Optimal Unit Allocation Policies of Standby Systems

By

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Abstract

In this paper, we consider optimal allocation policies for standby systems in which each unit stochastically fails. When all the units in the system fail, a system failure occurs. Each time the system fails, we determine the optimal number of units in the system to be next used so as to minimize the total expected replacement cost over a finite time horizon. The problems are formulated by dynamic programming, and are solved by a successive approximation method. Moreover, some properties of optimal policies are derived. That is, the optimal number of units of the system decreases in the remaining time, and there exists a critical point in time after which we leave the failed system.

1. Introduction

Since R.E. Barlow and F. Proschan [1] introduced the concept of age replacement, much effort has been devoted to the study of replacement problems for stochastically failing systems. C. Derman, G.J. Lieberman and S.M. Ross [7] analyzed the optimal sequential allocation problem as follows. There are n types of replacement units. The problem is to assign the initial unit and subsequent replacements from among the n types so as to minimize the total expected cost of providing an operative unit for t units of time. Y. Tabata and T. Nishida [9] introduced the concept of idle time. They discussed sequential unit allocation problems for a m-unit parallel system, and gave some properties of an optimal policy when the failure time distribution of the unit is exponential.

In this paper, two kinds of optimal replacement problems for a cold standby system with m units are discussed. First, we consider a m-unit cold standby system in which each unit has an identical failure time distribution. Only one unit is in an operating state and the others are in a standby state. If the unit in the operating state fails, a unit in the standby state takes over the operation. Here we assume that the switch-over is perfect, that is, the switchover consumes zero time.

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When all units in the standby system fail, we say that a systemdown has occurred or the system fails. For such a system, we consider two cases.

Case 1 ; We start the operation of the system at time 0, and we must operate for T units of time, that is, whenever a systemdown occurs, we must replace the failed system with a new one.

Case 2 ; We have a planning period of length T. We start the operation of the system at time 0. When a systemdown occurs, we can leave the failed system until the end of the planning time.

In case 1, we select an action from the actions mentioned below, considering the remaining planning time.

Action i ; We purchase an i-unit system and replace the failed system by the i-unit system.

The times of purchasing and replacement are assumed to be neglected. We must pay the purchasing cost of the new system, which increases in proportion to the number of the units in the new system. Also, a replacement cost is incurred.

In case 2, we consider the system with the same structures as were mentioned before. Now we start the operation of this system at time 0. When a systemdown occurs, that is, when all units of the system fail, we can select one of the following actions, considering the remaining time.

Action i ; We purchase an i-unit system and replace the failed system by the i-unit system.

Action 0 ; We do not purchase a new system, and leave the failed system until the end of the planning time.

When we choose action i, we assume that the replacement time for a new system can be neglected, and must pay the purchasing cost and replacement cost. When we select action 0, we must pay the penalty cost per unit time due to the loss of production.

In both cases, our purpose is to determine an optimal allocatilon policy that minimizes a total expected cost over a planning time. The problems are formulated by dynamic programming, and an optimal policy is derived. Moreover, some properties of a near optimal policy are obtained for each case.

2. Notations

For the behavior of the standby system, we define the following notations.

$F(x)$: failure time distribution function of a unit.

We assume that the failure time of each unit is exponentially distributed with expected time $1/\lambda$, namely

$F(x) = 1-e^{-\lambda x}$.

$f(x)$: density function of $F(x)$,

$F_i(x)$: failure time distribution function of an i-unit standby system. This is easily given by

$$F_i(x) = \int_0^x \frac{\lambda^i}{(i-1)!} e^{-\lambda t} t^{i-1} dt .$$

$\bar{F}_i(x)$: $1-F_i(x)$,

$f_i(x)$: density function of $F_i(x)$

In our system, we consider the following costs:

K : Purchasing cost for one unit.

L : Peplacement cost.

c : Penalty cost per unit time incurred when we leave the failed system as it is.
(This notation is used only in case 2.)

As for an optimal allocation policy, we introduce the following quantity.

$V(x)$: The optimal total expected cost when x units of time remain until the end of the planning time.

