A Method for Analysing Parametrically Excited System by Matrix Function

By

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Abstract

This paper describes a method for analysing parametrically excited system of higher order. The method is based on the theory of matrix function and the discrete Fourier transform. As a numerical example, we deal with a kind of Hill's equation derived from the synchronous generator circuit with unbalanced capacitive load and give its stability charts.

1. Introduction

When parameters in a linear circuit or system are varied periodically with time, the corresponding equations can be written by a set of ordinary differential equations with periodic coefficients. In this paper, we call such a set of equations a periodic system. Conventionally, the properties of a periodic system of the second order have been investigated in detail, aimed at studying its stability. However, there appears to have been little attention paid to the periodic system of a higher order and the wave forms of its solutions. In practical problems, we encounter many periodic systems of a higher order, such as various systems of synchronous machines, resonant transfer circuits and so forth. For these systems, we need to investigate the wave forms as well as the stability. Here, we propose a method for numerically analysing periodic systems of a higher order, using the theory of a matrix function, and an algorithm is given.

2. Periodic System[2)]

We deal with the periodic system described by

$$\left.\begin{aligned} \frac{d\boldsymbol{x}}{d\tau} &= \boldsymbol{P}(\tau)\boldsymbol{x} \\ \boldsymbol{P}(\tau+2\pi) &= \boldsymbol{P}(\tau) \end{aligned}\right\} \qquad (1)$$

where $\boldsymbol{P}(\tau)$ is an $n\times n$ real matrix periodic with respect to τ, and $\boldsymbol{x}$ is n real vector.

Let the matrix $\boldsymbol{\Phi}(\tau)$ be the normalized fundamental matrix of Eq. (1). Then $\boldsymbol{\Phi}(\tau)$ is a solution of the matrix differential equation

$$\frac{d\boldsymbol{X}}{d\tau} = \boldsymbol{P}(\tau)\boldsymbol{X} \qquad \boldsymbol{X}(0) = \mathbf{1} \tag{2}$$

where $\mathbf{1}$ is an $n \times n$ unit matrix. This equation is called the associated equation of Eq. (1). Following the Floquet theorem, we obtain

$$\boldsymbol{\Phi}(\tau) = \boldsymbol{L}(\tau) \exp(\tau \boldsymbol{W}), \quad \boldsymbol{L}(0) = \mathbf{1} \tag{3}$$

where $\boldsymbol{L}(\tau)$ is called the Liapunov matrix with period 2π, and $\boldsymbol{W}$ is an $n \times n$ constant matrix. Because $\boldsymbol{\Phi}(\tau+2\pi)$ also satisfies Eq. (2), there is a nonsingular matrix $\boldsymbol{V}$ such that

$$\boldsymbol{\Phi}(\tau+2\pi) = \boldsymbol{\Phi}(\tau)\boldsymbol{V}. \tag{4}$$

From Eq. (3) we obtain

$$\boldsymbol{\Phi}(\tau+2\pi) = \boldsymbol{\Phi}(\tau) \exp(2\pi \boldsymbol{W}) \tag{5}$$

Because $\boldsymbol{V}$ is a nonsingular matrix, there is the matrix $\boldsymbol{W}$ satisfying

$$\boldsymbol{V} = \exp(2\pi \boldsymbol{W}) \tag{6}$$

or

$$\boldsymbol{W} = \frac{1}{2\pi} \log \boldsymbol{V} \tag{7}$$

Therefore, if we can obtain the logarithm of $\boldsymbol{V}$, we can determine $\boldsymbol{L}(\tau)$ from Eq. (3).

3. Evaluation of log V

Let the eigenvalues of $\boldsymbol{V}$ be $\lambda_1, \lambda_2, \cdots, \lambda_s$. The function $f(\lambda)$ is assumed to be analytic with λ in the open interval of λ including $\lambda_1, \lambda_2, \cdots, \lambda_s$. Under this assumption, we can determine the function $f(\boldsymbol{V})$ by the fundamental formula of the matrix function derived from the Lagrange-Sylvester interpolation formula[1].

