Thermal Stress Slip Flow Induced in Rarefied Gas between Noncoaxial Circular Cylinders

By

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Abstract

The behavior of a slightly rarefied gas between noncoaxial circular cylinders with different temperatures is investigated by the generalized slip flow theory, which is derived systematically from the Boltzmann equation. When the temperature of the inner cylinder is higher than that of the outer, a flow is induced in the gas toward the narrower region along the inner cylinder and toward the wider along the outer, and the inner cylinder is subject to a force toward the axis of the outer cylinder. In the opposite case, the flow and the force are reversed. The corresponding results for the system where a cylinder lies in a semi-infinite expanse of the gas bounded by an infinite plane wall are derived as the limiting case where the radius of the outer cylinder tends to infinity.

I. Introduction

Consider a slightly rarefied gas between noncoaxial circular cylinders with different temperatures (the inner cylinder is kept at a temperature and the outer at another temperature). In the framework of the classical fluid dynamics, we cannot expect any flow to occur in the gas under the present situation. Further, the thermal creep flow[1,2] (of the order of the Knudsen number), one of the well-known rarefaction effects of gas, does not occur either because the temperature of each cylinder is uniform. According to Ref. 3, there is a velocity slip (of the second order of the Knudsen number) over a solid boundary when the temperature gradient of the gas normal to the boundary is nonuniform. A flow is to be induced in the gas between the two cylinders owing to this slip, since the temperature gradient is larger where the gap between the cylinders is narrower. This was pointed out by the senior author (Y.S.),[4] but the flow has not been analyzed yet. Here, we investigate the velocity field induced in the gas and the force acting on the inner cylinder. The corresponding results for the system where a cylinder kept at a temperature lies in a semi-infinite expanse of the gas bounded by an

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infinite plane wall at another temperature are derived as the limiting case where the radius of the outer cylinder tends to infinity.

We analyze the problem under the assumptions:

(i) the behavior of the gas is described by the Boltzmann-Krook-Welander equation[5];

(ii) the gas molecules make diffuse reflection[6,7] on the surface of the cylinders;

(iii) the temperature difference between the two cylinders is so small that the governing equation and the boundary condition may be linearized around a uniform equilibrium state at rest.

Since we are concerned with a slightly rarefied gas, which means that the mean free path of the gas is small but not negligibly small compared with the smaller of the two: the narrowest gap between the cylinders and the radius of the inner cylinder, we can make use of the result of the asymptotic theory (generalized slip flow theory) developed in Refs. 3 and 4 on the basis of the linearized Boltzmann-Krook-Welander equation. It is a general theory that describes the behavior of a gas for the Knudsen number of the system (the mean free path of the gas divided by the characteristic length of the system) being small. We can, thus, avoid the trouble to deal with the kinetic equation (Boltzmann-Krook-Welander equation) directly.

II. Asymptotic Theory

We summarize the result of the asymptotic theory in Refs. 3 and 4 (See also Ref. 8)*.

The behavior of a gas around solid bodies for the Knudsen number being small is described as follows.

(i) A quantity such as the velocity, the temperature, or the density of the gas (say f) can be split into two parts:

$$f=f_H+f_K\,, \tag{1}$$

where f_H is the Hilbert part, whose length scale of variation is of the order of the characteristic length of the system (e.g., the radius of the inner cylinder, the gap between the cylinders), and f_K is the Knudsen-layer part, which is appreciable only in a thin layer (with thickness of the order of the mean free path) adjacent to the solid boundary. The length scale of variation of f_K normal to the boundary is of the order of the mean free path. Each part is expanded in a power series of

* More details of the process of analysis and refined numerical data are given in Ref. 8, where the asymptotic theory for a two phase system of a gas and its condensed phase is developed.

the Knudsen number, namely

$$f_H = f_{H0} + f_{H1}k + f_{H2}k^2 + \cdots, \tag{2a}$$

$$f_K = f_{K1}k + f_{K2}k^2 + \cdots. \tag{2b}$$

The k is defined as

$$k = \frac{\sqrt{\pi}}{2}\frac{l}{L} = \frac{\sqrt{\pi}}{2}Kn, \tag{3}$$

where L is a reference length, l is the mean free path of the gas at our reference state, and, thus, Kn is the Knudsen number.

(ii) The Hilbert parts of the velocity $(2RT_0)^{1/2}u_i$, the pressure $p_0(1+p)$, and the temperature $T_0(1+\tau)$ of the gas,* where T_0 is a reference temperature, p_0 is a reference pressure, and R is the gas constant, obey the Stokes system of partial differential equations:

$$\frac{\partial p_{H0}}{\partial x_i} = 0, \tag{4}$$

$$\frac{\partial u_{iHm}}{\partial x_i} = 0, \tag{5}$$

$$\frac{\partial p_{Hm+1}}{\partial x_i} - \frac{\partial^2 u_{iHm}}{\partial x_j^2} = 0, \tag{6}$$

$$\frac{\partial^2 \tau_{Hm}}{\partial x_j^2} = 0, \tag{7}$$

$$(m = 0, 1, 2, \cdots),$$

where Lx_i is a Cartesian coordinate of the physical space. The same differential system occurs at each order of m.

The density $p_0(RT_0)^{-1}(1+\omega)$ of the gas is given by the linearized form of the equation of state as

$$\omega = p - \tau. \tag{8a}$$

Thus,

$$\omega_{Hm} = p_{Hm} - \tau_{Hm}. \tag{8b}$$

(iii) The boundary conditions on the solid boundary for the Stokes system are given in the form of slip conditions. That is, the boundary values of u_{iHm} and τ_{Hm} are determined by the data up to the previous stage of approximation. They are listed in Appendix 1.

* We can adopt the nondimensional perturbed quantities u_i, p, τ, etc. as f in Eq. (1).

(iv) The Knudsen-layer parts, also listed in Appendix 1, are derived from the slip conditions by replacing the slip coefficients by universal functions of the Knudsen-layer variable (a stretched coordinate normal to the boundary).

Now, our problem is reduced to a boundary-value problem of the Stokes system, which is much simpler than that of the kinetic equation.