3. An Optimal Policy in Case 1

In this section, we discuss an optimal policy when we must operate the system for T units of time. By the principle of optimality, the minimum cost $V(x)$ satisfies the following equations:

$$V(x) = \min_i H^i(x) . \tag{1}$$

$$H^i(x) = L+iK+\int_0^x f_i(t) V(x-t)dt . \tag{2}$$

$H^i(x)$ is the expected total cost over time x when we replace the failed system with a new standby system with i units, and adopt an optimal policy from that time on.

We solve these equations by a successive approximation method. That is, we consider the following recursive equations:

$$V_n(x) = \min_i H_n^i(x) . \tag{3}$$

$$H_n^i(x) = L+iK+\int_0^x f_i(t) V_{n-1}(x-t)dt . \tag{4}$$

$$V_0(x) = L+K+\lambda x(L+K) . \tag{5}$$

It is easily shown that $\lim_{n\to\infty} V_n(x) = V(x)$, and the proof is omitted.

From here, we investigate the characteristics of each $H_n^i(x)$ and $V_n(x)$.

Lemma 1. $H_n^i(x)$ is a non-decreasing function in x.

Proof. When $n=1$, we see that

$$\begin{aligned}H_1^i(x+y)-H_1^i(x) &= \int_0^{x+y} f_i(t)\{L+K+\lambda(x+y-t)(L+K)\}dt \\ &\quad -\int_0^x f_i(t)\{L+K+\lambda(x-t)(L+K)\}dt \\ &= (L+K)\{F_i(x+y)-F_i(x)\} \\ &\quad +(L+K)\int_x^{x+y} F_i(t)dt \geq 0\,.\end{aligned}$$

Consequently, $V_1(x)=\min_i H_1^i(x)$ is a non-negative non-decreasing function in x. Suppose that $V_k(k)$ is a non-negative and non-decreasing function in x, and for $n=k+1$, we show that

$$\begin{aligned}H_{k+1}^i(x+y)-H_{k+1}^i(x) &= \int_0^{x+y} f_i(t)V_k(kx+y-t)dt \\ &\quad -\int_0^x f_i(t)V_k(x-t)dt \\ &= \int_0^x f_i(t)\{V_k(x+y-t)-V_k(x-t)\}dt \\ &\quad +\int_x^{x+y} f_i(t)V_k(x+y-t)dt \geq 0\,.\end{aligned}$$

Thus, $V_{k+1}(x)$ is also a non-decreasing function in x. Repeating the same arguement, we can prove the lemma for all n.

Lemma 2. $$H_n^1(x)-L-V_{n-1}(x) \leq -L\bar{F}(x)\,. \tag{6}$$

Proof. The proof is done through two steps by a mathematical induction method.

(i) If $V_{n-1}(x)=H_{n-1}^i(x)$ for $2\leqq i\leqq m$, we see

$$\begin{aligned}H_n^1(x)-L-V_{n-1}(x) &= H_n^i(x)-L-H_{n-1}^i(x) \\ &= -L-(i-1)K+\int_0^x f(x-t)V_{n-1}(t)dt \\ &\quad -\int_0^x f_i(x-t)V_{n-2}(t)dt \\ &= -\{L+(i-1)K\}\bar{F}(x) \\ &\quad +\int_0^x f(x-t)\{V_{n-1}(t)-H_{n-1}^{i-1}(t)\}dt \\ &\leq -L\bar{F}(x)\,.\end{aligned}$$

(ii) If $V_{n-1}(x)=H_{n-1}^{i}(x)$,

$$\begin{aligned} H_n^1(x)-L-V_{n-1}(x) &= H_n^1(x)-L-H_{n-1}^{i}(x) \\ &= -L+\int_0^x f(x-t)\{V_{n-1}(t)-V_{n-2}(t)\}d \\ &\leq -L\bar{F}(x)\,. \end{aligned}$$

From (i) and (ii), the proof is done.

Lemma 3. For each n, $H_n^2(x)-H_n^1(x)$ changes its sign at most once in x, and if a change occurs, it is from positive.

Proof. We see that

$$H_n^2-H_n^1(x) = K\bar{F}(x)+\int_0^x f(x-t)\{H_n^1(t)-V_{n-1}(t)-L\}dt\,.$$

Substituting $f(t)=\lambda e^{-\lambda t}$ and $\bar{F}(t)=e^{-\lambda t}$, we have

$$e^{\lambda x}\{H_n^2(x)-H_n^1(x)\} = K+\int_0^x \lambda e^{\lambda t}\{H_n^1(t)-V_{n-1}(t)-L\}\,dt\,. \qquad (7)$$

Considering lemma 2, the right hand side of equation (7) is less than $K-\lambda Lx$. Therefore, the proof is over.