Let the minimal polynomial $\psi(\lambda)$ of $\boldsymbol{V}$ be

$$\begin{aligned} \psi(\lambda) &= (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_s)^{m_s} \\ & m_1+m_2+\cdots+m_s = m \quad (\leq n) \end{aligned} \tag{8}$$

where the integer m is the minimal degree of $\boldsymbol{V}$, and $m_i (i=1, \cdots, s)$ is the minimal multiplicity of λ_i. Then the matrix function $f(\boldsymbol{V})$ is expressed by

$$f(\boldsymbol{V}) = \sum_{i=1}^{s} \sum_{k=1}^{m_i-1} f^{(k)}(\lambda_i)\boldsymbol{Z}_{ik} \tag{9}$$

where $f^{(0)}(\lambda_i) \triangleq f(\lambda_i)$, and $f^{(k)}(\lambda_i) \triangleq \left.\frac{d^k f(\lambda)}{d\lambda^k}\right|_{\lambda=\lambda_i}$. The matrix $\boldsymbol{Z}_{ik}$ is the constituent matrix for $\boldsymbol{V}$, and its coefficient $f^{(k)}(\lambda_i)$ is the value of $f(\lambda)$ on the spectrum of $\boldsymbol{V}$. The matrix $\boldsymbol{Z}_{ik}$ depends not on $f(\lambda)$ but on $\boldsymbol{V}$. Therefore, it is important to determine $\boldsymbol{Z}_{ik}$ so as to be easily computed without using tedious programming techniques. In the following section, we show a method for determining $\boldsymbol{Z}_{ik}$ using linear independent test functions.

3–1 When V has a simple structure

In this case, we have $m=s$, $m_i=1$ $(i=1,\cdots,m)$. Let m linear independent test functions be $g_r(\lambda)$ $(r=1,\cdots,m)$. These functions are assumed to have the same property as $f(\lambda)$. Using the definition of the matrix function, we have

$$\boldsymbol{g}_r(\boldsymbol{V}) = \sum_{i=1}^{m} g_r(\lambda_i)\boldsymbol{Z}_{i0} \qquad r = 1, \cdots, m\,. \tag{10}$$

We must give the form of $g_r(\lambda)$ so that we can compute the value of $g_r(\boldsymbol{V})$ as easily as possible. Therefore we choose $g_r(\lambda)$ as

$$g_1(\lambda) = 1, \cdots, g_m(\lambda) = (\lambda-\lambda_1)\,(\lambda-\lambda_2)\cdots(\lambda-\lambda_{m-1})\,. \tag{11}$$

Then, from Eq. (10) we can obtain the simultaneous matrix equation with $\boldsymbol{Z}_{i0}$ $(i=1,\cdots,m)$. Solving it, we have

$$\boldsymbol{Z}^{\sharp} = (\boldsymbol{T}^{\sharp})^{-1}\boldsymbol{G}^{\sharp} \tag{12}$$

where

$$\boldsymbol{Z}^{\sharp} \triangleq \{\boldsymbol{Z}_{10}, \boldsymbol{Z}_{20}, \cdots, \boldsymbol{Z}_{m0}\}$$

$$\boldsymbol{G}^{\sharp} \triangleq \{\boldsymbol{1}, \boldsymbol{V}-\lambda_1\boldsymbol{1}, (\boldsymbol{V}-\lambda_1\boldsymbol{1})(\boldsymbol{V}-\lambda_2\boldsymbol{1}), \cdots, (\boldsymbol{V}-\lambda_1\boldsymbol{1})(\boldsymbol{V}-\lambda_2\boldsymbol{1})\cdots(\boldsymbol{V}-\lambda_{m-1}\boldsymbol{1})\}$$

$$\boldsymbol{T}^{\sharp} \triangleq \boldsymbol{T}\otimes\boldsymbol{1}$$

$$\boldsymbol{T} \triangleq \begin{pmatrix} 1 & 1 & \cdots\cdots & 1 \\ \cdot & \lambda_2-\lambda_1 & \cdots\cdots & \lambda_m-\lambda_1 \\ \cdot & \cdot & \ddots & \\ & & \ddots & \\ & & & \ddots \\ \cdot & \cdot & \cdots\cdots\ \cdot & (\lambda_m-\lambda_1)(\lambda_m-\lambda_2)\cdots(\lambda_m-\lambda_{m-1}) \end{pmatrix}.$$

Because $\boldsymbol{T}$ is the upper traingular matrix, $\boldsymbol{Z}^{\sharp}$ is easily obtained by the backward substitution.