III. Flow Induced in the Gas

In this section we investigate the flow field induced in the gas between the cylinders by the asymptotic theory summarized in the previous section. Since the temperature field (τ_{Hm}, τ_{Km}) is not of our main concern, its analysis is limited to the extent that is necessary to obtain the flow field. In our analysis we adopt the following reference quantities, notations, and coordinate system (Fig. 1).

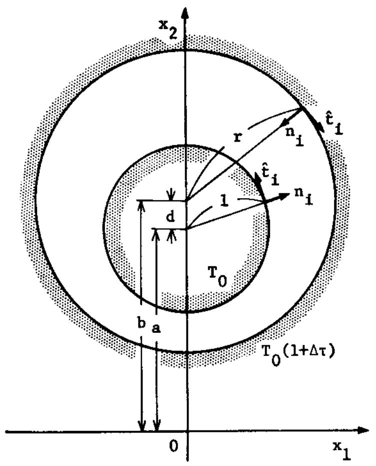

Fig. 1. Coordinate system.

(i) The radius of the inner cylinder is taken as the reference length L, and that of the outer is denoted by Lr; the distance between the axes of the two cylinders is denoted by Ld.

(ii) The temperature of the inner cylinder is taken as the reference temperature T_0 and that of the outer is denoted by $T_0(1+\Delta\tau)$.

(iii) The x_3-axis is taken parallel to the axes of the two cylinders; the axis of the inner cylinder is assumed to be located at $x_1=0$, $x_2=a$, and that of the outer at $x_1=0$, $x_2=b$, where

$$a=\frac{1}{2d}(r^2-1-d^2)\,, \quad b=a+d\,. \tag{9}$$

Now, we will analyze the Stokes system from the lowest order ($m=0$).

From Eq. (4),

$$p_{H0} = c_0 \,, \tag{10}$$

where c_0 is a constant. The determination of c_0 and the other constants in p_{Hm}'s, which will appear, is postponed until the end of the analysis.

From the (non-)slip condition (A1a),

$$u_{iH0} = 0 \,, \tag{11}$$

on each cylinder; from Eq. (A1b),

$$\tau_{H0} = 0 \,, \tag{12a}$$

on the inner cylinder and

$$\tau_{H0} = \Delta\tau \,, \tag{12b}$$

on the outer cylinder. From Eqs. (5), (6) with $m=0$ and Eq. (11), we have

$$u_{iH0} = 0 \,, \quad p_{H1} = c_1 \,, \tag{13}$$

where c_1 is a constant. The harmonic function τ_{H0} [Eq. (7) with $m=0$] that takes the boundary values (12a), (12b) is given by the logarithmic potential:

$$\tau_{H0} = \frac{\Delta\tau}{M}\left[\ln\frac{a+s}{a-s} - \ln\frac{x_1^2+(x_2+s)^2}{x_1^2+(x_2-s)^2}\right], \tag{14}$$

where

$$s = [(a+1)(a-1)]^{1/2} \,, \tag{15a}$$

$$M = \ln\frac{(a+s)(b-s)}{(a-s)(b+s)} \,. \tag{15b}$$

Both cylinders are so arranged that they are on coordinate lines of a bipolar coordinate system; namely,

$$x_1^2+(x_2\pm s)^2 = 2x_2(a\pm s) \,, \tag{16a}$$

on the inner cylinder and

$$x_1^2+(x_2\pm s)^2 = 2x_2(b\pm s) \,, \tag{16b}$$

on the outer cylinder.

Since $u_{iH0}\equiv 0$ and τ_{H0} is constant on each cylinder, Eqs. (A2a, b) show that

$$u_{iH1} = 0 \,, \tag{17}$$

on each cylinder (no thermal creep). From Eqs. (5), (6) with $m=1$ and Eq. (17), we have

$$u_{iH1} = 0\,, \quad p_{H2} = c_2\,, \tag{18}$$

where c_2 is a constant.

Noting that $u_{iH0}\equiv 0$, $u_{iH1}\equiv 0$ and that τ_{H0} is constant on each cylinder, we find that all the terms except one vanish in the second-order slip condition for the velocity u_{iH2} [Eqs. (A3a, b)].* Thus, on each cylinder

$$u_{iH2}\hat{t}_i = -e\frac{\partial^2 \tau_{H0}}{\partial x_i \partial x_j} n_i \hat{t}_j = \frac{2e\Delta\tau s}{M\delta^2}\frac{x_1}{x_2^2}\,, \tag{19a}$$

$$u_{iH2} n_i = 0\,, \tag{19b}$$

$$u_{3H2} = 0\,, \tag{19c}$$

$$e = 0.27922\,,$$

where $\delta=1$ on the inner cylinder and $\delta=r$ on the outer, n_i is the direction cosine of the normal vector to the boundary, pointed to the gas, and $\hat{t}_i$ is that of the tangential vector to the boundary such that (n_i, $\hat{t}_i$, the x_3-direction) forms an orthogonal right-hand system (Fig. 1).** The tangential velocity distributions on the two cylinders are plotted for $r=2$, $d=0.25$, 0.5, 0.75 in Fig. 2, where θ is the angular

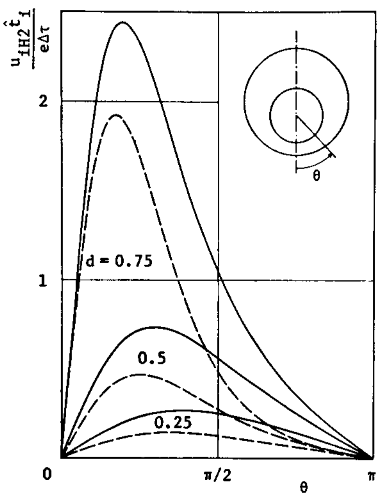

Fig. 2. Tangential velocity distributions on the two cylinders.
($r=2$; ——— on the inner cylinder, – – – – on the outer cylinder.)

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* The vanishing of the second term of $u_{iH2}n_i$ in Eq. (A3b) is not obvious. It is shown in Appendix 2.