Theorem 4. For each n, $i\geqq 2$, $H_n^i(x)-H_n^{i-1}(x)$ changes its sign at most once in x, and if a change occurs, it is from positive.

Proof. We see that,

$$\begin{aligned} H_n^i(x) &= L+iK+\int_0^x f_i(x-t)V_{n-1}(t)dt \\ &= L+iK-\{L+(i-1)K\}\bar{F}(x)+\int_0^x f(x-t)H_n^{i-1}(t)dt\,. \end{aligned}$$

Hence, the following equation holds as in the proof of lemma 3.

$$e^{\lambda x}\{H_n^i(x)-H_n^{i-1}(x)\} = K+\int_0^x \lambda e^{\lambda t}\{H_n^{i-1}(t)-H_n^{i-2}(t)\}dt\,. \qquad (8)$$

Repeating the result of lemma 3, the sign of the integrand of the second term of the right hand side in equation (8) is either positive for all $0\leq x\leq T$. Or, it changes its sign only one time, and it is from positive to negative. In this way we can prove the theorem for all i.

From lemma 3 and theorem 4, we have the following theorems.

Theorem 5. Assume that $H_n^{i+1}(x)$ crosses $H_n^i(x)$ at a point x_n^i, then $H_n^{j+1}(x)$ crosses $H_n^j(x)$ for all $j>i$, and the following inequality holds,

$$x_n^1<x_n^2< \cdots <x_n^i < \cdots\,. \qquad (9)$$

Proof.

$$e^{\lambda x}\{H_n^{i+1}(x)-H_n^i(x)\} = K+\int_0^x \lambda e^{\lambda t}\{H_n^i(t)-H_n^{i-1}(t)\}dt. \tag{10}$$

Assume $H_n^{i+1}(x)$ crosses $H_n^i(x)$, and $H_n^i(x)$ does not cross $H_n^{i-1}(x)$. The right hand side of equation (10) is positive and the left hand side of equation (10) is negative. This is a contradiction. Consequently, inequality (9) holds.

Theorem 6. Assume $H_n^{i+1}(x)$ does not cross $H_n^i(x)$, that is, $H_n^{i+1}(x) \geq H_n^i(x)$ holds for all $0 \leq x \leq T$. Then for all $j > i$, $H_n^{j+1}(x)$ does not cross $H_n^j(x)$.

Proof. The proof is easily done as in the proof of theorem 5. From theorems 5 and 6, we can obtain a near optimal policy of the form,

$$\begin{bmatrix} \text{select action } 1 & 0 \leq x \leq x_n^1, \\ \text{select action } 2 & x_n^1 \leq x \leq {}_n^2, \\ \vdots & \vdots \\ \text{select action } i & x_n^{i-1} \leq x \leq T. \end{bmatrix}$$

4. An Optimal Policy in Case 2

In this section, we treat an optimal replacement problem which includes a concept of idle time. By the principle of optimality, the minimum cost $V(x)$ satisfies the following equations:

$$V(x) = \min \begin{cases} cx \\ \min_i H_n^i(x). \end{cases} \tag{11}$$

$$H^i(x) = L+iK+\int_0^x f_i(t)V(x-t)dt. \tag{12}$$

cx is incurred when we leave the failed system until the end of the planning time T. $H^i(x)$ is the expected total cost over time x, when we replace the failed system with a new standby system with i units and adopt an optimal policy from that time on.

We consider the following recursive equations.

$$V_n(x) = \min \begin{cases} cx \\ \min_i H_n^i(x). \end{cases} \tag{13}$$

$$H_n^i(x) = L+iK+\int_0^x f_i(t)V_{n-1}(x-t)dt. \tag{14}$$

$$V_0(x) = cx. \tag{15}$$

It is easily shown that $\lim_{n\to\infty} V_n(x) = V(x)$, and the proof is omitted.

Next, we investigate the characteristics of each $H_n^i(x)$ and $V_n(x)$. For the purpose, we have the following lemmas.