3–2 When V doesn't have a simple structure

Using Eq. (9), we obtain

$$g_r(\boldsymbol{V}) = \sum_{i=1}^{s}\sum_{k=0}^{m_i-1} g_r^{(k)}(\lambda_i)\boldsymbol{Z}_{ik} \qquad r = 1, \cdots, s \tag{13}$$

where $g_r^{(0)}(\lambda_i) \triangleq g_r(\lambda_i)$, $g_r^{(k)}(\lambda_i) \triangleq \left.\dfrac{d^k g_r(\lambda)}{d\lambda^k}\right|_{\lambda=\lambda_i}$ and $g_r(\lambda)$ is given by the same form as Eq. (11). We can obtain the matrix $\boldsymbol{T}$ by the same processes as in the case of a simple structure. Because the expression of $\boldsymbol{T}$ is so complicated we do not express it. The constituent matrix $\boldsymbol{Z}_{ik}$ can be computed by the same procedures.

3–3 Expression of log V

We divide m eigenvalues into two sets. Let the set $\{\lambda_1, \cdots, \lambda_r\}$ be the set of real numbers and the set $\{\lambda_{r+1}, \lambda_{r+1}^*, \cdots, \lambda_\sigma, \lambda_\sigma^*\}$ be that of complex numbers, where the asterisk indicates the complex conjugate values. Let us put

$$\boldsymbol{Z}_{ik} = \frac{1}{2}\,(\boldsymbol{R}_{ik} - j\boldsymbol{X}_{ik})\,. \tag{14}$$

Then we have from Eq. (9)

$$\begin{aligned} f(\boldsymbol{V}) &= \sum_{i=1}^{r}\sum_{k=0}^{m_i-1} f^{(k)}(\lambda_i)\boldsymbol{Z}_{ik} \\ &\quad + \sum_{i=r+1}^{\sigma}\sum_{k=0}^{m_i-1}\{Re[f^{(k)}(\lambda_i)]\boldsymbol{R}_{ik} + \mathrm{Im}[f^{(k)}(\lambda_i)]\boldsymbol{X}_{ik}\}\,. \end{aligned} \tag{15}$$

If we express λ_i by the polar coordinate, we have

$$\lambda_i = r_i(\cos\varphi_i + j\sin\varphi_i) \qquad 0 \le \varphi_j < 2\pi\,. \tag{16}$$

Let us take the principal branch of $\log\lambda_i$. Then we have

$$\log\lambda_i = \log r_i + j\varphi_i\,. \tag{17}$$

Furthermore, we divide the set of real eigenvalues into two sets: the set $\{\lambda_1, \cdots, \lambda_p\}$ is defined as the set of positive eigenvalues, and the set $\{\lambda_{p+1}, \cdots, \lambda_r\}$ as that of negative eigenvalues. Then from Eq. (15) we have

$$\begin{aligned} \log\boldsymbol{V} &= \sum_{i=1}^{p}\left\{\log\lambda_i\boldsymbol{Z}_{i0} + \sum_{k=0}^{m_i-1}(k-1)!\left(-\frac{1}{\lambda_i}\right)^{k+1}\boldsymbol{Z}_{ik}\right\} \\ &\quad + \sum_{i=p+1}^{r}\left\{(\log|\lambda_i| + j\pi)\boldsymbol{Z}_{i0} + \sum_{k=1}^{m_i-1}(k-1)!\left(-\frac{1}{\lambda_i}\right)^{k+1}\boldsymbol{Z}_{ik}\right\} \\ &\quad + \sum_{i=r+1}^{\sigma}\left[\log r_i\boldsymbol{R}_{i0} + \varphi_i\boldsymbol{X}_{i0} + \sum_{k=1}^{m_i-1}(k-1)!\left\{\mathrm{Re}\left(-\frac{1}{\lambda_i}\right)^{k+1}\boldsymbol{R}_{ik} + \mathrm{Im}\left(-\frac{1}{\lambda_i}\right)^{k+1}\boldsymbol{X}_{ik}\right\}\right]. \end{aligned} \tag{18}$$

As is easily seen, if $\boldsymbol{V}$ has negative eigenvalues, $\log\boldsymbol{V}$ becomes the complex matrix.