** $n_i=[x_1, (x_2-a), 0]$ and $\hat{t}_i=[-(x_2-a), x_1, 0]$ on the inner cylinder; $n_i=[-x_1/r, -(x_2-b)/r, 0]$ and $\hat{t}_i=[(x_2-b)/r, -x_1/r, 0]$ on the outer.

coordinate in the plane polar coordinate system (in the x_1, x_2-plane) with its origin at the center of the inner cylinder and with its polar direction in the direction of the $-x_2$ axis.

Since the problem can be considered to be two-dimensional ($\partial/\partial x_3=0$, $u_{3H}=0$), we can introduce the stream function ψ of x_1 and x_2, from which the solenoidal velocity field u_{iH2} [Eq. (5)] is derived:

$$u_{1H2}=\frac{\partial\psi}{\partial x_2}\,,\quad u_{2H2}=-\frac{\partial\psi}{\partial x_1}\,. \tag{20}$$

Eliminating the pressure p_{H3} from Eq. (6) with $m=2$ by taking its curl and replacing u_{iH2} by ψ with the aid of Eq. (20), we have

$$\frac{\partial^2}{\partial x_i^2}\frac{\partial^2}{\partial x_j^2}\psi=0\,. \tag{21}$$

The boundary conditions (19a, b) are rewritten in terms of ψ as

$$\frac{\partial\psi}{\partial x_i}n_i=-\frac{2e\Delta\tau s}{M\delta^2}\frac{x_1}{x_2^2}\,, \tag{22a}$$

$$\psi=0\,. \tag{22b}$$

Equation (19b) does not imply that ψ takes a common constant on the two cylinders. From the symmetry of the problem [Eqs. (5), (6), (19a, b)] with respect to x_1 and the uniqueness[9] of the solution u_{iH2}, we can show that Eq. (22b) holds on the two cylinders.

Any two-dimensional biharmonic function $\psi(x_1, x_2)$ can generally be written in the form.[10]

$$\psi=x_2\Phi_1(x_1,x_2)+\Phi_2(x_1,x_2)\,, \tag{23}$$

where Φ_1 and Φ_2 are two-dimensional harmonic functions. (This can be rewritten in the form $\psi=x_1\tilde{\Phi}_1+\tilde{\Phi}_2$ with other harmonic functions $\tilde{\Phi}_1$, $\tilde{\Phi}_2$.) On the other hand, a series of harmonic functions that take the form $x_1^n x_2^m$ on the two cylinders is derived from the logarithmic potential $\ln[x_1^2+(x_2\pm s)^2]$ by differentiations. The series seems to be appropriate to be matched with the boundary condition (22a). In fact, we can construct the solution of Eq. (21) subject to Eqs. (22a), (22b) with these harmonic functions as

$$\frac{\psi}{e\Delta\tau}=Ax_1\ln\frac{x_1^2+(x_2+s)^2}{x_1^2+(x_2-s)^2}+Bx_1+C_{(+)}\frac{x_1}{x_1^2+(x_2+s)^2}+C_{(-)}\frac{x_1}{x_1^2+(x_2-s)^2}$$

$$+D_{(+)}\frac{x_1x_2}{x_1^2+(x_2+s)^2}+D_{(-)}\frac{x_1x_2}{x_1^2+(x_2-s)^2}$$

(Continued on the next page.)

$$+E_{(+)}\frac{x_1x_2(x_2+s)}{[x_1^2+(x_2+s)^2]^2}+E_{(-)}\frac{x_1x_2(x_2-s)}{[x_1^2+(x_2-s)^2]^2}, \tag{24}$$

where

$$A=\frac{d(b-ra)}{srM[(ab-s^2)M-2sd]}, \tag{25a}$$

$$B=\frac{A}{d}\left(a\ln\frac{a+s}{a-s}+b\ln\frac{b-s}{b+s}\right)+\frac{r-1}{sd^2M}\left(\frac{ab-s^2}{r}-1\right), \tag{25b}$$

$$C_{(\pm)}=\pm\frac{s^2(r-1)(a+b\pm2s)+(b-ra)(ab-s^2)}{s^2d^2M}, \tag{25c}$$

$$D_{(\pm)}=\pm\frac{(a+b\pm2s)[\pm s(r-1)-(b-ra)]}{s^2d^2M}$$

$$\mp\frac{r(r-1)(ab-s^2)}{(a\mp s)(b\mp s)s^2d^2M}-\frac{r(b-ra)}{s(a\mp s)(b\mp s)[(ab-s^2)M-2sd]}, \tag{25d}$$

$$E_{(\pm)}=\mp\frac{2(a\pm s)(b\pm s)[\pm s(r-1)-(b-ra)]}{s^2d^2M}. \tag{25e}$$

With this velocity field, p_{H3} is determined by Eq. (6) with $m=2$ as

$$\frac{p_{H3}}{2e\Delta\tau}=4A\frac{s(x_1^2-x_2^2+s^2)}{[x_1^2+(x_2+s)^2][x_1^2+(x_2-s)^2]}$$

$$+D_{(+)}\frac{x_1^2-(x_2+s)^2}{[x_1^2+(x_2+s)^2]^2}+D_{(-)}\frac{x_1^2-(x_2-s)^2}{[x_1^2+(x_2-s)^2]^2}$$

$$+E_{(+)}\frac{(x_2+s)[3x_1^2-(x_2+s)^2]}{[x_1^2+(x_2+s)^2]^3}+E_{(-)}\frac{(x_2-s)[3x_1^2-(x_2-s)^2]}{[x_1^2+(x_2-s)^2]^3}+c_3, \tag{26}$$

where c_3 is a constant. The determination of the constants c_i's depends on the choice of the reference pressure p_0. If we choose the pressure of the gas at some point outside the Knudsen layer as p_0, then

$$c_0=c_1=c_2=0,$$

and c_3 is determined by the condition that $p_{H3}=0$ at the point.

From Eqs. (A2a), (A2b), (A3a), and (A3b), we find that the Knudsen-layer part of the velocity (u_{iK1}, u_{iK2}) vanishes. Thus ψ represents the total flow field.