Lemma 7.
$$0 < H_n^i(x+y) - H_n^i(x) < cy . \qquad (16)$$
$$0 \leq x \leq T;\ y > 0 .$$

Proof. The proof is done by a mathematical induction method. For $n=1$,

$$\begin{aligned} H_1^i(x+y) - H_1^i(x) &= \int_0^{x+y} f_i(t) c(x+y-t)dt - \int_0^x f_i(t) c(x-t)dt \\ &= \int_x^{x+y} cF_i(t)dt . \end{aligned}$$

Hence, it is clear that $0 < \int_x^{x+y} cF_i(t)dt < c\int_x^{x+y} dt = dy$, which implies that inequality (16) holds. For $n=k$, we assume that inequality (16) holds. Then, for $n=k+1$, we can show the following inequality:

$$V_k(x+y-t) \leq \begin{cases} V_k(x-t) + cy & 0 \leq t \leq x \\ c(x+y-t) . & x < t \leq x+y . \end{cases} \qquad (17)$$

Considering inequality (17),

$$\begin{aligned} \text{(i)} \quad H_{k+1}^i(x+y) - H_{k+1}^i(x) &= \int_0^{x+y} f_i(t) V_k(x+y-t)dt - \int_0^x f_i(t) V_k(x-t)dt \\ &\leq \int_0^x f_i(t) \{V_k(x-t) + cy\} dt \\ &\quad + \int_x^{x+y} f_i(t) c(x+y-t)dt - \int_0^x f_i(t) V_k(x-t)dt \\ &= c\int_x^{x+y} F_i(t)dt < c\int_x^{x+y} dt = cy . \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad H_{k+1}^i(x+y) - H_{k+1}^i(x) &= \int_0^{x+y} f_i(t) V_k(x+y-t)dt - \int_0^x f_i(t) V_k(x-t)dt \\ &= \int_0^x f_i(t) \{V_k(x+y-t) - V_k(x-t)\} dt \\ &\quad + \int_x^{x+y} f_i(t) V_k(x+y-t)dt \geq 0 . \end{aligned}$$

From (i) and (ii), inequality (16) holds for $n=k+1$. That is, $H_n^i(x)$ are non-decreasing functions in x for all n and i. The proof is over.

Lemma 8. For each n and i, $H_n^i(x) - cx$ changes its sign at most once in x, and if a change occurs, it is from positive.

Proof. We put $g_n^i(x) = H_n^i(x) - cx$. Clearly, it holds that $g_n^i(0) = L + iK > 0$.

From lemma 7, we obtain that

$$g_n^i(x+y)-g_n^i(x) = \{H_n^i(x+y)-H_n^i(x)\}-cy<0\,.$$

From (i) and (ii), we can show $g_n^i(x)$ is a non-increasing function in x and is positive when $x=0$, which implies that the lemma holds.

Lemma 9. $H_n^1(x)-L-V_{n-1}(x)$ changes its sign at most once, and if a change occurs, it is from positive.

Proof. For $n=1$, we see that

$$H_1^1(x)-L-V_0(x) = K-c\int_0^x \bar{F}(t)dt\,.$$

Therefore, lemma 9 holds for $n=1$. Then, for $n\geqq 2$:

(i) If $V_{n-1}(x)=cx$, then we see

$$H_n^1(x)-L-V_{n-1}(x) = H_n^1(x)-L-cx = K-c\int_0^x \bar{F}(t)dt\,.$$

(ii) If $V_{n-1}(x)=H_{n-1}^1(x)$, then

$$\begin{aligned} H_n^1(x)-L-V_{n-1}(x) &= H_n^1(x)-L-H_{n-1}^1(x) \\ &= -L+\int_0^x f(x-t)\{V_{n-1}(t)-V_{n-2}(t)\}dt\leq 0\,. \end{aligned}$$

(iii) If $V_{n-1}(x)=H_{n-1}^i(x)$, $i\geq 2$, then

$$\begin{aligned} H_n^1(x)-L-V_{n-1}(x) &= H_n^1(x)-L-H_{n-1}^i(x) = -\{L+(i-1)K\}\bar{F}(x) \\ &\quad +\int_0^x f(x-t)\{V_n(t)-H_{n-1}^{i-1}(t)\}dt\leq 0\,. \end{aligned}$$

From (i), (ii), (iii) and lemma 8, lemma 9 holds for $n\geq 2$. The proof is over.