4. Computation of $L(\tau)$

Using the technique of numerical integration such as the Runge-Kutter-Gill method, we can obtain the numerical solution of Eq. (2) in the interval $[0, 2\pi]$.

Let us denote the sequence of the numerical solution of Eq. (2) by $\{\boldsymbol{\Phi}(\tau_p)\}$, where $\tau_p=2\pi p/N\,(p=0,1,\cdots,N)$. From Eq. (4), we have $\boldsymbol{V}=\boldsymbol{\Phi}(2\pi)$. Therefore, we have

$$\boldsymbol{\Phi}(\tau_p)=\boldsymbol{L}(\tau_p)\exp\left(\frac{\tau_p}{2\pi}\log\boldsymbol{V}\right)\qquad p=0,1,\cdots,N-1 \tag{19}$$

Solving Eq. (19), we obtain

$$\begin{aligned}\boldsymbol{L}(\tau_p)&=\boldsymbol{\Phi}(\tau_p)\exp\left(-\frac{p}{N}\log\boldsymbol{V}\right)\\&=\boldsymbol{\Phi}(\tau_p)\,\boldsymbol{V}^{-p/N}\qquad\qquad p=0,1,\cdots,N-1\,.\end{aligned} \tag{20}$$

Because $\boldsymbol{V}^{-p/N}$ is the non-integer power of $\boldsymbol{V}$, we obtain numerically

$$\begin{aligned}\boldsymbol{V}^{-1/N}&=\exp\left(-\frac{1}{N}\log\boldsymbol{V}\right)\\&\simeq\sum_{k=0}^{K}\frac{1}{k!}\left(-\frac{1}{N}\log\boldsymbol{V}\right)^k.\end{aligned} \tag{21}$$

Therefore, $\boldsymbol{V}^{-p/N}$ is computed by

$$\boldsymbol{V}^{-p/N}=(\boldsymbol{V}^{-1/N})^p\qquad p=0,1,\cdots,N-1\,. \tag{22}$$

and the numerical sequence $\{\boldsymbol{L}(\tau_p)\}$ is given by

$$\boldsymbol{L}(\tau_p)=\boldsymbol{\Phi}(\tau_p)\,\boldsymbol{V}^{-p/N}\qquad p=0,1,\cdots,N-1\,. \tag{23}$$

Applying the discrete Fourier transform (abbreviated as DFT) to $\{\boldsymbol{L}(\tau_p)\}$ we can compute the harmonic components of $\boldsymbol{L}(\tau)$.

5. Algorithm

The above results lead us to the following algorithm.

Step 0 : Divide the period 2π of $\boldsymbol{P}(\tau)$ by N, and put $\tau_p=p\frac{2\pi}{N}$.

Step 1 : Compute the numerical sequence $\{\boldsymbol{\Phi}(\tau_p)\}$ for $p=0,1,\cdots,N$ by solving Eq. (2) in terms of a numerical integration.

Step 2 : Compute the eigenvalues of $\boldsymbol{V}=\boldsymbol{\Phi}(2\pi)$.
Comment: At this step, we can nvestigate the stability. If the absolute values of all eigenvalues are less than unity, the periodic system is asymptotically stable.

Step 3 : Compute the constituent matrix $\boldsymbol{Z}_{ik}$ by Eq. (12). Use the modified form of Eq. (12) if $\boldsymbol{V}$ does not have a simple structure.

Step 4 : Compute $\log\boldsymbol{V}$ by Eq. (18).

Step 5 : Compute $\boldsymbol{V}^{-1/N}$ by Eq. (21).

Step 6 : Compute the numerical sequence $\{V^{-p/N}\}$ by Eq. (22).
Step 7 : Compute the numerical sequence $\{L(\tau_p)\}$ by Eq. (23).
Step 8 : Compute the harmonic components of $L(\tau)$ by applying the DFT to the numerical sequence $\{L(\tau_p)\}$.
Step 9 : Stop.