If we take the limit of the results obtained in this section as r and $d\rightarrow\infty$ with $r-d$ being kept at a fixed value h, we obtain the corresponding results for the case where a circular cylinder (radius L, temperature T_0) lies in a semi-infinite expanse of the gas bounded by an infinite plane wall [temperature $T_0(1+\Delta\tau)$] at $x_2=0$. The Lh represents the distance of the center of the cylinder from the wall. The limiting expressions of Eqs. (14), (24), (26) are obtained by making the following replacement.

$$a \rightarrow h\,, \tag{27a}$$

$$s \rightarrow (h^2-1)^{1/2}\,, \tag{27b}$$

$$M \rightarrow 2|\cosh^{-1}h|\,, \tag{27c}$$

$$A \rightarrow -\frac{h-1}{4(h^2-1)^{1/2}|\cosh^{-1}h|\,[h|\cosh^{-1}h|-(h^2-1)^{1/2}]}\,, \tag{27d}$$

$$B \rightarrow 0\,, \tag{27e}$$

$$C_{(\pm)} \rightarrow \pm\frac{1}{2(h+1)\,|\cosh^{-1}h|}\,, \tag{27f}$$

$$D_{(\pm)} \rightarrow \mp\frac{1+(h-1)[h\pm(h^2-1)^{1/2}]}{2(h^2-1)\,|\cosh^{-1}h|} + \frac{(h-1)[h\pm(h^2-1)^{1/2}]}{2(h^2-1)^{1/2}[h|\cosh^{-1}h|-(h^2-1)^{1/2}]}\,, \tag{27g}$$

$$E_{(\pm)} \rightarrow \mp\frac{[h\pm(h^2-1)^{1/2}][h-1\pm(h^2-1)^{1/2}]}{(h^2-1)\,|\cosh^{-1}h|}\,. \tag{27h}$$

To derive the tangential velocity of the gas on the plane wall from Eq. (19a) or (22a) with $\delta=r$, the replacement

$$\frac{1}{\delta^2 x_2^2} \rightarrow \frac{4}{(x_1^2+h^2-1)^2}\,,$$

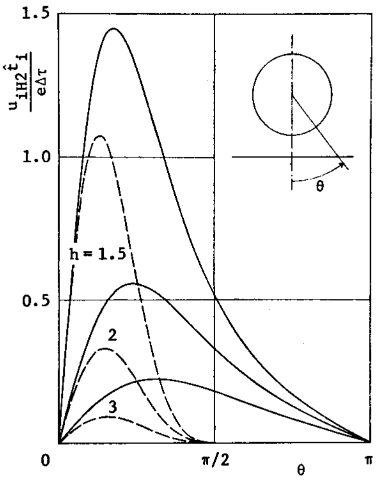

Fig. 3. Tangential velocity distributions on the cylinder and on the plane wall.
(——— on the cylinder, – – – – on the plane wall.)

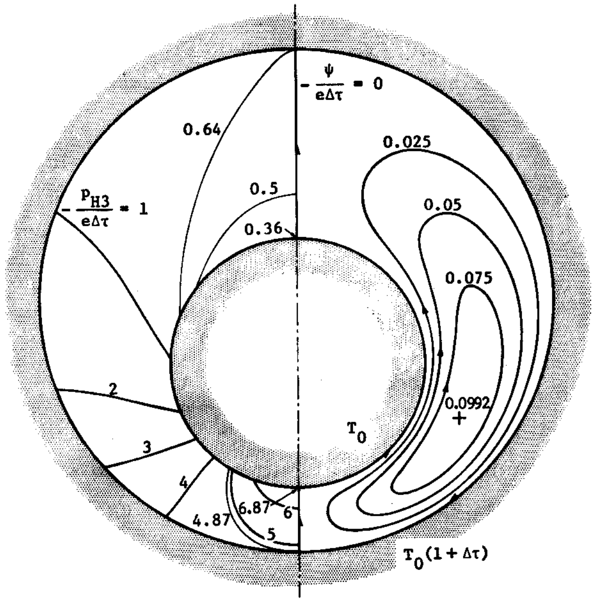

Fig. 4. Streamlines and isobars.
(r=2, d=0.5; the arrows show the direction of flow for $\Delta\tau>0$.)

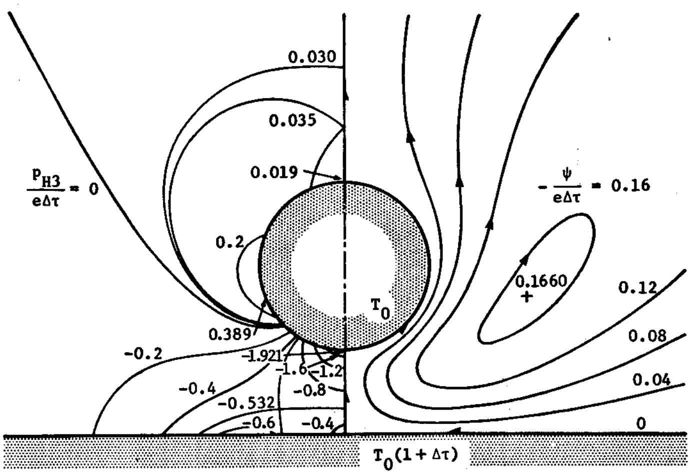

Fig. 5. Streamlines and isobars.
($h=r-d=2$, $r\to\infty$, $d\to\infty$; the arrows show the direction of flow for $\Delta\tau>0$.)

in addition to Eqs. (27b, c) is necessary since $r\to\infty$ and $x_2\to 0$ on the plane wall.

The tangential velocity distributions on the cylinder and on the plane wall are plotted for $h=1.5$, 2, 3 in Fig. 3. Streamlines (ψ=constant) and isobars (more exactly, p_{H3}=constant) for $r=2$, $d=0.5$ are shown in Fig. 4 and those for $h=r-d=2$, $r\to\infty$, $d\to\infty$ in Fig. 5. The right half of each of Figs. 4 and 5 shows ψ=constant and the left half p_{H3}=constant. The arrows show the direction of flow for $\Delta\tau>0$. For $\Delta\tau<0$, the flow is reversed. The difference between isobar and p_{H3}=constant is important only inside the Knudsen layer. In both figures, c_3 is taken to be zero; in Fig. 5 this corresponds to the pressure at infinity being taken as p_0.

