which may not be orthonormal. (Note that $\psi_k=\phi_k=\sqrt{2}\cos k\pi x$ for $k\geq n$.) Therefore,

$$\begin{aligned}||S(t)u|| &= ||\sum_{k=0}^{n-1} e^{\mu_k t}u_k\psi_k+\sum_{k=n}^{\infty} e^{-k^2\pi^2 t}u_k\phi_k||^2\\ &\leq e^{-Lt||\Sigma u_k\psi_k||}\\ &= M_u e^{-Lt}\,.\end{aligned}$$

This completes the proof.

6. Conclusion

The present article provides an effective approach for the formulation and solution of the pole assignment, and also the stabilization by the state feedback problem. By rather simple but powerful tools, we prescribed the method of solution by clarifying the conditions under which the state feedback exists to shift the poles of the closed loop system. Although we applied it specifically to the one dimensional heat equation, this method can treat more genearl cases e.g., when the operator A is an elleptic partial differential operator, and also the wave equation, without any additional flexibilty.

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