6. Some applications

6–1 Application to Mathieu equation

To assure our method, we apply it to the Mathieu equation. We deal with the equation

$$\frac{d^2x}{d\tau^2}+(1-0.32\cos 2\tau)x = 0 \tag{24}$$

The parameters lie in an unstable region. The first solution is given by

$$x(\tau) = Ke^{\mu\tau}\sum_{r=1}^{\infty}\{C_{2r-1}\cos(2r-1)\tau+S_{2r-1}\sin(2r-1)\tau\} \tag{25}$$

where K is a constant. The second solution is obtained by changing the sign of τ in Eq. (25). Omitting the external multiplier, we show our results in Table 1 together with McLachlan's result[3]. Both are in good agreement. This result justifies our method.

Table 1. Numerical results. $N=2^5+1$

	McLachlan	Our method
μ	±0.08	±0.07976
S_1	0.94	0.94
S_3	-1.75×10^{-2}	-1.7558×10^{-2}
S_5	1.12×10^{-4}	1.1230×10^{-4}
C_1	1.0	1.0
C_3	-2.10×10^{-2}	-2.1072×10^{-2}
C_5	1.44×10^{-4}	1.4426×10^{-4}

6–2 Application to self-excitation of synchronous generator

We deal with a periodic system which describes the phenomenon of a synchronous generator connected to an unsymmetrical capacitive load. Under some assumptions, the periodic system can be written by

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\{\theta_0+2\sum_{k=1}^{n}\theta_k\cos 2k\tau\} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{26}$$

where $\theta_k=\theta_0\alpha^k$, $0<\alpha<1$. The parameters θ_0 and α are given by

$$\theta_0 = \frac{x_c}{2\sqrt{x_d x_q}}, \quad \alpha = \frac{\sqrt{x_d}-\sqrt{x_q}}{\sqrt{x_d}+\sqrt{x_q}} \tag{27}$$

where x_d, x_q and x_c are the generator reactances in the direct and quadrature axes and the reactance of the capacitor, respectively. The derivation of Eq. (26) is given in the Appendix. For the numerical case of $n=2$, we show the results for the stable and unstable parametric points in Table 2. Also, for the numerical cases of $n=2, \cdots, 5$ the regions of self-excitation are shown in Figs. 1–a to 1–d. The shaded area is a region of self-excitation. The dashed lines in the stable regions

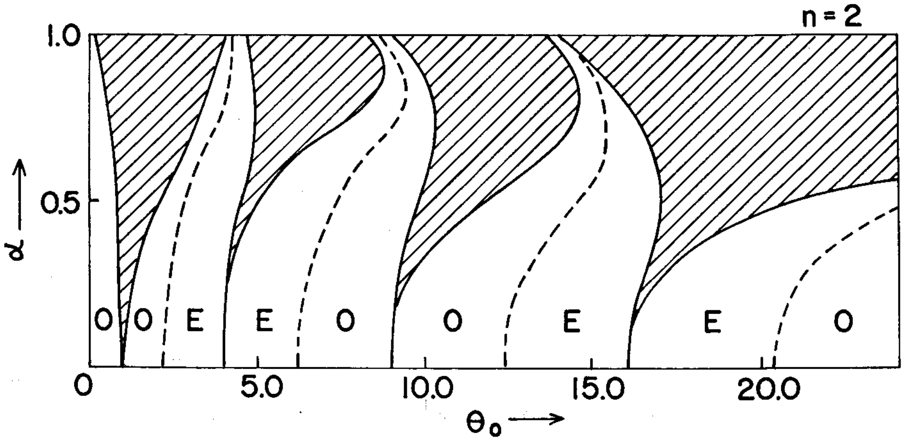
Table 2-a. Harmonic components of $\boldsymbol{L}(\tau)$. $\alpha=0.5$, $\theta_0=1.0$ (unstable point)

μ	$\boldsymbol{L}_1$	$\boldsymbol{L}_3$	$\boldsymbol{L}_5$
± 0.25740	$\begin{bmatrix} 0.45644 & -j0.51428 \\ j0.60070 & 0.40648 \end{bmatrix}$	$\begin{bmatrix} 0.03848 & -j0.02349 \\ j0.12202 & 0.06137 \end{bmatrix}$	$\begin{bmatrix} 0.00483 & -j0.00634 \\ j0.02594 & 0.03058 \end{bmatrix}$