IV. Force on the Inner Cylinder

In this section we will calculate the total force acting on the inner cylinder by applying the momentum theorem to a control surface that encloses the inner cylinder. The stress tensor defined in Ref. 6 Sec. 17 being denoted by $p_0(\delta_{ij}+P_{ij})$, where δ_{ij} is the Kronecker's delta, the total force F_i, per unit length of the cylinder, can be written by a contour integral of P_{ij} in the x_1, x_2-plane as

$$F_i = -p_0 L\int_c P_{ij}\tilde{n}_j ds\,, \tag{28a}$$

where the path of integration c is a closed curve that encloses the inner cylinder, ds is the line element, and $\tilde{n}_j$ is the outward unit normal to the contour c. In the linearized problem the momentum flux due to gas motion is negligible. From the symmetry of the problem, $F_1=F_3=0$. If we take the path c outside the Knudsen layer, P_{ij} in Eq. (28a) may be replaced by its Hilbert part P_{ijH}. Thus,

$$\frac{F_i}{p_0 L} = -\int_c P_{ijH}\tilde{n}_j ds\,, \tag{28b}$$

where P_{ijH} is given by the formula:

$$P_{ijH} = P_{ijH0}+P_{ijH1}k+P_{ijH2}k^2+\cdots\,, \tag{29}$$

$$P_{ijH0} = p_{H0}\delta_{ij}\,,$$

$$P_{ijH1} = p_{H1}\delta_{ij}-\left(\frac{\partial u_{iH0}}{\partial x_j}+\frac{\partial u_{jH0}}{\partial x_i}\right),$$

$$P_{ijH2} = p_{H2}\delta_{ij}-\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right)+\frac{\partial^2\tau_{H0}}{\partial x_i\partial x_j},$$

$$P_{ijH3} = p_{H3}\delta_{ij}-\left(\frac{\partial u_{iH2}}{\partial x_j}+\frac{\partial u_{jH2}}{\partial x_i}\right)+\frac{\partial^2\tau_{H1}}{\partial x_i\partial x_j}-\frac{\partial^2}{\partial x_l^2}\left(\frac{\partial u_{iH0}}{\partial x_j}+\frac{\partial u_{jH0}}{\partial x_i}\right).$$

The stress tensor P_{ij} that is calculated from a solution of the linearized Boltzmann equation (including the linearized Boltzmann-Krook-Welander equation) satisfies the equation of conservation of momentum (Ref. 6, Sec. 17):

$$\frac{\partial P_{ij}}{\partial x_j} = 0\,. \tag{30a}$$

Thus,

$$\frac{\partial P_{ijH}}{\partial x_j} = 0\,, \tag{30b}$$

since the Hilbert part is a class of solutions of the Boltzmann equation.[3,8] Equation (30b) corresponds to Eq. (6). From the Gauss's theorem and Eq. (30b) it turns out that the integral (28b) is independent of the choice of the contour c. (c may be in the Knudsen layer.) This shows that the Knudsen-layer part of P_{ij} does not contribute to the total force. [This statement, as is obvious from the above derivation, holds for any closed body (in two or three dimensional problem).]

As will be shown in Appendix 3, the thermal stress $\partial^2\tau_{Hm}/\partial x_i\partial x_j$ in Eq. (29), which is derived from the harmonic function τ_{Hm}, does not contribute to the total force. Now, we can calculate the force F_i from Eq. (28b) with the aid of Eqs. (29), (24), (20), and (26). The result is:

$$\begin{aligned} &F_2 = 8\pi eAp_0L\varDelta\tau k^3\,, \\ &F_1 = F_3 = 0\,. \end{aligned} \tag{31}$$

The force F_i for the limiting case where r and $d\to\infty$ with $r-d=h$ fixed (a cylinder and a plane wall) is obtained by making the replacement given by Eq. (27d). Thus,

$$F_2 = -2\pi e\frac{h-1}{(h^2-1)^{1/2}|\cosh^{-1}h|\,[h|\cosh^{-1}h|-(h^2-1)^{1/2}]}p_0L\varDelta\tau k^3\,. \tag{32}$$

$-F_2/(ep_0L\varDelta\tau k^3)$ of Eq. (31) is plotted as a function of $d/(r-1)$ (the eccentricity or the distance between the axes of the cylinders divided by the difference of their radii) for various values of r (the ratio of the radii of the cylinders) in Fig. 6. When $\varDelta\tau<0$, the inner cylinder is subject to a force toward the axis of the outer. $-F_2/(ep_0L\varDelta\tau k^3)$ of Eq. (32) is plotted as a function of $h-1$ in Fig. 7. The force [Eq. (31) or (32)] becomes infinitely large as the two cylinders or the cylinder and the plane wall approach each other ($d\to r-1$ or $h\to 1$); the formula, however, is no longer reliable then since the asymptotic theory is not applicable

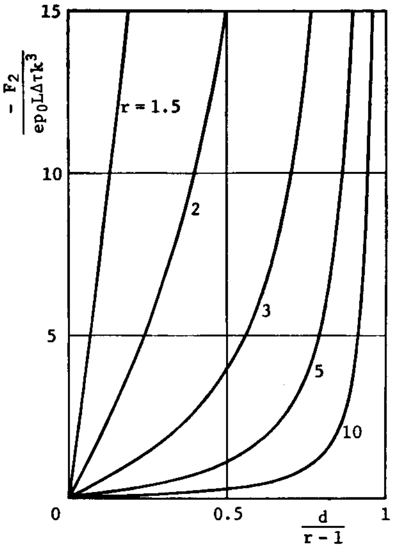

Fig. 6. Force on the inner cylinder.