Lemma 10. For each n, $H_n^2(x)-H_n^1(x)$ changes its sign at most once, and if a change occurs, it is from positive.

Proof. We see that

$$H_n^2(x)-H_n^1(x) = K\bar{F}(x)+\int_0^x f(x-t)\{H_n^1(t)-V_{n-1}(t)-L\}dt\,.$$

Substituting $f(t)=\lambda e^{-\lambda t}$ and $\bar{F}(t)=e^{-\lambda t}$, we see that

$$e^{\lambda x}\{H_n^1(x)-H_n^2(x)\} = K+\int_0^x \lambda e^{\lambda t}\{H_n^1(t)-L-V_{n-1}(t)\}dt\,. \tag{18}$$

From lemma 9, the sign of the integrand in equation (18) changes at most once, and if a chenge occurs, it is from positive, which implies that the lemma holds.

Theorem 11. For each n, i, $H^{i+1}_n(x)-H^i_n(x)$ changes its sign at most once, and if a change occurs, it is from positive.

Proof. We see that

$$H^i_n(x) = L+iK+\int_0^x f_i(x-t)V_{n-1}(t)dt$$
$$= L+iK-\{L+(i-1)K\}F(x)+\int_0^x f(x-t)H^{i-1}_n(t)dt\,.$$

Therefore,

$$H^{i+1}_n(x)-H^i_n(x) = KF(x)+\int_0^x f(x-t)\{H^i_n(t)-H^{i-1}_n(t)\}dt\,.$$

Substituting $f(t)=\lambda e^{-\lambda t}$ and $F(t)=e^{-\lambda t}$, we see that

$$e^{\lambda x}\{H^{i+1}_n(x)-H^i_n(x)\} = K+\int_0^x \lambda e^{\lambda t}\{H^i_n(t)-H^{i-1}_n(t)\}dt\,. \tag{19}$$

Here, repeating the same arguments as in lemma 10, the sign of the integrand in equation (19) changes at most once, and if a change occurs, it is from positive. Hence, the sign of the left hand (19) has the same property, which implies that the lemma holds.

Theorem 12. If $H^{i+1}_n(x)$ does not cross $H^i_n(x)$, then $H^{j+1}_n(x)$ does not cross $H^j_n(x)$ for $j>i$.

Proof. If $H^{i+1}_n(x)$ does not cross $H^i_n(x)$, then it holds that $H^{i+1}_n(x)\geq H^i_n(x)$, from theorem 11. Considering equation (19), we show that $H^{j+1}_n(x)>H^j_n(x)$, for all $j>i$. The proof is over.

Theorem 13. Suppose $H^{i+1}_n(x)$ crosses $H^i_n(x)$ at x^i_n. Then, the following relationship between each x^i_n holds;

$$x^1_n<x^2_n<\cdots<x^i_n<\cdots\,. \tag{20}$$

Proof. From theorems 11 and 12, we can easily show inequality (20). From lemma 8 and theorem 13, we can obtain a near optimal policy of the form.

(i) If $\lambda(L+K)\geq c$,
it is optimal to take action 0 whenever a systemdown occurs.

(ii) If $\lambda(L+K)<c$,

$$\begin{bmatrix} \text{select action } 0 & 0 \leq x\leq x^*_n\,, \\ \text{select action } i & x^*_n \leq x\leq x^i_n\,, \\ \vdots & \vdots \\ \text{select action } j & x^{j-1}_n\leq x\leq T\,. \end{bmatrix} \qquad 1\leq i\leq j\leq m\,.$$

5. Conclusion

In this paper, we consider optimal allocation policies for standby systems in which each unit stohcastically fails. In both cases 1 and 2, the problems are formulated by dynamic programming and are solved by a successive approximation method. We can conclude that the optimal number of units of the system, which minimizes the total expected cost, decreases as the remaining time decreases, and in case 2, there exists a critical time point after which we should leave the failed system.

Here, we assume that each unit fails exponentially. However, there are many cases where their failure time distributions are general. We sohuld study an optimal allocation problem in these cases, which will be postponed to a future opportunity.

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