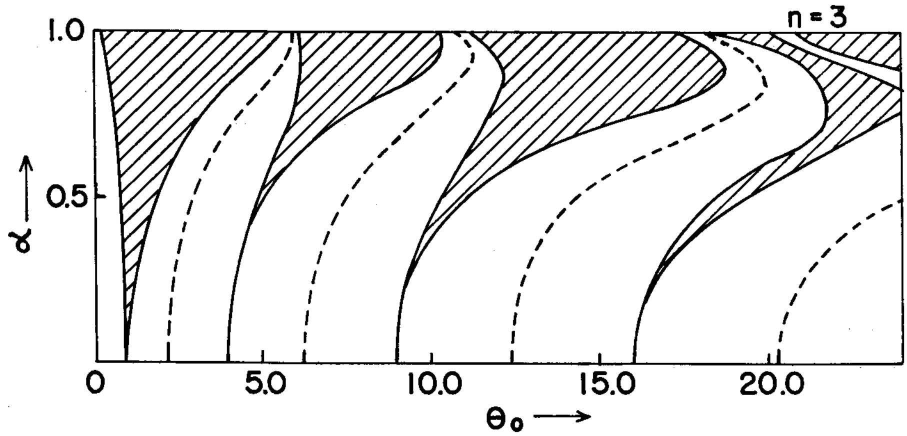
Table 2-b. Harmonic components of $\boldsymbol{L}(\tau)$. $\alpha=0.5$, $\theta_0=6.0$ (stable point)

μ	$\boldsymbol{L}_0$	$\boldsymbol{L}_2$	$\boldsymbol{L}_4$	$\boldsymbol{L}_6$
$\pm j0.24361$	$\begin{bmatrix} -0.82590 & 0.0 \\ 0.0 & -0.04826 \end{bmatrix}$	$\begin{bmatrix} 0.77211 & -j0.12781 \\ j1.6740 & 0.30074 \end{bmatrix}$	$\begin{bmatrix} 0.09126 & -j0.03681 \\ j0.40243 & 0.15257 \end{bmatrix}$	$\begin{bmatrix} 0.04439 & -j0.00916 \\ j0.27567 & 0.05756 \end{bmatrix}$

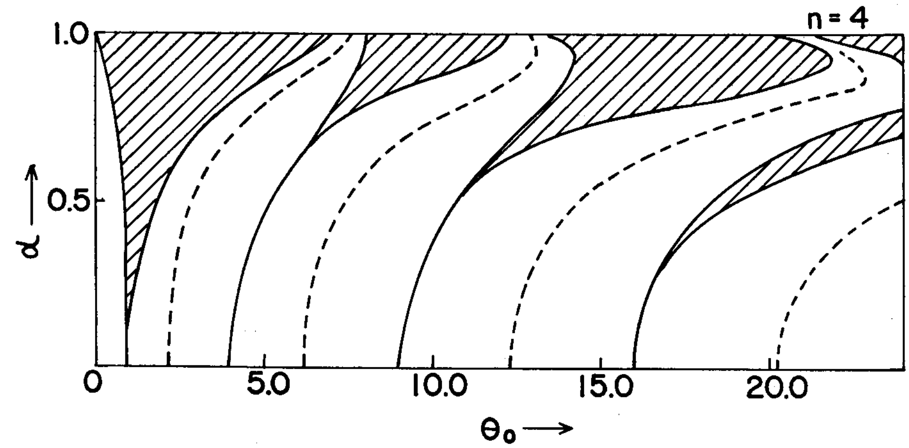
* $\boldsymbol{L}(\tau)=\boldsymbol{L}_0+\sum_{k=1}^{K}(\boldsymbol{L}_k e^{jk\tau}+\boldsymbol{L}_k^* e^{-jk\tau})$, $n=2$, $N=2^7+1$