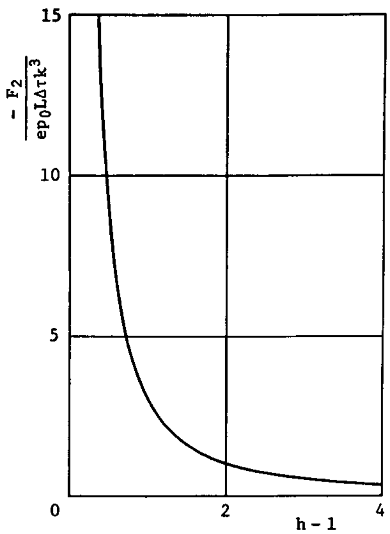

Fig. 7. Force on the cylinder. ($r\to\infty$, $d\to\infty$, $r-d=h$)

where the mean free path of the gas is not small compared with the gap between the two boundaries.

We have not chosen the reference pressure p_0 yet. In the present analysis, the pressure of the gas is denoted by $p_0(1+p)$ and p is assumed to be so small that its square may be negligible. Therefore, the choice of p_0 is limited to a narrow range around the pressure at some point in the gas. Then the choice of p_0 affects p_H and ω_H by additive constants, l, and also k through l, but the errors that are introduced through p_0 and k in τ_H, u_{iH}, and F_i can be neglected since they are of the higher order of smallness that is neglected in the linearized problem.

V. Discussion

Equation (29) shows that thermal stress, in addition to viscous stress, is acting in a slightly rarefied gas where the temperature gradient is nonuniform. The thermal stress, however, does not contribute at all to the equation of conservation of momentum [Eq. (30b)→Eq. (6)]. Thus, the thermal stresses integrated over a control surface in the gas vanish; that is, the thermal stresses on the control surface are balanced with themselves without the help of the pressures and the viscous stresses. In the absence of boundary, therefore, the gas remains at rest under a uniform pressure as if the thermal stresses were absent. This balance is violated in the neighborhood of a solid boundary.

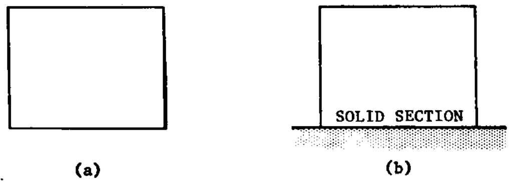

Fig. 8. Control surface and solid section.

Suppose that a part of the control surface is replaced by a solid boundary without changing the surrounding temperature field [Fig. 8(a)→Fig. 8(b)]. (Let it be plane and be called solid section.) The thermal stress and the pressure on the control surface except the solid section do not suffer any abrupt change by this replacement. Neither does the contribution to those on the solid section by the molecules reaching the section. But the contribution to the tangential stress on the solid section by the molecules leaving it suffers an abrupt and considerable change, because their velocity distribution is quite different from that in the gas. For example, when the molecules make diffuse reflection on the solid section, the leaving molecules do not contribute at all to the tangential stress. The tangential stress on the solid section, therefore, is reduced to half by the replacement. That violates the balance of the thermal stresses acting on the control surface. Thus, a flow is induced along the solid section, and it grows until the tangential momentum transferred by the reaching molecules, which now have tangential velocity as a whole, compensates for the reduction of the thermal stress. The final steady flow is determined by the slip $en_j t_i \partial G_{i0}/\partial x_j$ in Eq. (A3a) and is called thermal stress slip flow. Other examples of thermal stress slip flow are given in Refs. 11 and 12.

Kogan et al.[13] pointed out the existence of another kind of flow which is also induced by thermal stress. When the temperature variation in a gas is so large that the linearized approximation is not applicable, the thermal stress, which is not so simple as that given in Eq. (29), has a contribution in the equation of conservation of momentum. Then the thermal stresses are no longer balanced with themselves. Further, except some special cases the thermal stress cannot be incorporated into the pressure term in the momentum equation; that is, the thermal stress cannot be made balanced with the pressure. Therefore a flow is induced to maintain the conservation of momentum. This kind of flow also occurs in our problem when the temperature difference between two cylinders is large. In this case the solid boundary plays only an indirect role in inducing the flow that it establishes the temperature field, which produces thermal stress, in contrast with the case of the thermal stress slip flow.

The difference between the thermal creep flow and the thermal stress slip flow is made clear when we consider a particle in a slightly rarefied gas with a temperature gradient.[12,14] When the thermal conductivity of the particle is of the same order of that of the gas, the side of the particle that faces toward the hotter region is heated more and a temperature gradient comparable to that of the gas arises on the surface of the particle. Thus, the thermal creep flow, determined by the second term of the velocity slip in Eq. (A2a), is induced from the colder to the hotter region and the particle is subject to a force in the direction opposite to the temperature gradient. When the thermal conductivity of the particle is much higher than that of gas, the temperature of the particle is nearly uniform and, correspondingly, the thermal creep flow is negligibly small. Then the thermal stress slip flow, a higher-order effect, determines the flow field. It is from the hotter to the colder region and the particle is subject to a force in the direction of the temperature gradient. Thus, a particle left free in the gas drifts in the direction of the temperature gradient or in the opposite direction depending on its thermal conductivity. Therefore we can separate particles in a slightly rarefied gas *by their thermal conductivity* by imposing a temperature gradient in the gas. These phenomena are called thermophoresis and the force on the particle thermal force.

According to Bakanov and Derjaguin[15] and Waldmann[16], the thermal force is independent of the thermal conductivity of the particle and in the direction opposite to the temperature gradient when the Knudsen number is infinitely large (highly rarefied gas). Thus, for a particle of high thermal conductivity, the direction of the thermal force is reversed at some Knudsen number. Therefore we can separate particles of high thermal conductivity in a rarefied gas *by their size* by imposing a temperature gradient in the gas.

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The paper was read as a sectional lecture at the 15th International Congress Theoretical and Applied Mechanics held in Toronto in 1980. Several technically incorrect expressions (fortunately not in mathematical expressions) are found in the proceedings of our paper of the Congress because several changes were made on our final photoready manuscript *without our consent* in the editorial process.

Appendix

1. Slip boundary condition and Knudsen-layer part

$$u_{iH0} = u_{wi} \,, \tag{A1a}$$

$$\tau_{H0} = \tau_w \,, \tag{A1b}$$

$$\begin{bmatrix} u_{iH1}t_i \\ u_{iK1}t_i \end{bmatrix} = S_{ij0}n_i t_j \begin{bmatrix} k_0 \\ Y_0 \end{bmatrix} + G_{i0}t_i \begin{bmatrix} K_1 \\ \frac{1}{2}Y_1 \end{bmatrix}, \tag{A2a}$$

$$\begin{bmatrix} u_{iH1}n_i \\ u_{iK1}n_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{A2b}$$

$$\begin{bmatrix} \tau_{H1} \\ \tau_{K1} \\ \omega_{K1} \end{bmatrix} = -G_{i0}n_i \begin{bmatrix} d_1 \\ \Theta_1 \\ \Omega_1 \end{bmatrix}, \tag{A2c}$$

$$\begin{aligned} \begin{bmatrix} u_{iH2}t_i \\ u_{iK2}t_i \end{bmatrix} = {} & S_{ij1}n_i t_j \begin{bmatrix} k_0 \\ Y_0 \end{bmatrix} + \frac{\partial S_{ij0}}{\partial x_r} n_j n_r t_i \begin{bmatrix} a \\ -Y_1 \end{bmatrix} + \bar{\kappa} S_{ij0} n_i t_j \begin{bmatrix} b \\ -2\tilde{Y}_0 \end{bmatrix} \\ & + \kappa_{ij} S_{jr0} n_r t_i \begin{bmatrix} c \\ -Y_1 - k_0 Y_0 \end{bmatrix} + \frac{\partial G_{i0}}{\partial x_j} n_j t_i \begin{bmatrix} e \\ 0 \end{bmatrix} \\ & + \bar{\kappa} G_{i0} t_i \begin{bmatrix} f \\ Y_2 - \tilde{Y}_1 \end{bmatrix} + \kappa_{ij} G_{j0} t_i \begin{bmatrix} g \\ \frac{1}{2} Y_2 - \left(K_1 + \frac{1}{4}\right) Y_0 \end{bmatrix}, \end{aligned} \tag{A3a}$$

$$\begin{aligned} \begin{bmatrix} u_{iH2}n_i \\ u_{iK2}n_i \end{bmatrix} = {} & \frac{1}{2} \frac{\partial S_{ij0}}{\partial x_r} n_i n_j n_r \begin{bmatrix} \alpha \\ \int_\infty^\eta Y_0(y)\,dy \end{bmatrix} \\ & + \left(\frac{\partial G_{i0}}{\partial x_j} n_i n_j + 2\bar{\kappa} G_{i0} n_i \right) \begin{bmatrix} \beta \\ \frac{1}{2} \int_\infty^\eta Y_1(y)\,dy \end{bmatrix}, \end{aligned} \tag{A3b}$$

$$\begin{bmatrix} \tau_{H2} \\ \tau_{K2} \\ \omega_{K2} \end{bmatrix} = -G_{i1}n_i \begin{bmatrix} d_1 \\ \Theta_1 \\ \Omega_1 \end{bmatrix} - \frac{\partial S_{ij0}}{\partial x_r} n_i n_j n_r \begin{bmatrix} d_4 \\ \Theta_4 \\ \Omega_4 \end{bmatrix} - \bar{\kappa} G_{i0} n_i \begin{bmatrix} d_5 \\ \Theta_5 \\ \Omega_5 \end{bmatrix}. \tag{A3c}$$

The notations are:

$$S_{ijm} = -\left(\frac{\partial u_{iHm}}{\partial x_j} + \frac{\partial u_{jHm}}{\partial x_i} \right),$$

$$G_{im} = -\frac{\partial \tau_{Hm}}{\partial x_i}.$$

$$\left.\begin{array}{llllll} k_0 &= -1.01619, & K_1 &= -0.38316, & a &= 0.76632, \\ b &= 0.50000, & c &= -0.26632, & e &= 0.27922, \\ f &= 0.26693, & g &= -0.76644, & \alpha &= 0.23368, \\ \beta &= 0.26693, & d_1 &= 1.30272, & d_4 &= 0.11169, \\ d_5 &= 1.82181. & & & & \end{array}\right\} \quad \text{(A4)*}$$

$(2RT_0)^{1/2}u_{wi}$ is the velocity of the boundary, $T_0(1+\tau_w)$ is the temperature of the boundary, n_i is the direction cosine of the normal vector (pointed to the gas) to the boundary, and t_i is the direction cosine of a tangential vector to the boundary. The $\bar{\kappa}$ and κ_{ij} are defined as

$$\bar{\kappa} = \frac{1}{2}(\kappa_1+\kappa_2), \quad \kappa_{ij} = \kappa_1 l_i l_j + \kappa_2 m_i m_j,$$

where $L^{-1}\kappa_1$ and $L^{-1}\kappa_2$ are the principal curvatures of the boundary and l_i and m_i are the direction cosines of the corresponding principal directions; we take $\kappa_i < 0$ when the corresponding center of the curvature lies on the side of the gas. Y_0, Y_1, Y_2, $\tilde{Y}_0$, $\tilde{Y}_1$, Θ_1, Θ_4, Θ_5, Ω_1, Ω_4, and Ω_5 are functions of the Knudsen-layer variable η:

$$x_i = n_i k\eta + x_{wi}(\zeta_1, \zeta_2),$$

where $x_{wi}(\zeta_1, \zeta_2)$ is the boundary surface, and ζ_1 and ζ_2 are coordinates within a surface η=const. These functions are plotted in Figs. 9 and 10 by the data obtained in Ref. 8.** All the quantities with the subscript H including S_{ijm}, G_{im} are evaluated on the boundary.

2. The vanishing of the second term of $u_{iH2}n_i$ in Eq. (A3b)

The second term of $u_{iH2}n_i$ in Eq. (A3b) generally vanishes on the boundary of a uniform temperature.

Proof: We introduce s_i defined by the equations $s_is_i=1$, $s_it_i=0$, $s_in_i=0$; then $n_in_j+t_it_j+s_is_j=\delta_{ij}$. Therefore

* The slip coefficients except e do not appear in the main text. Thus, the double role of the symbols a, b, f will not introduce any misunderstanding. The notations of the slip coefficients (a, b, etc.) are the same as those in Refs. 3, 4 except for K_1, i.e., K_1(here)$=(2k_1+1)/4$ (in Ref. 3).