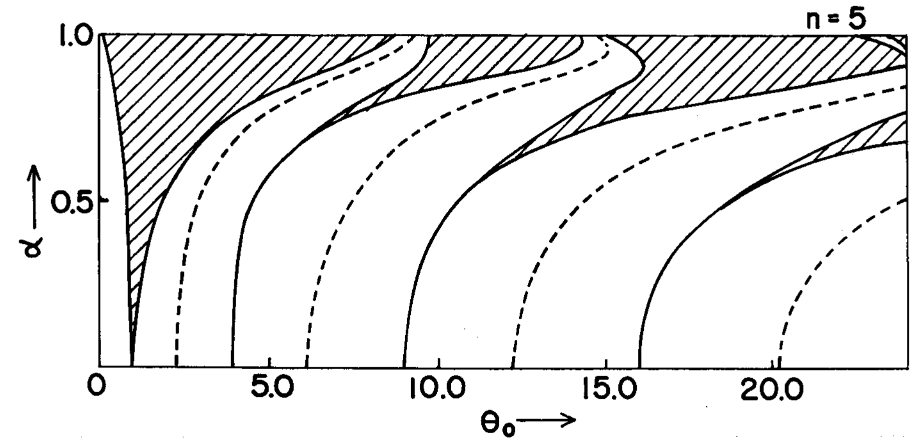
1–a.



1-b.



1-c.



1-d.

Fig. 1. The stability charts for a kind of Hill's equation.

demonstrate the boundaries which divide the even and odd harmonic components of $\boldsymbol{L}(\tau)$. The symbols O and E denote the odd and even harmonics, respectively. The above computations are carried out with double precision.

7. Conclusion

We have presented a method for obtaining the numerical solution of a periodic system of a higher order by a matrix function. When we applied it to some periodic systems, we had good numerical results. This method is based on the R.K.G-method. Therefore, a large storage of memories becomes necessary when it is applied to a system of a higher order, as the numbers of the sampling points increase. We also emphasize that the numerical sequence of the Liapunov matrix $\boldsymbol{L}(\tau)$ can be computed by this method.

For running the algorithm, we have used the computer FACOM–M200 at the Data Processing Center of Kyoto University. The authors wish to express their gratitude to Mr. Katsumi Sato, a student of Kyoto University, for his co-oporation with the numerical computation.

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Appendix

We assume that a three-phase synchronous generator is ideal[4]. The 0-α-β expression of the performance equations of the generator is effective when it is connected to an unsymmetrical load[5]. We normalize the parameters, variables and time. Here, the variables v, i, and ψ stand for the voltage, current and flux interlinkage, respectively.

The α, β components of the terminal voltages of the generators are given by

$$\left.\begin{aligned} v_\alpha &= p\psi_\alpha - ri_\alpha \\ v_\beta &= p\psi_\beta - ri_\beta \end{aligned}\right\} \tag{A.1}$$

where $p \triangleq \dfrac{d}{d\tau}$, and r is the resistance of the armature windings. The flux relations are given by

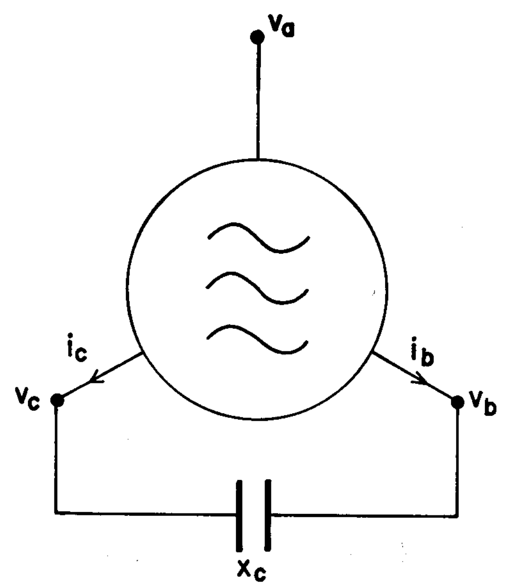

Fig. A1. The synchronous generator with unbalanced capacitive load.