** The notations of the Knudsen-layer functions (Y_0, Y_1, etc.) are the same as those in Refs. 3, 4. Thus (Ω_4, Θ_4, d_4) (here)$=\left(-\frac{1}{4}\Omega_4^*, -\frac{1}{4}\Theta_4^*, -\frac{1}{4}d_4^*\right)$ (in Ref. 8). (See the list of notations in Appendix B of Ref. 8.)

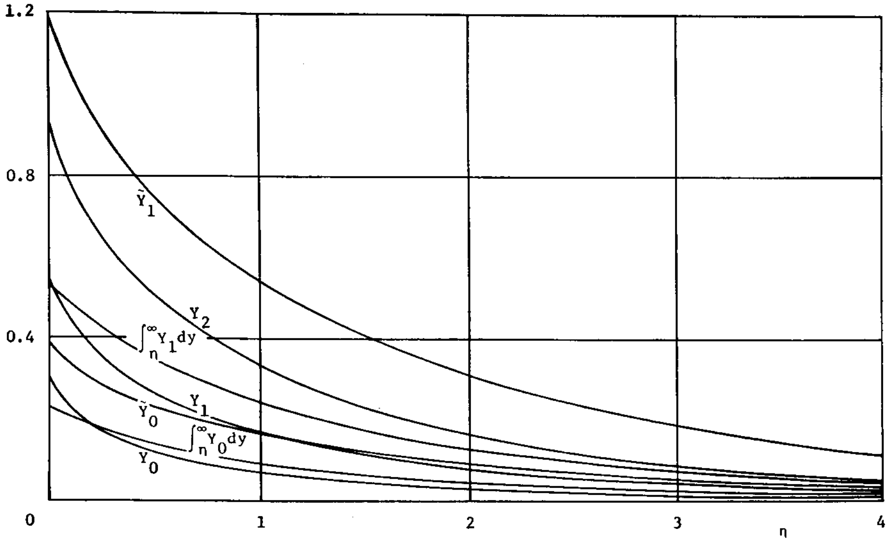

Fig. 9. Knudsen-layer functions I.

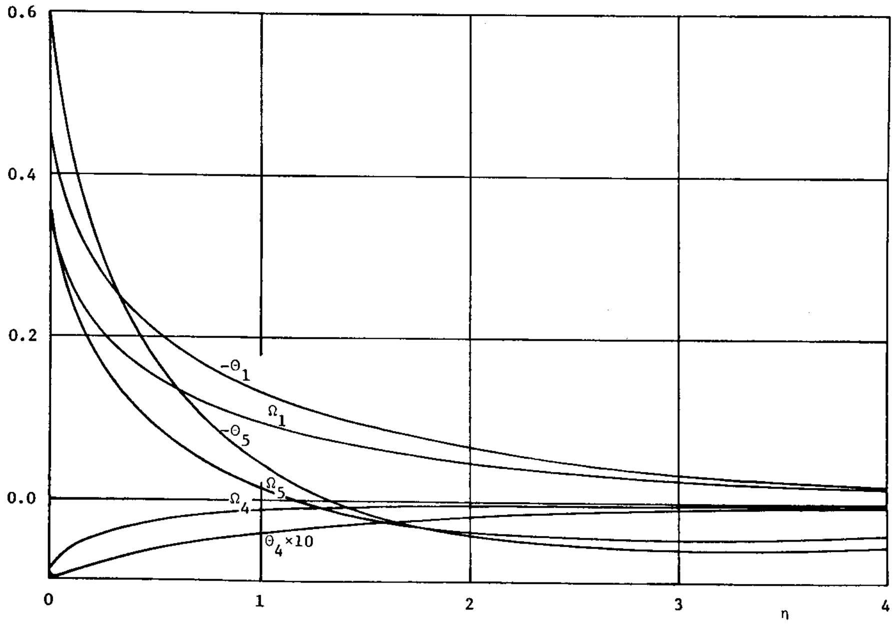

Fig. 10. Knudsen-layer functions II.

$$\frac{\partial G_{i0}}{\partial x_j}n_i n_j = \frac{\partial G_{i0}}{\partial x_i} - \frac{\partial G_{i0}}{\partial x_j}t_i t_j - \frac{\partial G_{i0}}{\partial x_j}s_i s_j$$

$$= -t_j\frac{\partial G_{i0}t_i}{\partial x_j} - s_j\frac{\partial G_{i0}s_i}{\partial x_j} + G_{i0}\left(t_j\frac{\partial t_i}{\partial x_j} + s_j\frac{\partial s_i}{\partial x_j}\right)$$

[Eq. (7); a proper extension of t_i, s_i in the neighborhood of the point under consideration[17]]

$$= G_{i0}\left(t_j\frac{\partial t_i}{\partial x_j} + s_j\frac{\partial s_i}{\partial x_j}\right)$$

[Eq. (A1b) and τ_w=const.]

$$= -2G_{i0}n_i\bar{\kappa}\,,$$

(The definition and property of the mean curvature[17]).

3. The vanishing of the thermal stress term in Eq. (28b)

The thermal stress term in Eq. (28b) for $i=2$

$$= -\int_c\left[\frac{\partial}{\partial x_1}\left(\frac{\partial\tau_{Hm}}{\partial x_2}\right)\tilde{n}_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial\tau_{Hm}}{\partial x_2}\right)\tilde{n}_2\right]ds$$

$$= -\int_c\left[\frac{\partial}{\partial x_2}\left(\frac{\partial\tau_{Hm}}{\partial x_1}\right)\tilde{t}_2 + \frac{\partial}{\partial x_1}\left(\frac{\partial\tau_{Hm}}{\partial x_1}\right)\tilde{t}_1\right]ds$$

[τ_{Hm} is harmonic $(\partial^2\tau_{Hm}/\partial x_2^2 = -\partial^2\tau_{Hm}/\partial x_1^2)$; $(\tilde{t}_1, \tilde{t}_2)$ is the unit tangential vector to the path c and points to the direction encircling c counterclockwise.]

$$= 0\,,$$

($\because$ τ_{Hm} is a *single valued* harmonic function).

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