$$\left.\begin{aligned}\psi_\alpha &= \cos\tau\cdot G(p)E-\{\cos\tau\, x_d(p)\cos\tau+\frac{1}{2}x_q(1-\cos 2\tau)\}i_\alpha\\ &\quad -\{\cos\tau\, x_d(p)\sin\tau-\frac{1}{2}x_q\sin 2\tau\}i_\beta\\ \psi_\beta &= \sin\tau\cdot G(p)E-\{\sin\tau\, x_d(p)\cos\tau-\frac{1}{2}x_q\sin 2\tau\}i_\alpha\\ &\quad -\{\sin\tau\, x_d(p)\sin\tau+\frac{1}{2}x_q(1+\cos 2\tau)\}i_\beta\end{aligned}\right\} \quad (A.2)$$

where $x_d(p)$, x_q and E represent the operational impedance in the direct axis, the reactance in the quadrature axis and the field voltage of the generator, respectively. $G(p)$ and $x_d(p)$ are operational functions given by

$$G(p)=\frac{1}{T'_{d0}p+1}\,,\quad x_d(p)=\frac{x'_dT'_{d0}p+x_d}{T'_{d0}p+1} \quad (A.3)$$

where T'_{d0}, x'_d and x_d are the time constant of the field circuit, the transient reactance in the direct axis and the synchronous reactance, respectively.

As shown in **Fig. A1**, the terminal conditions are given by

$$i_a=0\,,\quad i_b+i_c=0\,,\quad p(v_b-v_c)=x_ci_b\,. \quad (A.4)$$

These conditions are expressed by the $0-\alpha-\beta$ components

$$i_0 = i_\alpha = 0\,, \quad pv_\beta = \frac{1}{2}x_c i_\beta\,. \tag{A.5}$$

Therefore, from Eqs. (A.1) (A.2) and (A.5), we have

$$\left.\begin{aligned} p\psi_\beta &= v_\beta + r i_\beta \\ pv_\beta &= \frac{1}{2}x_c i_\beta \\ \psi_\beta &= \sin\tau\, G(p)E - \{\sin\tau\, x_d(p)\sin\tau + \frac{1}{2}x_q(1+\cos 2\tau)\} i_\beta\,. \end{aligned}\right\} \tag{A.6}$$

If we assume $x_d(p) = x_d = \text{const.}$, and $G(p)E = E = \text{const.}$, we have

$$\left\{(x_d+x_q) - (x_d-x_q)\cos 2\tau + 2r\frac{1}{p} + x_c\frac{1}{p^2}\right\} i_\beta = 2E\sin\tau\,. \tag{A.7}$$

Here, we define a new variable by

$$v \triangleq \{(x_d+x_q) - (x_d-x_q)\cos 2\tau\} i_\beta\,. \tag{A.8}$$

Therefore, we have

$$\begin{aligned} i_\beta &= \frac{1}{(x_d+x_q)-(x_d-x_q)\cos 2\tau}\, v \\ &= \frac{1}{x_d+x_q}\,\frac{1+\alpha^2}{1-\alpha^2}\{1+2\sum_{n=1}^{\infty}\alpha^n\cos 2n\tau\} v \\ \alpha &= (\sqrt{x_d}-\sqrt{x_q})\,/\,(\sqrt{x_d}+\sqrt{x_q})\,. \end{aligned} \tag{A.8}$$

Then, Eq. (A.7) can be written as

$$\begin{aligned} &\frac{d^2v}{d\tau^2} + \delta\frac{d}{d\tau}\{1+2\sum_{m=1}^{\infty}\alpha^m\cos 2m\tau\} v \\ &\quad + k^2\{1+2\sum_{m=1}^{\infty}\alpha^m\cos 2m\tau\} v = -2E\sin\tau \\ &\delta = \frac{r}{\sqrt{x_d x_q}}\,, \quad k^2 = \frac{x_c}{2\sqrt{x_d x_q}}\,. \end{aligned} \tag{A.9}$$

If r is negligibly small, the homogeneous equation associated with Eq. (A.9) becomes a kind of Hill's equation

$$\frac{d^2v}{d\tau^2} + (\theta_0 + 2\sum_{n=1}^{\infty}\theta_n\cos 2n\tau) v = 0 \tag{A.10}$$

where $\theta_0 = k^2$, $\theta_n = k^2\alpha^n$.